ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 39, Number 5, 2009

A NEW FAMILY OF CURVATURE HOMOGENEOUS PSEUDO-RIEMANNIAN MANIFOLDS

COREY DUNN

ABSTRACT. We construct a new family of curvature homogeneous pseudo-Riemannian manifolds modeled on $\mathbf{R}^{3k+2}$ for integers $k \geq 1$. In contrast to previously known examples, the signature may be chosen to be $(k+1+a, k+1+b)$ where $a, b \in \mathbf{N} \cup \{0\}$ and $a+b=k$. The structure group of the 0-model of this family is studied, and is shown to be indecomposable. Several invariants that are not of Weyl type are found which will show that, in general, the members of this family are not locally homogeneous.

1. Introduction.

Let (M, g) be a smooth pseudo-Riemannian manifold of signature (p, q), and let $P \in M$. Using the Levi-Civita connection ∇, one can compute the Riemann curvature tensor $R \in \otimes^4 T_P^* M$ as follows:

$$R(X,Y,Z,W) := g(\nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z, W),$$
$$\text{for } X, Y, Z, W \in T_P M.$$

One similarly defines the tensors $\nabla^i R$, for $i = 0, 1, 2, \ldots$. For convenience, we write $\nabla^0 R = R$. Let g_P, R_P and $\nabla^i R_P$ denote the evaluation of these tensors at the point P.

The manifold (M, g) is *r-curvature homogeneous* if, for all points $P, Q \in M$ and $i = 0, 1, \ldots, r$, there exists a linear isomorphism $\Phi_{PQ} : T_P M \to T_Q M$ so that $\Phi_{PQ}^* g_Q = g_P$ and $\Phi_{PQ}^* \nabla^i R_Q = \nabla^i R_P$.

There is an equivalent characterization of r-curvature homogeneous manifolds that will be of use. Let V be a finite-dimensional real vector space, let the dual vector space $V^* := \mathrm{Hom}_{\mathbf{R}}(V, \mathbf{R})$, and let $(\cdot, \cdot)$ be a

2000 AMS *Mathematics subject classification.* Primary 53C50, Secondary 53C21, 53B30.
Keywords and phrases. Curvature homogeneous, pseudo-Riemannian, isometry invariants.
The research of the author is partially supported by a CSUSB faculty research grant.
Received by the editors on January 16, 2007, and in revised form on April 5, 2007.

DOI:10.1216/RMJ-2009-39-5-1443

symmetric nondegenerate inner product on V. An element $A^0 \in \otimes^4 V^*$ is called an *algebraic curvature tensor* on V if it satisfies the following three properties for all $v_1, \dots, v_4 \in V$:

$$\begin{aligned} A^0(v_1, v_2, v_3, v_4) &= -A^0(v_2, v_1, v_3, v_4), \\ A^0(v_1, v_2, v_3, v_4) &= A^0(v_3, v_4, v_1, v_2), \text{ and} \\ 0 &= A^0(v_1, v_2, v_3, v_4) + A^0(v_2, v_3, v_1, v_4) \\ &\quad + A^0(v_3, v_1, v_2, v_4). \end{aligned}$$

An element $A^1 \in \otimes^5 V^*$ is called an *algebraic covariant derivative curvature tensor* on V if it satisfies the following four properties for all $v_1, \dots, v_5 \in V$:

$$\begin{aligned} A^1(v_1, v_2, v_3, v_4; v_5) &= -A^1(v_2, v_1, v_3, v_4; v_5), \\ A^1(v_1, v_2, v_3, v_4; v_5) &= A^1(v_3, v_4, v_1, v_2; v_5), \\ 0 &= A^1(v_1, v_2, v_3, v_4; v_5) + A^1(v_2, v_3, v_1, v_4; v_5) \\ &\quad + A^1(v_3, v_1, v_2, v_4; v_5), \\ 0 &= A^1(v_1, v_2, v_3, v_4; v_5) + A^1(v_1, v_2, v_4, v_5; v_1) \\ &\quad + A^1(v_1, v_2, v_5, v_1; v_4). \end{aligned}$$

Let $A^i \in \otimes^{4+i} V^*$ for $i = 2, 3, \dots, r$. The tensors A^0 and A^1 are algebraic analogues of R and ∇R. The symmetries of the tensors $\nabla^2 R$, $\nabla^3 R, \dots$ are more difficult to express and are not relevant to our discussion. Thus, we will not impose any restrictions on the tensors A^i for $i = 2, 3, \dots, r$. We define an *r-model* to be a tuple $\mathcal{V}_r := (V, (\cdot,\cdot), A^0, \dots, A^r)$. A *weak r-model* is an r-model without the bilinear form. Thus, a pseudo-Riemannian manifold (M, g) is r-curvature homogeneous if and only if, for each $P \in M$ there exists a linear isometry $\Phi_P : T_P M \to V$, with $\Phi_P^* A^i = \nabla^i R_P$ for $i = 0, 1, \dots, r$. In such an event we say that (M, g) is r-modeled on $\mathcal{V}_r$, or that $\mathcal{V}_r$ is an r-model for (M, g). The *structure group* $\mathcal{G}_{\mathcal{V},r}$ of the r-model $\mathcal{V}_r$ is the group of isomorphisms of $\mathcal{V}_r$. For an r-curvature homogeneous space, this group is independent of P.

It is clear that a locally homogeneous manifold is r-curvature homogeneous for all r. The converse, however, is not always true: There exist pseudo-Riemannian manifolds which are r-curvature homogeneous for some r, and not (locally) homogeneous. The study of curvature homogeneity in the Riemannian setting began with a paper by Singer [**27**] in 1960. His result was extended by Podesta and Spiro [**22**] to the pseudo-Riemannian setting in 1996:

Theorem 1.1. *Let (M, g) be a smooth, simply connected, complete manifold of dimension n.*

(1) [**27**]. *If (M, g) is Riemannian, then there exists an integer $k_{0,n}$ so that if (M, g) is $k_{0,n}$-curvature homogeneous, then it is homogeneous.*

(2) [**22**]. *If (M, g) is a pseudo-Riemannian manifold of signature (p, q), then there exists an integer $k_{p,q}$ so that if (M, g) is $k_{p,q}$-curvature homogeneous then it is homogeneous.*

Since then, many authors have studied curvature homogeneous manifolds, both in the Riemannian and higher signature settings; indeed, the list of references is becoming quite large and we only summarize the results pertinent to our goal (for more details see [**1, 10**]). Opozda [**21**] has obtained a result similar to Theorem 1.1 in the affine case.

In the Riemannian setting, it is clear that $k_{0,2} = 0$, and the efforts of Gromov [**18**] and Yamato [**30**] have established bounds on $k_{0,n}$ which are linear in n. The work of Sekigawa, Suga and Vanhecke [**25, 26**] shows $k_{0,3} = k_{0,4} = 1$. There are examples of 0-curvature homogeneous Riemannian manifolds which are not locally homogeneous, see [**8, 19, 28**]. There are no known examples of 1-curvature homogeneous Riemannian manifolds which are not locally homogeneous.

In the pseudo-Riemannian setting, the situation is somewhat similar. There are many known examples of 0-curvature homogeneous pseudo-Riemannian manifolds which are not locally homogeneous, see for example [**2, 12**] in the Lorentzian setting and [**6, 13, 15, 17**] in the higher signature setting. It is clear that $k_{1,1} = 0$. The work of Bueken and Djorić [**3**] and the work of Bueken and Vanhecke [**4**] show that $k_{1,2} \geq 2$, while the work in [**7**] shows $k_{2,2} \geq 2$. Derdzinski [**5**] has also studied isometry invariants in signature $(2, 2)$. In contrast to the Riemannian setting, however, examples exist of higher curvature homogeneity in the higher signature setting. For instance, examples constructed by Gilkey and Nikčević [**15**] show that balanced signature pseudo-Riemannian manifolds exist which are r-curvature homogeneous and not locally homogeneous for any r (although the dimension of these manifolds is roughly twice r). With exception to the remarkable four-dimensional

Lorentzian manifolds recently found in [**20**], if $m := \min\{p, q\}$, then there are no known examples of $(m+1)$-curvature homogeneous manifolds of signature (p, q) which are not locally homogeneous. These considerations have led Gilkey to conjecture [**16**] that $k_{p,q} = m + 1$. The authors in [**20**] produce an exceptional family of four-dimensional Lorentzian manifolds and show that $k_{1,3} = 3$.

With exception to [**20**], the examples in the higher signature setting above were not originally constructed for the study of curvature homogeneity, and this leads us to a motivation for this study. In fact, the manifolds in [**6, 7, 15**] appeared in [**11**], and the manifolds in [**13**] appeared in [**14**]; they were used as counterexamples to the Osserman conjecture [**9, 24**] in the higher signature setting. As a result, the known examples have very rigid signatures. The manifolds in [**6, 7**] have balanced signature, and the manifolds in [**14**] have signature $(2s, s)$ for $s \geq 1$. It is the aim of this article to provide examples in the higher signature setting of a more arbitrary signature.

The following is an example of a 0-model that will be central to our discussion.

Definition 1.2. Let $k \geq 1$ be an integer, and choose $a, b \in \mathbf{N} \cup \{0\}$ so that $a + b = k$. Let ε_i be a choice of signs. Let $\{U_0, \dots, U_k, V_0, \dots, V_k, S_1, \dots, S_k\}$ be a basis for $\mathbf{R}^{3k+2}$. For $i = 1, \dots, k$, we define the nonzero entries of a symmetric nondegenerate bilinear form $(\cdot, \cdot)$ and algebraic curvature tensor R on the basis above as:

$$\begin{gathered}(U_0, V_0) = (U_i, V_i) = 1, \quad (S_i, S_i) = \varepsilon_i, \\ \text{and } R(U_0, U_i, U_i, S_i) = 1.\end{gathered} \tag{1.a}$$

We define the 0-model $\mathcal{V} := (\mathbf{R}^{3k+2}, (\cdot, \cdot), R)$. Let $\mathcal{G}_{\mathcal{V}}$ be the structure group of this 0-model. We define a *normalized basis* for V to be a basis that preserves the normalizations given in equation (1.a). Thus the structure group $\mathcal{G}_{\mathcal{V}}$ can be viewed as the set of normalized bases for $\mathcal{V}$.

Using the same k, a, b, and ε_i in Definition 1.2, we now define a family of pseudo-Riemannian manifolds.

Definition 1.3. Put coordinates $(u_0, \dots, u_k, v_0, \dots, v_k, s_1, \dots, s_k)$ on the Euclidean space $M := \mathbf{R}^{3k+2}$. Let $F := (f_1(u_1), \dots, f_k(u_k))$ where $f_i(u_i)$ are a collection of smooth functions with $f_i(u_i) + 1 \neq 0$ for all u_i. Define the nonzero entries of a symmetric metric g_F on the coordinate frames as follows:

$$\begin{aligned} g_F(\partial_{u_0}, \partial_{u_i}) &= 2f_i(u_i)s_i, & g_F(\partial_{u_i}, \partial_{u_i}) &= -2u_0 s_i, \\ g_F(\partial_{u_i}, \partial_{v_j}) &= \delta_{ij}, & g_F(\partial_{s_i}, \partial_{s_i}) &= \varepsilon_i. \end{aligned}$$

Let $\mathcal{M}_F := (\mathbf{R}^{3k+2}, g_F)$. If we choose a of the ε_i to be -1 and $k - a = b$ of the ε_i to be $+1$, then this is a manifold of signature $(k+1+a, k+1+b)$. □

We shall show that the manifolds $\mathcal{M}_F$ are 0-curvature homogeneous:

Theorem 1.4. *Adopt the notation of Definition* 1.2 *and of Definition* 1.3. *The manifolds $\mathcal{M}_F$ are* 0*-modeled on $\mathcal{V}$.*

Define the subspaces of the model space V as follows:

$$A_V := \{\xi \in V \mid R(\xi, *, *, *) = 0\} = \ker(R), \qquad A_{S,V} := A_V^\perp. \tag{1.b}$$

These spaces are necessarily preserved by any isomorphism of the structure group because they are defined in a basis-free fashion. We will prove the following result involving the group of permutations Sym_k of k objects that reflects the rigid nature of this group:

Theorem 1.5. *Adopt the notation of Definition* 1.2. *If A is an isomorphism of $\mathcal{V}$, then there exists a permutation $\sigma \in \mathrm{Sym}_k$ and constants a_0, b_i with $|a_0|b_i^2 = 1$ so that*

$$\begin{aligned} AU_0 &= a_0 U_0 + \Xi_0 && \text{for some } \Xi_0 \in A_V, \\ AU_i &= b_i U_{\sigma(i)} + \Xi_i && \text{for some } \Xi_i \in A_{S,V}, \\ AS_i &= \operatorname{sign}(a_0) S_{\sigma(i)} + \overline{\Xi}_i && \text{for some } \overline{\Xi}_i \in A_V. \end{aligned}$$

A natural question to ask is whether or not the manifolds $\mathcal{M}_F$ are really built from smaller dimensional manifolds with the same properties. We recall some basic definitions relevant to this question.

Definition 1.6. We say that a k-model $\mathcal{V}_k = (V, (\cdot,\cdot), A^0, \dots, A^k)$ is *decomposable* if there exists a nontrivial orthogonal decomposition $V = V_1 \oplus V_2$ which induces an orthogonal decomposition $A^i = A^i_1 \oplus A^i_2$ for $0 \le i \le k$; in this setting, we shall write $\mathcal{V} = \mathcal{V}^1 \oplus \mathcal{V}^2$ where the k-model $\mathcal{V}^p := (V_p, (\cdot,\cdot)|_{V_p}, A^0_p, \dots, A^k_p)$ for $p = 1$ and 2. One says that $\mathcal{V}_k$ is *indecomposable* if $\mathcal{V}_k$ is not decomposable. One says that a smooth pseudo-Riemannian manifold $\mathcal{M}$ is *locally decomposable* at a point $P \in M$ if there exists a neighborhood $\mathcal{O}$ of P so that $(\mathcal{O}, g_M) = (\mathcal{O}_1 \times \mathcal{O}_2, g_1 \oplus g_2)$ decomposes as a Cartesian product. We say $\mathcal{M}$ is *locally indecomposable* at P if this does not happen. □

It is easy to see that if $\mathcal{V}_k(\mathcal{M}, P)$ is indecomposable for some k then $\mathcal{M}$ is locally indecomposable at P. We shall show that the manifolds $\mathcal{M}_F$ are locally indecomposable at every point in Theorem 1.7:

Theorem 1.7. *Adopt the notation of Definitions* 1.2 *and* 1.3*.*

(1) *The model space $\mathcal{V}$ is indecomposable.*

(2) *The manifolds $\mathcal{M}_F$ are locally indecomposable at every point.*

Using Theorem 1.5, we can produce new isometry invariants which are not of Weyl type. For example, in Section 5 we prove the following:

Theorem 1.8. *Adopt the notation of Definition* 1.3*.*

(1) *The following quantity is an ℓ-model invariant*:

$$\beta_\ell = \sum_{j=1}^{k} \frac{f_j^{(\ell+1)}(1 + f_j')^{\ell-1}}{\left[f_j^{(2)}\right]^\ell}.$$

(2) *If the manifold $\mathcal{M}_F$ is ℓ-curvature homogeneous, then β_p is constant for all $p = 1, 2, \dots, \ell$.*

(3) *If $\mathcal{M}_F$ is locally homogeneous, then β_ℓ is constant for all ℓ.*

Using this theorem and a similarly defined ℓ-model invariant (see Theorem 5.5), it is possible to prove:

Theorem 1.9. *Suppose $f_i'(u_i)+1\neq 0$ for $1\leq i\leq k$. If $f_i''(u_i)\neq 0$, then $\mathcal{M}_F$ is not 2-curvature homogeneous.*

The following is a brief outline of the paper. We will compute the entries of tensors R and ∇R, and prove Theorem 1.4 in Section 2. In Section 3 we study the structure group $\mathcal{G}_{\mathcal{V}}$ and establish Theorem 1.5. We study the notion of indecomposability in Section 4 and prove Theorem 1.7. In Section 5 we conclude the paper by establishing Theorems 1.8 and 1.9.

2. Curvature homogeneity.

We begin this section with a calculation of the Christoffel symbols of the Levi-Civita connection of the manifolds $\mathcal{M}_F$.

Lemma 2.1. *Let $\partial_{u_i}, \partial_{s_i}$ and ∂_{v_i} be coordinate vector fields on $\mathcal{M}_F$.*

(1) *The nonzero covariant derivatives of the coordinate vector fields are*

$$\begin{aligned}
\nabla_{\partial_{u_0}}\partial_{u_i} &= \nabla_{\partial_{u_i}}\partial_{u_0} = -s_i\partial_{v_i} - f_i(u_i)\varepsilon_i\partial_{s_i},\\
\nabla_{\partial_{u_i}}\partial_{u_i} &= (2f_i'(u_i)+1)s_i\partial_{v_0} + u_0\varepsilon_i\partial_{s_i},\\
\nabla_{\partial_{u_0}}\partial_{s_i} &= \nabla_{\partial_{s_i}}\partial_{u_0} = f_i(u_i)\partial_{v_i},\\
\nabla_{\partial_{u_i}}\partial_{s_i} &= \nabla_{\partial_{s_i}}\partial_{u_i} = f_i(u_i)\partial_{v_0} - u_0\partial_{v_i}.
\end{aligned}$$

(2) *The only nonzero entries of the Riemannian curvature tensor R (up to the usual $\mathbf{Z}_2$ symmetries) are*

(a) $R_0(i) := R(\partial_{u_0}, \partial_{u_i}, \partial_{u_i}, \partial_{u_0}) = f_i(u_i)^2\varepsilon_i$, *and*

(b) $R_s(i) := R(\partial_{u_0}, \partial_{u_i}, \partial_{u_i}, \partial_{s_i}) = f_i'(u_i)+1$.

(3) *The only nonzero entries of the covariant derivative tensor ∇R (up to the usual symmetries) are*:

(a) $\nabla R(\partial_{u_0}, \partial_{u_i}, \partial_{u_i}, \partial_{u_0}; \partial_{u_i}) = 2f_i(u_i)\varepsilon_i(2f_i'(u_i)+1)$

(b) $\nabla R(\partial_{u_0}, \partial_{u_i}, \partial_{u_i}, \partial_{s_i}; \partial_{u_i}) = f_i''(u_i)$.

(4) *The following assertions are equivalent*:

(a) *For each i with $1\leq i\leq s$, either $f_i(u_i)=0$ or $f_i'(u_i)=-1/2$.*

(b) *$\mathcal{M}_F$ is a local symmetric space.*

Proof. We compute the nonzero components of the covariant derivatives of the coordinate vector fields, the curvature tensor R and its covariant derivative ∇R. Note that $g(\partial_{u_j}, \partial_{s_i}) = g(\partial_{v_j}, \partial_{s_i}) = 0$ and $g(\partial_{s_i}, \partial_{s_i}) = \varepsilon_i$ is constant. So if X and Y are any coordinate vector fields, we have

$$g(\nabla_{\partial_{s_i}} X, Y) = g(\nabla_X \partial_{s_i}, Y) = -g(\nabla_X Y, \partial_{s_i}) = \frac{1}{2}(\partial_{s_i} g(X,Y)).$$

We let the index i range from 1 to k.

$$\begin{aligned}
g(\nabla_{\partial_{u_0}} \partial_{u_i}, \partial_{u_i}) &= \frac{1}{2}\partial_{u_0} g(\partial_{u_i}, \partial_{u_i}) \\
&= \frac{1}{2}(-2s_i) = -s_i, \\
g(\nabla_{\partial_{u_0}} \partial_{u_i}, \partial_{s_i}) &= \frac{1}{2}(\partial_{u_0} g(\partial_{u_i}, \partial_{s_i}) + \partial_{u_i}(\partial_{u_0}, \partial_{s_i}) - \partial_{s_i} g(\partial_{u_0}, \partial_{u_i})) \\
&= \frac{1}{2}(2f_i) = f_i, \\
g(\nabla_{\partial_{u_i}} \partial_{u_i}, \partial_{u_0}) &= \frac{1}{2}(2\partial_{u_i} g(\partial_{u_i}, \partial_{u_0}) - \partial_{u_0} g(\partial_{u_i}, \partial_{u_i})) \\
&= \frac{1}{2}(2 \cdot 2f_i' s_i - (-2s_i) = s_i(2f_i' + 1), \\
g(\nabla_{\partial_{u_i}} \partial_{u_i}, \partial_{s_i}) &= -\frac{1}{2}(\partial_{s_i} g(\partial_{u_i}, \partial_{u_i}) = u_0, \\
g(\nabla_{\partial_{u_0}} \partial_{s_i}, \partial_{u_i}) &= \frac{1}{2}(\partial_{s_i} g(\partial_{u_0}, \partial_{u_i})) = f_i, \\
g(\nabla_{\partial_{u_i}} \partial_{s_i}, \partial_{u_0}) &= \frac{1}{2}(\partial_{s_i} g(\partial_{u_i}, \partial_{u_0})) = f_i, \\
g(\nabla_{\partial_{u_i}} \partial_{s_i}, \partial_{u_i}) &= \frac{1}{2}(\partial_{s_i} g(\partial_{u_i}, \partial_{u_i})) = -u_0.
\end{aligned}$$

We may then use this computation to see that:

$$\begin{aligned}
&R(\partial_{u_0}, \partial_{u_i})\partial_{u_i} \\
&\qquad = (\nabla_{\partial_{u_0}} \nabla_{\partial_{u_i}} - \nabla_{\partial_{u_i}} \nabla_{\partial_{u_0}})\partial_{u_i} \\
&\qquad = \nabla_{\partial_{u_0}}[(2f_i' + 1)s_i \partial_{v_0} + u_0 \varepsilon_i \partial_{s_i}] - \nabla_{\partial_{u_i}}[-s_i \partial_{v_i} - f_i \varepsilon_i \partial_{s_i}] \\
&\qquad = \varepsilon_i \partial_{s_i} + u_0 \varepsilon_i \nabla_{\partial_{u_0}} \partial_{s_i} + f_i' \varepsilon_i \partial_{s_i} + f_i \varepsilon_i \nabla_{\partial_{u_i}} \partial_{s_i} \\
&\qquad = (1 + f_i')\varepsilon_i \partial_{s_i} + f_i^2 \varepsilon_i \partial_{v_0}.
\end{aligned}$$

The covariant derivative of R is given by:

$$\begin{aligned}
&\nabla R(\partial_{u_0},\partial_{u_i},\partial_{u_i},\partial_{u_0};\partial_{u_i})\\
&\qquad\qquad\qquad = \partial_{u_i}(f_i^2\varepsilon_i) - 2R(\nabla_{\partial_{u_i}}\partial_{u_0},\partial_{u_i},\partial_{u_i},\partial_{u_0})\\
&\qquad\qquad\qquad\quad - 2R(\partial_{u_0},\nabla_{\partial_{u_i}}\partial_{u_i},\partial_{u_i},\partial_{u_0})\\
&\qquad\qquad\qquad = 2f_if_i'\varepsilon_i + 2f_i\varepsilon_i(f_i'+1) = 2f_i\varepsilon_i(2f_i'+1),
\end{aligned}$$

$$\begin{aligned}
&\nabla R(\partial_{u_0},\partial_{u_i},\partial_{u_i},\partial_{s_i};\partial_{u_i})\\
&\qquad = \partial_{u_i}(f_i'+1) - R(\nabla_{\partial_{u_i}}\partial_{u_0},\partial_{u_i},\partial_{u_i},\partial_{s_i})\\
&\qquad\quad - R(\partial_{u_0},\nabla_{\partial_{u_i}}\partial_{u_i},\partial_{u_i},\partial_{s_i})\\
&\qquad\quad - R(\partial_{u_0},\partial_{u_i},\nabla_{\partial_{u_i}}\partial_{u_i},\partial_{s_i}) - R(\partial_{u_0},\partial_{u_i},\partial_{u_i},\nabla_{\partial_{u_i}}\partial_{s_i})\\
&\qquad = f_i''.
\end{aligned}$$

The lemma now follows. □

We establish Theorem 1.4 after a brief remark.

Remark 2.2. Let index μ range from 1 to k, and let index ν range from 0 to k. If we relabel coordinates $x_\nu = u_\nu$, $x_{k+\mu} = s_\mu$, and $x_{2k+1+\nu} = v_\nu$, the above calculations show that $\nabla_{\partial_{x_i}}\partial_{x_j} = \sum_{k>\max\{i,j\}} \Gamma_{ij}{}^k(x_0,\dots,x_{k-1})\partial_{x_k}$. Thus by definition, $\mathcal{M}_F$ is a family of *generalized plane wave manifolds*. By the results of Gilkey and Nikčević [**17**], we conclude that members of the family $\mathcal{M}_F$ are Ricci-flat, complete, $\exp : T_PM \to M$ is a diffeomorphism for all P, and all Weyl scalar invariants vanish. We will see in Section 5 that there are members of the family $\mathcal{M}_F$ which are not locally homogeneous. This is not possible in the Riemannian setting as Prüfer, Tricerri and Vanhecke [**23**] showed that if all local scalar Weyl invariants up to order $n(n-1)/2$ are constant on a Riemannian manifold (N,h) of dimension n, then (N,h) is locally homogeneous and determined up to local isometry by these invariants. □

Proof of Theorem 1.4. To show that $\mathcal{M}_F$ are 0-modeled on $\mathcal{V}$, we will produce a normalized basis for $(T_PM, g|_P, R|_P)$ for any $P \in M$ (see

Definition 1.2). We have that $f_i(u_i)+1\neq 0$ for $1\leq i\leq k$. We set

$$\begin{aligned}
&U_0 := \partial_{u_0} + \sum_j a_j\partial_{s_j}, &&U_i := b_i\partial_{u_i} + \beta_i\partial_{v_0} + \widetilde{\beta}_i\partial_{v_i},\\
&S_i := \kappa_i\partial_{s_i} + \gamma_i\partial_{v_i}, &&V_0 := \partial_{v_0},\\
&V_i = b_i^{-1}\partial_{v_i},
\end{aligned}$$

where b_i, β_i, $\widetilde{\beta}_i$, κ_i and γ_i will be specified presently. The potentially nonzero curvatures are then:

$$\begin{aligned}
R(U_0,U_i,U_i,U_0) &= b_i^2\{f_i(u_i)^2\varepsilon_i + 2a_i(f_i'(u_i)+1)\},\\
R(U_0,U_i,U_i,S_i) &= b_i^2(f_i'(u_i)+1)\varepsilon_i\kappa_i.
\end{aligned}$$

To ensure that $R(U_0,U_i,U_i,U_0)=0$ and $R(U_0,U_i,U_i,S_i)=+1$, we set

$$\begin{aligned}
a_i &:= -\frac{f_i(u_i)^2\varepsilon_i}{2(f_i'(u_i)+1)},\\
\kappa_i &:= \varepsilon_i \operatorname{sign}(f_i'(u_i)+1),\\
b_i &:= |f_i'(u_i)+1|^{-1/2}.
\end{aligned}$$

The potentially nonzero inner products are

$$\begin{aligned}
&(U_0,V_0)=1, &&(U_0,S_i)=\kappa_i a_i+\gamma_i,\\
&(U_0,U_i)=b_i g_F(\partial_{u_0},\partial_{u_i})+\beta_i, &&(S_i,S_i)=1,\\
&(U_i,U_i)=b_i^2 g_F(\partial_{u_i},\partial_{u_i})+2b_i\widetilde{\beta}_i, &&(U_i,V_i)=1.
\end{aligned}$$

We complete the proof by setting:

$$\begin{aligned}
&\gamma_i := -\kappa_i a_i, \qquad \beta_i := -b_i g_F(\partial_{u_0},\partial_{u_i}),\\
&\widetilde{\beta}_i := -\tfrac{1}{2}b_i g_F(\partial_{u_i},\partial_{u_i}). \qquad \square
\end{aligned}$$

It will be convenient to compute several values of the curvature tensor and its covariant derivatives on a normalized basis, see Theorems 5.2 and 5.5. We list these quantities below for future reference.

Lemma 2.3. *Adopt the notation of Definitions* 1.2 *and* 1.3. *Suppose that* $\{U_0,\ldots,U_k,V_0,\ldots,V_k,S_1,\ldots,S_k\}$ *is the normalized basis found in the previous theorem.*

(1) $\nabla R(U_0, U_i, U_i, U_0; U_i) = (f_i\varepsilon_i)/((f_i'+1)^{5/2})[2(2f_i'+1)(f_i'+1) - f_if_i'']$.

(2) $\nabla R(U_0, U_i, U_i, S_i; U_i) = (f_i''\kappa_i)/(|f_i'+1|^{3/2})$.

(3) $\nabla^\ell R(U_0, U_i, U_i, S_i; U_i, \dots, U_i) = \kappa_i f_i^{(\ell+1)}|f_i'+1|^{-(2+\ell)/2}$.

(4) $\nabla^2 R(U_0, U_i, U_i, U_0; U_i, U_i) = (\varepsilon_i)/((f_i'+1)^2)(4(f_i')^2 + 2f_i' + 6f_if_i'' - ((f_i)^2 f_i''')/(f_i'+1))$.

Proof. We use the normalized basis found in the proof of Theorem 1.4 and the calculations of Lemma 2.1 to compute these directly–the calculations are omitted. □

3. The structure group $\mathcal{G}_\mathcal{V}$. In this section we study the structure group $\mathcal{G}_\mathcal{V}$. For convenience, we establish notation as follows for the normalized bases $\mathcal{B}$ and $\widetilde{\mathcal{B}}$:

$$\mathcal{B} = \{U_0, \dots, U_k, V_0, \dots, V_k, S_1, \dots S_k\},$$
$$\widetilde{\mathcal{B}} = \{\widetilde{U}_0, \dots, \widetilde{U}_k, \widetilde{V}_0, \dots, \widetilde{V}_k, \widetilde{S}_1, \dots \widetilde{S}_k\}.$$

We adopt the notation of equation (1.b). For any normalized basis $\mathcal{B}$, one has

$$A_V = \operatorname{Span}\{V_0, \dots, V_k\}, \text{ and}$$
$$A_{S,V} = \operatorname{Span}\{S_1, \dots, S_k, V_0, \dots, V_k\}.$$

Let Sym_k be the group of permutations of the numbers $\{1, \dots, k\}$.

Proof of Theorem 1.5. Note that $AS_i \in A_{S,V}$. We expand:

$$\begin{aligned} AU_0 &= a_0U_0 + \sum_j (b_{0j}U_j + d_{0j}S_j) + A_V, \\ AS_i &= \sum_j f_{ij}S_j + A_V, \\ AU_i &= a_iU_0 + \sum_j b_{ij}U_j + A_{S,V}. \end{aligned} \tag{3.a}$$

For any $\xi_1, \xi_2 \in V$, we have that:

$$0 = R(\xi_1, U_0, U_0, \xi_2) = R(A\xi_1, AU_0, AU_0, A\xi_2). \tag{3.b}$$

Choose ξ_i so that $A\xi_1 = U_0$ and $A\xi_2 = S_j$. We then have

$$0 = R(U_0, AU_0, AU_0, S_j) = b_{0j}^2.$$

Consequently, $b_{0j} = 0$. We have $A \cdot A_V = A_V$. As $1 = (U_0, V_0) = (AU_0, AV_0)$, there exists a $v \in A_V$ so $(AU_0, v) \neq 0$. Since $AU_0 = a_0U_0 + A_{S,V}$, we conclude $a_0 \neq 0$. Choosing $A\xi_1 = A\xi_2 = U_i$ in equation (3.b) we have:

$$0 = R(U_i, AU_0, AU_0, U_i) = 2a_0 d_{0j}.$$

Since $a_0 \neq 0$, $d_{0j} = 0$, display (3.a) becomes

$$\begin{aligned} AU_0 &= a_0U_0 + A_V, \qquad AS_i = \sum_j f_{ij}S_j + A_V, \\ AU_i &= a_iU_0 + \sum_j b_{ij}U_j + A_{S,V}. \end{aligned}$$

Since $AV_i \in A_V$, the matrix $[b_{ij}]$ is invertible. Suppose that the matrix element $b_{ij} \neq 0$. Choose ξ_1 so $A\xi_1 = S_j$. Since $k \geq 2$, we may choose positive induces $l \neq i$. Then

$$0 = R(U_0, U_i, U_l, \xi_1) = R(AU_0, AU_i, AU_l, A\xi_1) = a_0 b_{ij} b_{lj}.$$

Thus, if $b_{ij} \neq 0$, $b_{lj} = 0$ for $i \neq l$. So, in the matrix b_{ij}, each column has at most one nonzero entry. Since b_{ij} is invertible, each column has exactly one nonzero entry. So one has:

$$\begin{aligned} AU_0 &= a_0U_0 + A_V, \quad AS_i = \sum_j f_{ij}S_j + A_V, \\ AU_i &= a_iU_0 + b_iU_{\sigma(i)} + A_{S,V}. \end{aligned}$$

The relation $\delta_{ij} = R(AU_0, AU_i, AU_i, AS_j)$ shows $f_{ij} = 0$ for $j \neq \sigma(i)$. Since AS_j is a unit vector, this coefficient is ± 1. Thus,

$$\begin{aligned} AU_0 &= a_0U_0 + A_V, \qquad AS_i = \pm S_{\sigma(i)} + A_V, \\ AU_i &= a_iU_0 + b_iU_{\sigma(i)} + A_{S,V}. \end{aligned}$$

Since $1 = R(AU_0, AU_i, AU_i, AS_i)$, we have $\pm b_i^2 a_0 = 1$. Finally, since $k \geq 2$ and since $0 = R(AU_i, AU_j, AU_j, AS_i)$, we have $a_i b_j = 0$ and hence $a_i = 0$. The relation $|a_0|b_i^2 = 1$ and $AS_i = \operatorname{sign}(a_0)S_{\sigma(i)}$ now follow. This establishes the theorem. □

Remark 3.1. Theorem 1.5 does not apply when $k = 1$, although a similar statement is true in that case: If A is an isomorphism of $\mathcal{V}$, then

$$\begin{aligned} AU_0 &= a_0U_0 + \Xi_0 && \text{for some } \Xi_0 \in A_V, \\ AU_1 &= a_1U_0 + b_1U_1 + \Xi_1 && \text{for some } \Xi_1 \in A_{S,V}, \\ AS_1 &= \operatorname{sign}(a_0)S_1 + \overline{\Xi}_1 && \text{for some } \overline{\Xi}_1 \in A_V. \end{aligned}$$

Notice the extra freedom in choosing a_1. Since Sym_1 is the trivial group, the symmetric group action is not so evident as when $k \geq 2$. □

The crucial part of the previous result is that any change of basis will permute the interesting information, single out the vector U_0 and $A \cdot A_{S,V} \subseteq A_{S,V}$. This will be important when defining invariants in the next section. The extra information one has when $k = 1$ will not create any ambiguity in the development of any of our invariants.

4. Indecomposability. Since $\mathbf{R}^{3k+2}$ is contractible, any real vector bundle over $\mathbf{R}^{3k+2}$ is trivial, in particular, the tangent bundle is trivial. With the added structure of a metric and a curvature tensor, however, more information is available.

A natural question to ask is if these manifolds are really products of manifolds of smaller dimension. More specifically, is $\mathbf{R}^{3k+2} = M_1 \times M_2$ and $g_F = g_{M_1} \oplus g_{M_2}$? If this were the case, then $T\mathbf{R}^{3k+2} = TM_1 \oplus TM_2$, and one has that the curvature tensor $R_M = R_{M_1} \oplus R_{M_2}$. This is a more algebraic notion of indecomposability which we briefly study. The motivation comes from the main result in [**29**]: any family of Riemannian manifolds 0-modeled on an irreducible symmetric space is homogeneous (in fact, symmetric). In the pseudo-Riemannian setting, the notion of irreducibility seems more elusive, and although we do not show that the 0-model $\mathcal{V}$ is irreducible, we prove the weaker Theorem 1.7. Although the main step of the result in [**29**] is to use the hypothesis to establish that the manifolds in question are Einstein. We recall Remark 2.2: the manifolds $\mathcal{M}_F$ are not only Einstein, but Ricci-flat. Thus, this family of manifolds provides interesting insight into the distinction between Riemannian and pseudo-Riemannian manifolds.

Recall the notation established in Definitions 1.2 and 1.3. We show in this section that the manifolds $\mathcal{M}_F$ are locally indecomposable at every point, and thus locally $\mathcal{M}_F$ is not the direct product of smaller dimensional manifolds, answering the above question in the negative.

We fix a normalized basis $\mathcal{B}$ for this section. Using the subspace A_V defined in the introduction, denote $V/A_V = B_{U,S}$, and $\pi : V \to B_{U,S}$ the projection. A basis for $B_{U,S}$ is the image of $U_0, \dots, U_k, S_1, \dots, S_k$ under π. Write $\overline{U}_i = \pi U_i$ and similarly for the other vectors. Since $A_V \subset \ker(R)$, we have a well-defined algebraic curvature tensor $\overline{R}$ defined on $B_{U,S}$, characterized by the relation $\pi^*\overline{R} = R$. We have the same relations for $\overline{R}$ on the image of the normalized basis as we do for R on the original normalized basis for V, although of course the projection of such a basis to $B_{U,S}$ is no longer linearly independent. We recall that, on V, we have the relations

$$(U_i, V_j) = \delta_{ij}, \qquad (S_i, S_i) = \varepsilon_i, \qquad R(U_0, U_i, U_i, S_i) = 1.$$

Lemma 4.1. *The weak* 0*-model* $(B_{U,S}, \overline{R})$ *is indecomposable for* $k \geq 1$.

Proof. We assume to the contrary there exists a nontrivial decomposition of the model space $(W, R) = (\overline{W}_1 \oplus \overline{W}_2, R_1 \oplus R_2)$ and argue for a contradiction. We begin by expressing $\overline{U}_0 = \xi_1 + \xi_2$, for $\xi_i \in W_i$.

Case I. One of ξ_i is 0 (suppose without loss of generality that $\xi_2 = 0$). This means that we can write $\overline{U}_0 \in \overline{W}_1$. Let $0 \neq \eta \in \overline{W}_2$. Consequently, we may express $\eta = \gamma_0 \overline{U}_0 + \sum_{j=1}^{k} \gamma_j \overline{U}_j + \gamma_j' \overline{S}_j$. Then for $i > 0$,

$$\begin{aligned}\overline{R}(\overline{U}_0, \overline{U}_i, \overline{U}_i, \eta) &= \gamma_i' = 0, \text{ and}\\ \overline{R}(\overline{U}_0, \overline{U}_i, \eta, \overline{S}_i) &= \gamma_i = 0.\end{aligned}$$

So $\eta = \gamma_0 \overline{U}_0$, and $\eta \neq 0$ means that $\eta \in W_2$ and $U_0 \in \overline{W}_1$ are not linearly independent, and so $W_1 \cap W_2 \neq \{0\}$. This contradiction permits us to eliminate this case from consideration.

Case II. $\overline{U}_0 = \xi_1 + \xi_2$ and both $\xi_i \neq 0$. We express these vectors as

$$\xi_1 = \alpha_0 \overline{U}_0 + \sum_j \alpha_j \overline{U}_j + \alpha'_j \overline{S}_j,$$
$$\xi_2 = \beta_0 \overline{U}_0 + \sum_j \beta_j \overline{U}_j + \beta'_j \overline{S}_j.$$

Since $\xi_1 + \xi_2 = \overline{U}_0$, we must have $\alpha_0 + \beta_0 = 1$, $\alpha_j + \beta_j = \alpha'_j + \beta'_j = 0$. For $j = 1, 2$ and $i = 1, \dots, k$, we compute $\overline{R}(\overline{U}_0, \xi_j, \xi_j, \overline{S}_i)$ in two ways. First, we could have only the $\overline{U}_i$ coefficients of ξ_j, so $\overline{R}(\overline{U}_0, \xi_j, \xi_j, \overline{S}_i) = \alpha_i^2$, $j = 1$, or β_i^2, $j = 2$. On the other hand (for $j = 1$),

$$\begin{aligned} \overline{R}(\overline{U}_0, \xi_1, \xi_1, \overline{S}_i) &= \overline{R}(\xi_1 + \xi_2, \xi_1, \xi_1, \overline{S}_i) \\ &= \overline{R}(\xi_1, \xi_1, \xi_1, \overline{S}_i) + \overline{R}(\xi_2, \xi_1, \xi_1, \overline{S}_i) \\ &= 0. \end{aligned}$$

Similarly for $j = 2$. Thus, $\alpha_i = \beta_i = 0$ for all i.

Now we go to work on the other coefficients. Since $\alpha_0 + \beta_0 = 1$, at least one of these must be nonzero. Suppose without loss of generality that $\alpha_0 \neq 0$. Compute $0 = \overline{R}(\xi_1, \overline{U}_j, \overline{U}_j, \xi_2) = \alpha_0 \beta'_j + \beta_0 \alpha'_j$. Since $\alpha_0 \neq 0$, we can solve for $\beta'_j = (-\beta_0 \alpha'_j)/\alpha_0$. Imposing the condition $\alpha'_j + \beta'_j = 0$ gives us $\alpha'_j(\alpha_0 - \beta_0) = 0$ for all $j = 1, 2, \dots, k$. These equations could be solved by having either $\alpha'_j = 0$ for all j or $\alpha_0 = \beta_0$.

Case II.a. Suppose we have $\alpha'_j = 0$ for all j. Then we again impose the condition $\alpha'_j + \beta'_j = 0$ to see that $\beta'_j = 0$ for all j as well. This gives us $\xi_1 = \alpha_0 \overline{U}_0$ and $\xi_2 = \beta_0 \overline{U}_0$, and at this point there are several contradictions: by assumption, both ξ_i are nonzero, and we have $\xi_1 = \lambda \xi_2$, not linearly independent, but living in different subspaces. This is false.

Case II.b. Suppose $\alpha_0 = \beta_0$. Then $\alpha_0 + \beta_0 = 1$ implies $\alpha_0 = \beta_0 = 1/2$. Unfortunately, we must go into further cases and consider where another vector lives. The analysis of this new vector is similar to the previous technique. Since $k \geq 1$, there exists an $\overline{U}_1 \in B_{U,S}$, and we proceed by studying $\overline{U}_1$. Write $\overline{U}_1 = \eta_1 + \eta_2$, and $\eta_i \in \overline{W}_i$.

Case II.b.i. One of $\eta_i = 0$. Without loss of generality, assume $\eta_2 = 0$. Then $\overline{U}_1 \in \overline{W}_1$. Then $\overline{R}(\xi_2, \overline{U}_1, \overline{U}_1, \overline{S}_1) = 1/2$, but since $\xi_2 \in \overline{W}_2$ and $\overline{U}_1 \in \overline{W}_1$, we must have $\overline{R}(\xi_2, \overline{U}_1, \overline{U}_1, \overline{S}_1) = 0$ which gives us a contradiction.

Case II.b.ii. Both $\eta_i \neq 0$. We write $\eta_i = a_i\overline{U}_1 + v_i$ for $v_i \in \overline{W}_i$. Then $a_1 + a_2 = 1$ and hence both a_i cannot be 0 simultaneously. We compute

$$\overline{R}(\xi_2, \overline{U}_1, \eta_1, \overline{S}_1) = \frac{1}{2}a_1 = 0,$$
$$\overline{R}(\xi_1, \overline{U}_1, \eta_2, \overline{S}_1) = \frac{1}{2}a_2 = 0.$$

This yields a contradiction; this final contradiction completes the proof. □

Proof of Theorem 1.7. We have shown in Lemma 4.1 that the weak model space $B_{U,S}$ is indecomposable. In addition, $\ker R = \operatorname{Span}\{V_0, \dots, V_k\}$ is a totally isotropic subspace. Thus, according to [**10**], the model space $\mathcal{V}$ is indecomposable.

We now prove assertion (2). We have shown that $\mathcal{V}$ is a 0-model for the tangent space T_PM at any point $P \in M$. Such a decomposition of T_PM would induce a decomposition of the 0-model $\mathcal{V}$. But $\mathcal{V}$ is indecomposable by assertion (1), and no such decomposition of the tangent bundle is possible. □

5. Isometry invariants and local homogeneity. Since all Weyl scalar invariants vanish (see Remark 2.2) we use the determination of the structure group $\mathcal{G}_{\mathcal{V}}$ given in Theorem 1.5 to define new isometry invariants. We build invariants involving normalized bases and only the tensors $\nabla R, \dots, \nabla^\ell R$; these are so-called ℓ-model invariants. This will aid us in studying the question of ℓ-curvature homogeneity for $\ell \geq 2$ for the manifolds $\mathcal{M}_F$. We will need a technical lemma describing the behavior of the higher covariant derivatives on a normalized basis.

Lemma 5.1. *For the manifolds defined above, the following assertions hold. Let $\ell \geq 1$ and $i = 1, 2, \dots, k$.*

(1) $\nabla^\ell R(\partial_{u_0}, \partial_{u_i}, \partial_{u_i}, \partial_{s_i}; \partial_{u_i}, \dots, \partial_{u_i}) = f_i^{(\ell+1)}(u_i)$.

(2) $\nabla^\ell R(\partial_{u_0}, \partial_{u_i}, \partial_{u_i}, \partial_{u_0}; \partial_{u_i}, \dots \partial_{u_i})$ *is a function of u_i, expressible as an algebraic combination of the derivatives of f_i.*

(3) $\nabla^\ell R(*, *, *, *; *, \dots, *, \partial_{s_i}) = 0$.

(4) $\nabla^\ell R(*, *, *, *; *, \dots, *, \partial_{u_0}) = 0$.

(5) *The only possible nonzero entries of the covariant derivatives of R on any normalized basis are*

$$\nabla^\ell R(U_0, U_i, U_i, S_i; U_i, \dots, U_i) \text{ and } \nabla^\ell R(U_0, U_i, U_i, U_0; U_i, \dots U_i).$$

Proof. Assertions 1 and 2 follow from Lemma 2.1, assertion 3. Note that, in these terms, both are functions of only the u_i. Hence, to uncover any other nonzero terms of the higher covariant derivatives other than those ending in only ∂_{u_i}, we must look to our calculation of ∇ on the coordinate frames (see Lemma 2.1, assertion 1). Assertion 3 is now obvious, and since $\nabla_{\partial_{u_0}} \partial_{u_0} = 0$, we see assertion 4 follows as well. As we may only build higher covariant derivatives from ∂_{u_i} with those relations in assertion 3 of Lemma 1.1, and any change of normalized basis will permute the same positive U_* and S_* induces, the only nonzero higher covariant derivatives on any normalized basis are only those listed. ◻

Let $\mathcal{B} = \{U_0, \dots, U_k, V_0, \dots V_k, S_1, \dots, S_k\}$ be the normalized basis found in Theorem 2.1. We define below the functions $(\beta_\ell)_{\mathcal{B}}$ for $\ell \geq 2$, which a priori depends on the choice of normalized basis. Assume for now that all denominators are nonzero. Define

$$(\beta_\ell)_{\mathcal{B}} := \sum_{j=0}^{k} \frac{\nabla^\ell R(U_0, U_j, U_j, S_j; U_j, \dots, U_j)}{\left(\nabla R(U_0, U_j, U_j, S_j; U_j)\right)^\ell}.$$

Lemma 5.2. *Adopt the notation of Definitions* 1.2 *and* 1.3. *If $f_i'' \neq 0$ and $\ell \geq 2$, then $(\beta_\ell)_{\mathcal{B}}$ is independent of the normalized basis chosen.*

Remark 5.3. The hypothesis $f_i' + 1 \neq 0$ is required for a normalized basis to exist. The condition that $f_i'' \neq 0$ is required for the invariants β_ℓ to exist at all, as we divide by the quantity f_i'' in the definition of β_ℓ. These two hypothesis are needed only for these reasons, i.e., we need everything to "make sense." Later, we remove the restriction $f_i'' \neq 0$ in the definition of another invariant (see Theorem 5.5). ◻

Proof of Lemma 5.2. Let $\widetilde{\mathcal{B}}$ be another normalized basis, and let $\sigma \in \mathrm{Sym}_k$ be the corresponding permutation of the induces found in

Theorem 1.5. By Lemma 5.1, we know how a normalized change of basis affects the entries of the higher covariant derivatives. Essentially, the only change of basis possible is a permutation of the U_* and S_* basis vectors with a (nonzero) scaling factor. So,

$$\begin{aligned}\nabla^\ell R(&\widetilde{U}_0, \widetilde{U}_j, \widetilde{U}_j, \widetilde{S}_j; \widetilde{U}_j, \dots, \widetilde{U}_j)\\ &= \left(\frac{\pm 1}{\sqrt{|a_0|}}\right)^\ell \nabla^\ell R(U_0, U_{\sigma(j)}, U_{\sigma(j)}, S_{\sigma(j)}; U_{\sigma(j)}, \dots, U_{\sigma(j)}),\end{aligned}$$

and

$$\begin{aligned}(\nabla R(&\widetilde{U}_0, \widetilde{U}_j, \widetilde{U}_j, \widetilde{S}_j; \widetilde{U}_j))^\ell\\ &= \left(\frac{\pm 1}{\sqrt{|a_0|}}\right)^\ell \nabla R(U_0, U_{\sigma(j)}, U_{\sigma(j)}, S_{\sigma(j)}; U_{\sigma(j)})^\ell.\end{aligned}$$

The permutation σ is a bijection of a finite set of induces, and so if we put

$$I = \{\sigma^{-1}(1), \dots, \sigma^{-1}(k)\} = \{\ell_1, \dots, \ell_k\},$$

we get the rearranged (but equal) sum

$$\begin{aligned}(\beta_\ell)_{\widetilde{\mathcal{B}}} &= \sum_{j=1}^{k} \frac{\nabla^\ell R(\widetilde{U}_0, \widetilde{U}_{\ell_j}, \widetilde{U}_{\ell_j}, \widetilde{S}_{\ell_j}; \widetilde{U}_{\ell_j}, \dots, \widetilde{U}_{\ell_j})}{\left(\nabla R(\widetilde{U}_0, \widetilde{U}_{\ell_j}, \widetilde{U}_{\ell_j}, \widetilde{S}_{\ell_j}; \widetilde{U}_{\ell_j})\right)^\ell}\\ &= \sum_{j=0}^{k} \frac{\nabla \ell R(U_0, U_j, U_j, S_j; U_j, U_j)}{(\nabla R(U_0, U_j, U_j, S_j; U_j))^\ell}\\ &= (\beta_\ell)_{\mathcal{B}}.\end{aligned}$$

Hence, $(\beta_\ell)_{\mathcal{B}} = (\beta_\ell)_{\widetilde{\mathcal{B}}} = \beta_\ell$ is independent of the basis chosen and is an invariant of the manifolds $\mathcal{M}_F$. □

Proof of Theorem 1.8. Evaluating these tensors on a normalized basis and using Theorem 5.1 and Lemma 2.3 establishes the first assertion of Theorem 1.8.

If $\mathcal{M}_F$ were ℓ-curvature homogeneous, then there exists a p-model for every $p = 0, 1, \dots, \ell$, along with a normalized basis for $T_P M$ so that the metric and curvature entries up to order ℓ are constant. Since β_p is built from these entries, β_p must be constant for all $p = 0, \dots, \ell$. This establishes assertion 2 of Theorem 1.8.

If $\mathcal{M}_F$ is locally homogeneous, then it is ℓ-curvature homogeneous for all ℓ. Applying the previous assertion shows that β_ℓ has to be constant for all ℓ in this case. $\square$

The next lemma presents exactly the family of functions for which β_ℓ is constant; this technical result will be used in the proof of Theorem 1.9.

Lemma 5.4. *Let $\mathcal{O} \subseteq \mathbf{R}$, and denote $\mathcal{O}^p$ as the product of $\mathcal{O}$ with itself p times.*

(1) *Let $g_i : \mathcal{O} \to \mathbf{R}$. Let $g_i \in C^\infty(\mathcal{O})$ for $1 \le i \le p$. Suppose that $\sum_{i=1}^p g_i(u_i)$ is constant on $\mathcal{O}^p$. Then g_i is constant for $1 \le i \le p$.*

(2) *Suppose $f^{(2)}(0) \neq 0$, and $k \in \mathbf{R}$. Then the local solutions to the differential equation $\Omega(f) = (f^{(3)}(1+f'))/[f^{(2)}]^2 = k$ are as follows:*

(a) *$k = 0 \Rightarrow f$ is quadratic.*

(b) *$k = 1 \Rightarrow 1 + f' = e^{au+b}$ for some $0 < a \in \mathbf{R}$, and $b \in \mathbf{R}$.*

(c) *$k \neq 0$ and $k \neq 1 \Rightarrow 1 + f' = \sqrt[1-k]{(1-k)(au+b)}$ for some $0 < a \in \mathbf{R}$ and $b \in \mathbf{R}$.*

(3) *Any solution to $\beta_2 = k$ where k is constant is also a solution to $\beta_\ell = k'$ where k' is constant.*

Proof. Assertion 1 is obvious as each summand is a function of different variables. We apply the previous assertion to the differential equation $\beta_2 = k$ to note that each of the summands $(f_j^{(3)}(1+f_j'))/[f_j^{(2)}]^2$ is constant. We can solve this explicitly for all functions on which β_ℓ is defined. The hypotheses ensure that the given expression makes sense in a small neighborhood of $u = 0$. We consider each case given in the theorem:

Case I. $k = 0$. This is more or less obvious since the denominator of Ω is nonzero, and $(1 + f')$ is nonzero. Thus $f^{(3)} = 0$; this establishes assertion 2 (a). For the next cases, we compute

$$\begin{aligned}
\frac{f^{(3)}(1+f')}{\left[f^{(2)}\right]^2} &= k \iff \\
\frac{f'''}{f''} &= \frac{f''}{1+f'}k \iff \\
\log f'' &= k\log(1+f') + a' \iff \\
\frac{f''}{(1+f')^k} &= e^{a'} = .
\end{aligned} \tag{5.a}$$

Case II. $k = 1$. We integrate equation (5.a) to get

$$\begin{aligned}
\log(1+f') &= au + b \iff \\
1 + f' &= e^{au+b}.
\end{aligned}$$

Case III. $k \neq 0$ and $k \neq 1$. We integrate (5.a) to get

$$\begin{aligned}
\frac{1}{1-k}(1+f')^{1-k} &= au + b \iff \\
1 + f' &= \sqrt[1-k]{(1-k)(au+b)}.
\end{aligned}$$

One can simply check that each of the families found in the previous assertion are also solutions to $\beta_\ell =$ constant. Of course, more initial conditions will need to be given for higher values of ℓ to completely describe all solutions. $\square$

We will need another family of invariants which can be constructed in the same manner as β_ℓ using the other nonzero higher covariant derivatives of the curvature tensor R, as listed in Lemma 5.1. Here, we may remove the hypothesis that $f_i'' \neq 0$.

Theorem 5.5. *Adopt the notation of Definitions* $1\ell \geq 2$*, and set*

$$\gamma_\ell = \sum_j \nabla^\ell R(U_0, U_j, U_j, U_0; U_j, \dots, U_j) \cdot \nabla R(U_0, U_j, U_j, U_0; U_j)^{\ell-2}.$$

(1) *γ_ℓ is independent of the normalized basis chosen, and is an ℓ-model invariant.*

(2) $\gamma_2 = \sum_j \left[\frac{\varepsilon_j}{(f_j'+1)^2}\left(4(f_j')^2 + 2f_j' + 6f_jf_j'' - \frac{(f_j)^2 f_j'''}{f_j'+1}\right)\right]$.

(3) *If $\mathcal{M}_F$ is ℓ-curvature homogeneous, then γ_p is constant for $1 \leq p \leq \ell$.*

(4) *If $\mathcal{M}_F$ is locally homogeneous, then γ_ℓ is constant for all ℓ.*

Proof. Let $\widetilde{\mathcal{B}}$ be another normalized basis. By Theorem 1.5 there exists an $a_0 \neq 0$ and a $\sigma \in \mathrm{Sym}_k$ so that

$$\begin{aligned}\nabla^\ell R(&\widetilde{U}_0, \widetilde{U}_j, \widetilde{U}_j, \widetilde{U}_0; \widetilde{U}_j, \dots, \widetilde{U}_j)\\ &= \left(\frac{1}{\sqrt{|a_0|}}\right)^{\ell-2} \nabla^l \ell R(U_0, U_{j'}, U_{j'}, U_0; U_{j'}, \dots, U_{j'}),\end{aligned}$$

and

$$\nabla R(\widetilde{U}_0, \widetilde{U}_j, \widetilde{U}_j, \widetilde{U}_0; \widetilde{U}_j) = \sqrt{|a_0|}\nabla R(U_0, U_{j'}, U_{j'}, U_0; U_{j'}),$$

where $j' = \sigma(j)$. Combining the above according to the definition of γ_ℓ establishes assertion 1. Lemma 2.3 and Theorem 5.1 establish assertion 2.

Assertions 3 and 4 follow similarly as in the proof of assertions 2 and 3 of Theorem 1.8. □

We use the invariants described above to study the local homogeneity of the manifold $\mathcal{M}_F$ and establish Theorem 1.9.

Proof of Theorem 1.9. If $\mathcal{M}_F$ were 2-curvature homogeneous, then by assertion 3 of Theorem 5.2, β_2 is constant. By assertion 3 of Theorem 5.5, γ_2 must also be constant. None of the solutions to $\beta_2 =$ constant listed in Lemma 5.4 make γ_2 constant as well. □

In most cases, Theorem 1.9 tells us these manifolds are not 2-curvature homogeneous, and hence not generally locally homogeneous. One asks if any of the $\mathcal{M}_F$ are 1-curvature homogeneous. We will study this question in a subsequent paper.

Acknowledgments. The author would like to take this opportunity to thank P. Gilkey and E. Puffini for their valuable help while this research was conducted. Also, the author wishes to thank the referee for several valuable suggestions.

REFERENCES

1. E. Boeckx, O. Kowalski and L. Vanhecke, *Riemannian manifolds of conullity two*, World Scientific, River Edge, NJ, 1996.

2. P. Bueken, *On curvature homogeneous three-dimensional Lorentzian manifolds*, J. Geom. Physics **22** (1997), 349–362.

3. P. Bueken and M. Djorić, *Three-dimensional Lorentz metrics and curvature homogeneity of order one*, Ann. Global Anal. Geom. **18** (2000), 85–103.

4. P. Bueken and L. Vanhecke, *Examples of curvature homogeneous Lorentz metrics*, Classical Quantum Gravity **14** (1997), L93–L96.

5. A. Derdzinski, *Einstein metrics in dimension four*, in *Handbook of differential geometry*, Vol. I., Dillen and Verstaelen, eds., North-Holland Publishing Co., Amsterdam, 2000.

6. C. Dunn and P. Gilkey, *Curvature homogeneous pseudo-Riemannian manifolds which are not locally homogeneous*, in *Complex, contact and symmetric manifolds*, Birkhäuser, Boston, MA, 2005.

7. C. Dunn, P. Gilkey, and S. Nikčević, *Curvature homogeneous signature* $(2,2)$ *manifolds*, in *Differential geometry and its applications*, Proc. 9th International Conference, Matfyzpress, Prague, 2005.

8. D. Ferus, H. Karcher and H. Münzner, *Cliffordalgebren und neue isoparametrische Hyperflächen*, Math. Z. **177**, (1981), 479–502.

9. P. Gilkey, *Geometric properties of natural operators defined by the Riemann curvature tensor*, World Scientific, Singapore, 2001.

10. ———, *The geometry of curvature homogeneous pseudo-Riemannian manifolds*, Imperial College Press, London, 2007.

11. P. Gilkey, R. Ivanova and T. Zhang, *Szabo Osserman IP pseudo-Riemannian manifolds*, Publ. Math. Debrecen **62**, (2003), 387–401.

12. P. Gilkey and S. Nikčević, *Affine curvature homogeneous* 3*-dimensional Lorentz manifolds*, Int. J. Geom. Meth. Mod. Phys. **2** (2005), 737–749.

13. ———, *Complete curvature homogeneous pseudo-Riemannian manifolds*, Classical and Quantum Gravity **21**, (2004), 3755–3770.

14. ———, *Nilpotent spacelike Jordan Osserman pseudo-Riemannian manifolds*, Rend. Circ. Mat. Palermo, (2), Suppl. No. **72** (2004), 99–105.

15. ———, *Complete k-curvature homogeneous pseudo-Riemannian manifolds*, Annals of Global Analysis, **27** (2005), 87–100.

16. ———, *Isometry groups of k-curvature homogeneous manifolds*, Rend. del Circ. Mat. Pelermo, Proc. 25th Winter School Geometry and Physics, Smi 2005, to appear.

17. P. Gilkey and S. Nikčević, *Generalized plane wave manifolds*, Kragujevac J. Mathematics **28** (2005), 113–138.

18. M. Gromov, *Partial differential relations*, Ergeb. Math. Grenzgeb 3. Folge, Band 9, Springer-Verlag, Berlin, 1986.

19. O. Kowalski, F. Tricerri and L. Vanhecke, *Curvature homogeneous Riemannian manifolds*, J. Math. Pures Appl. **71** (1992), 471–501.

20. R. Milson and N. Pelavas, *The curvature homogeneity bound for Lorentzian four-manifolds*, Int. J. Geom. Methods Mod. Phys. **6** (2009), 99–127.

21. B. Opozda, *Affine versions of Singer's theorem on locally homogeneous spaces*, Ann. Global Anal. Geom. **15** (1997), 187–199.

22. F. Podesta and A. Spiro, *Introduzione ai Gruppi di Trasformazioni*, Preprint Series of the Mathematics Department "V. Volterra" of the University of Ancona, Via delle Brecce Bianche, Ancona, Italy, 1996.

23. F. Prüfer, F. Tricerri and L. Vanhecke, *Curvature invariants, differential operators and local homogeneity*, Trans. Amer. Math. Soc. **348** (1996), 4643–4652.

24. E. García Rio, D. Kupeli and R. Vázquez-Lorenzo, *Osserman manifolds in semi-Riemannian Geometry*, Lecture Notes Math., Springer-Verlag, Berlin, 2002.

25. K. Sekegawa, H. Suga and L. Vanhecke, *Four-dimensional curvature homogeneous spaces*, Comm. Math. Univ. Carol. **33** (1992), 261–268.

26. K. Sekegawa, H. Suga and L. Vanhecke, *Curvature homogeneity for four-dimensional manifolds*, J. Korean Math. Soc. **32** (1995), 93–101.

27. I.M. Singer, *Infinitesimally homogeneous spaces*, Commun. Pure Appl. Math. **13** (1960), 685–697.

28. H. Takagi, *On curvature homogeneity of Riemannian manifolds*, Tôhoku Math. J. **26** (1974), 581–585.

29. F. Tricerri and L. Vanhecke, *Variétés riemanniennes dont le tenseur de courbure est celui d'un espace symétrique riemannien irréductible*, C.R. Acad. Sci. Paris **302** (1986), 233–235.

30. K. Yamato, *Algebraic Riemann manifolds*, Nagoya Math. J. **115**, (1989), 87–104.

Mathematics Department, California State University@San Bernardino, San Bernardino, CA 92407
Email address: cmdunn@csusb.edu