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# AMBARZUMYAN-TYPE THEOREMS FOR THE STURM-LIOUVILLE EQUATION ON A GRAPH

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ABSTRACT. In this paper we consider the inverse spectral problem of small vibrations of a graph consisting of $d$, $d \geq 2$, $d \in \mathbf{N}$, joint inhomogeneous smooth strings which can be reduced to the Sturm-Liouville boundary value problem on a graph. This problem occurs also in quantum mechanics. An analog of Ambarzumyan's theorem is proved for the case of a Sturm-Liouville problem on the compact metric graph consisting of $d$ segments of equal length with the Neumann boundary conditions at the pendant vertices and Kirchhoff boundary conditions at the central vertex, which case is also exceptional. We also extend Ambarzumyan's theorem of a Sturm-Liouville problem to the compact metric graph with the Dirichlet boundary conditions at the pendant vertices, by imposing an additional condition on the potential functions. The proof is based on the Gelfand-Levitan equation and variational principle.

**1. Setting of the problem.** From an historical viewpoint, the paper [**1**] of Ambarzumyan may be thought to be the starting point of the inverse spectral theory aiming to reconstruct the potential from the spectrum (or spectra), Ambarzumyan proved the following theorem:

*If $q \in C[0,\pi]$, and $\{n^2 : n = 0,1,2,\dots\}$ is the spectra set of the boundary value problem*

$$-y'' + q(x)y = \lambda y, \quad y'(0) = y'(\pi) = 0,$$

*then $q \equiv 0$ in $[0,\pi]$.*

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Later it became clear that the case investigated by Ambarzumyan was exceptional; in general, two spectra are needed to determine the potential [**2, 3, 10**]. Various generalizations of Ambarzumyan's theorem can be found in [**1, 5, 6, 7, 9, 12**], etc.

Quantum graphs are differential (self-adjoint) operators on metric graphs determined on the functions satisfying certain boundary conditions at the vertices. Differential operators on metric graphs (quantum graphs) is a rather new and rapidly developing area of modern mathematical physics. Such operators can be used to model the motion of quantum particles confined to certain low-dimensional structures. Inverse spectral problems of the Sturm-Liouville equation on graphs were investigated by Brown and Weikard [**4**], Pivovarchik [**11, 12**], Wassel [**13**] and Yurko [**14**], etc.

In the present paper we consider inverse spectral problems for the Sturm-Liouville equation on a star-shaped metric graph consisting of $d$ segments of equal length. For their inverse spectral problems, in general one spectrum does not uniquely determine the potential on the edges of the graph [**11**]. However, as in the case of a single interval, there are exceptional cases, in which the potential is uniquely determined by its spectrum. We consider an exceptional case in which a part of one spectrum of a boundary value problem with the Neumann boundary conditions at the pendant vertices uniquely determines the set of potentials on the edges of the graph. We also investigate extensional Ambarzumyan-type theorems, that is, a part of one spectrum of a boundary value problem with the Dirichlet boundary conditions at the pendant vertices and an additional condition on the potential functions uniquely determine the set of potentials on the edges of the graph. The proof uses the Gelfand-Levitan equation and variational principle [**7, 9, 10**]. In paper [**12**] a star-shaped graph consisting of three segments was considered with the Neumann boundary conditions at the pendant vertices and Kirchhoff boundary conditions at the central vertex. The present paper extends the results in [**12**] to star graphs consisting of an arbitrary number of segments with the Neumann boundary conditions and the Dirichlet boundary conditions at the pendant vertices, respectively.

Consider the following boundary value problem:

$$(1.1_j) \qquad -y_j'' + q_j(x)y_j = \lambda y_j, \quad j = 1, 2, \dots, d;\ d \geq 2,\ d \in \mathbf{N}$$

subject to

$$(1.2_j)_1 \qquad y_j(0) = 0, \quad j = 1, 2, \dots, d$$

or

$$(1.2_j)_2 \qquad y_j'(0) = 0, \quad j = 1, 2, \dots, d,$$

$$(1.3) \qquad y_1(\lambda, \pi) = y_2(\lambda, \pi) = \cdots = y_d(\lambda, \pi),$$

$$(1.4) \qquad y_1'(\lambda, \pi) + y_2'(\lambda, \pi) + \cdots + y_d'(\lambda, \pi) = 0,$$

where the $q_j \in L^2[0, \pi]$, $j = 1, 2, \dots, d$, are real-valued functions. Equation (1.3) is called a continuity condition, (1.4) a Kirchhoff condition and the collection of both interface conditions. This problem occurs in the small vibrations of a graph of $d$ inhomogeneous smooth strings each having one end joint, and a quantum particle moving in a quasi one-dimensional graph domain.

For all

$$f, g \in L_d^2[0, \pi] =: \bigoplus_{i=1}^{d} L^2[0, \pi],$$

define an inner product and a norm

$$(f, g) = \sum_{j=1}^{d} \int_0^\pi f_j(x)\overline{g_j}(x)\, dx, \quad ||f|| = \left( \sum_{j=1}^{d} \int_0^\pi |f_j(x)|^2\, dx \right)^{1/2},$$

where $f = (f_1, \dots, f_d)^T$, $g = (g_1, \dots, g_d)^T$.

Give the operator interpretation of the above problems. By $A_1, A_2$ we denote the operator acting in Hilbert space $L_d^2[0, \pi]$ by the formulas, respectively,

$$A_1 \begin{pmatrix} y_1(x) \\ \vdots \\ y_d(x) \end{pmatrix} = \begin{pmatrix} -y_1''(x) + q_1(x)y_1(x) \\ \vdots \\ -y_d''(x) + q_d(x)y_d(x) \end{pmatrix},$$

$$D(A_1) = \left\{ \begin{array}{l} \begin{pmatrix} y_1(x) \\ \vdots \\ y_d(x) \end{pmatrix} : y_j \in L^2[0, \pi], -y_j'' + q_j y_j \in L^2[0, \pi],\ y_j(0) = 0 \\ j = 1, \dots, d,\ \ y_1(\lambda, \pi) = \cdots = y_d(\lambda, \pi),\ \sum_{j=1}^{d} y_j'(\lambda, \pi) = 0 \end{array} \right\};$$

and

$$A_2\begin{pmatrix} y_1(x) \\ \vdots \\ y_d(x) \end{pmatrix} = \begin{pmatrix} -y_1''(x) + q_1(x)y_1(x) \\ \vdots \\ -y_d''(x) + q_d(x)y_d(x) \end{pmatrix},$$

$$D(A_2) = \left\{ \begin{array}{l} \begin{pmatrix} y_1(x) \\ \vdots \\ y_d(x) \end{pmatrix} : y_j \in L^2[0,\pi],\ -y_j'' + q_j y_j \in L^2[0,\pi],\ y_j'(0) = 0 \\ j = 1, \dots, d,\ y_1(\lambda,\pi) = \cdots = y_d(\lambda,\pi),\ \sum_{j=1}^d y_j'(\lambda,\pi) = 0 \end{array} \right\}.$$

Here the operator $A_i$, $i = 1, 2$, corresponds to problems $(1.1_j)$, $(1.2_j)_i$, (1.3) and (1.4), respectively. It is easy to check that the operators $A_i$, $i = 1, 2$, are self-adjoint, and each operator's spectrum, which consists of eigenvalues, is real. Denote the spectrum of the operator $A_i$ by $\sigma(A_i)$, $i = 1, 2$.

The main results of this paper are as follows.

**Theorem 1.1.** *Let the real-valued functions $q_j \in L^2[0,\pi]$, $j = 1, 2, \dots, d$. If $\{0\} \cup \{m_k^2 : k = 1, 2, \dots\} \subset \sigma(A_2)$, where 0 is the first eigenvalue of $A_2$, and $m_k$ is a strictly ascending infinite sequence of positive integers, then $q_j(x) = 0$, $j = 1, 2, \dots, d$, almost everywhere in $L^2[0,\pi]$.*

**Theorem 1.2.** *Let the real-valued functions $q_j \in L^2[0,\pi]$, $j = 1, 2, \dots, d$. If $\{0\} \cup \{(m_k - (1/2))^2 : k = 1, 2, \dots\} \subset \sigma(A_2)$, where 0 is the first eigenvalue of $A_2$ and the multiplicity of each eigenvalue $(m_k - (1/2))^2$ is $d - 1$, $m_k$ is a strictly ascending infinite sequence of positive integers, then $q_j(x) = 0$, $j = 1, 2, \dots, d$, almost everywhere in $L^2[0,\pi]$.*

**Theorem 1.3.** *Let the real-valued functions $q_j \in L^2[0,\pi]$, $j = 1, 2, \dots, d$. Then $(1/4) \cup \{(m_k - (1/2))^2 : k = 1, 2, \dots\} \subset \sigma(A_1)$, where $1/4$ is the first eigenvalue of $A_1$, $m_k$ is a strictly ascending infinite sequence of positive integers, and the potential functions $q_j(x)$, $j = 1, 2, \dots, d$, satisfy*

$$\int_0^\pi \cos x \sum_{j=1}^d q_j(x)\,dx = 0$$

*if and only if* $q_j(x) = 0$, $j = 1, 2, \dots, d$, *almost everywhere in* $L^2[0, \pi]$.

**Theorem 1.4.** *Let the real-valued functions* $q_j \in L^2[0, \pi]$, $j = 1, 2, \dots, d$. *Then* $\{1/4\} \cup \{m_k^2 : k = 1, 2, \dots\} \subset \sigma(A_1)$, *where* $1/4$ *is the first eigenvalue of* $A_1$ *and the multiplicity of each eigenvalue* $m_k^2$ *is* $d-1$, $m_k$ *is a strictly ascending infinite sequence of positive integers, and the potential functions* $q_j(x)$, $j = 1, 2, \dots, d$, *satisfy*

$$\int_0^\pi \cos x \sum_{j=1}^d q_j(x)\,dx = 0$$

*if and only if* $q_j(x) = 0$, $j = 1, 2, \dots, d$, *almost everywhere in* $L^2[0, \pi]$.

The paper is organized as follows. In Section 2, we obtain the equation for eigenvalues of problems $(1.1_j)$, $(1.2_j)_1$ or $(1.2_j)_2$, (1.3) and (1.4), respectively. Finally, Section 3 proves theorems obtained in this paper.

**2. Equation for the eigenvalues.** In this section, by resorting to the Gelfand-Levitan equation developed in [**7, 10**], we derive the equation for eigenvalues of the problem $(1.1_j)$, $(1.2_j)_1$ or $(1.2_j)_2$, (1.3) and (1.4), respectively.

First we study the equation for eigenvalues of problems $(1.1_j)$, $(1.2_j)_1$, (1.3) and (1.4). Denote by $s_j(\lambda, x)$, $j = 1, 2, \dots, d$, the solution of $(1.1_j)$ that satisfies the conditions

$$(2.1_j) \qquad s_j(\lambda, 0) = 0, \quad s_j'(\lambda, 0) = 1;$$

then the solution of equation $(1.1_j)$ that satisfies the condition $(1.2_j)_1$ is

$$(2.2_j) \qquad y_j(\lambda, x) = c_j s_j(\lambda, x),$$

where $c_j$ are constants. Substituting $(2.2_j)$ into (1.3) and (1.4), we obtain the following equation for eigenvalues of the operator $A_1$: $\lambda$ is an eigenvalue of operator $A_1$ if and only if

(2.3)
$$\varphi_1(\lambda) =: \begin{vmatrix} s_1(\lambda,\pi) & -s_2(\lambda,\pi) & 0 & \cdots & 0 & 0 \\ 0 & s_2(\lambda,\pi) & -s_3(\lambda,\pi) & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & s_{d-1}(\lambda,\pi) & -s_d(\lambda,\pi) \\ s_1'(\lambda,\pi) & s_2'(\lambda,\pi) & s_3'(\lambda,\pi) & \cdots & s_{d-1}'(\lambda,\pi) & s_d'(\lambda,\pi) \end{vmatrix}$$
$$= \sum_{j=1}^{d} s_j'(\lambda,\pi) \prod_{j\neq l\in\{1,2,\dots,d\}} s_l(\lambda,\pi) = 0.$$

Making use of the formulas in [**7, 10**], we have

$$\begin{aligned} s_j(\lambda,x) &= \frac{\sin(\sqrt{\lambda}x)}{\sqrt{\lambda}} + \frac{1}{\sqrt{\lambda}}\int_0^x K_j(x,t)\sin(\sqrt{\lambda}t)\,dt \\ &= \frac{\sin(\sqrt{\lambda}x)}{\sqrt{\lambda}} - \frac{\cos(\sqrt{\lambda}x)}{\lambda}K_j(x,x) \\ &\quad + \frac{1}{\lambda}\int_0^x K_{j,t}(x,t)\cos(\sqrt{\lambda}t)\,dt; \\ s_j'(\lambda,x) &= \cos(\sqrt{\lambda}x) + \frac{K_j(x,x)}{\sqrt{\lambda}}\sin(\sqrt{\lambda}x) \\ &\quad + \frac{1}{\sqrt{\lambda}}\int_0^x K_{j,x}(x,t)\sin(\sqrt{\lambda}t)\,dt, \end{aligned} \tag{2.4$_j$}$$

where $K_j(x,t)$, $j = 1,2,\dots,d$, have first partial derivatives $K_{j,x}(x,t)$ and $K_{j,t}(x,t)$ belonging to $L^2[0,\pi]$ as functions of $t$ for fixed $x$, and

$$\begin{gathered} K_j(x,t) = 0 \ (t > x), \qquad K_j(0,0) = 0, \qquad K_j(x,0) = 0, \\ K_j(\pi,\pi) = \frac{1}{2}\int_0^\pi q_j(x)\,dx. \end{gathered} \tag{2.5}$$

For brevity, we set

$$a_j = \int_0^\pi K_{j,x}(\pi,t)\sin(\sqrt{\lambda}t)\,dt, \quad b_j = \int_0^\pi K_{j,t}(\pi,t)\cos(\sqrt{\lambda}t)\,dt;$$

then the Riemann-Lebesgue lemma implies

$$a_j \longrightarrow 0, \qquad b_j \longrightarrow 0 \quad (\text{as } \lambda \to \infty). \tag{2.6}$$

**Lemma 2.1.** *The function $\varphi_1(\lambda)$ defined by (2.3) corresponding to the operator $A_1$ can be presented as follows*:

(2.7)

$$\begin{aligned}\varphi_1(\lambda) = {}& \frac{d\cos(\sqrt{\lambda}\pi)\sin^{d-1}(\sqrt{\lambda}\pi)}{\sqrt{\lambda}^{d-1}} + \frac{(d-1)\cos(\sqrt{\lambda}\pi)\sin^{d-2}(\sqrt{\lambda}\pi)}{\sqrt{\lambda}^{d}} \\ & \times \sum_{j=1}^{d} b_j - \frac{(d-1)\cos^2(\sqrt{\lambda}\pi)\sin^{d-2}(\sqrt{\lambda}\pi)}{\sqrt{\lambda}^{d}} \\ & \times \sum_{j=1}^{d} K_j(\pi,\pi) + \frac{\sin^{d-1}(\sqrt{\lambda}\pi)\sum_{j=1}^{d} a_j}{\sqrt{\lambda}^{d}} \\ & + \frac{\sin^{d}(\sqrt{\lambda}\pi)}{\sqrt{\lambda}^{d}} \sum_{j=1}^{d} K_j(\pi,\pi) + \cos(\sqrt{\lambda}\pi)\sum_{j=1}^{d}\sum_{m=1}^{d-2}\frac{\alpha_m^j}{\sqrt{\lambda}^{d+m}} \\ & + \sin(\sqrt{\lambda}\pi)\sum_{j=1}^{d}\sum_{m=1}^{d-1}\frac{K_j(\pi,\pi)\beta_m^j}{\sqrt{\lambda}^{d+m}} + \sum_{j=1}^{d}\sum_{m=1}^{d-1}\frac{a_j\beta_m^j}{\sqrt{\lambda}^{d+m}},\end{aligned}$$

*where $\alpha_m^j$, $1 \le m \le d-2$, $\beta_m^j$, $1 \le m \le d-1$, $1 \le j \le d$, are entire functions in $\lambda$, which are the finite summation of the finite products with $b_j$, $\sin(\sqrt{\lambda}\pi)$ and $\cos(\sqrt{\lambda}\pi)$.*

*Proof.* By (2.3) and ($2.4_j$) we have

$$\begin{aligned}\varphi_1(\lambda) &= \sum_{j=1}^{d}\left[\cos(\sqrt{\lambda}\pi) + \frac{K_j(\pi,\pi)}{\sqrt{\lambda}}\sin(\sqrt{\lambda}\pi) + \frac{a_j}{\sqrt{\lambda}}\right] \\ &\quad \times \prod_{j\neq l\in\{1,2,\dots,d\}}\left[\frac{\sin(\sqrt{\lambda}\pi)}{\sqrt{\lambda}} + \frac{b_l - \cos(\sqrt{\lambda}\pi)K_l(\pi,\pi)}{\lambda}\right] \\ &= \sum_{j=1}^{d}\left[\cos(\sqrt{\lambda}\pi) + \frac{K_j(\pi,\pi)}{\sqrt{\lambda}}\sin(\sqrt{\lambda}\pi) + \frac{a_j}{\sqrt{\lambda}}\right] \\ &\quad \times \left[\frac{\sin^{d-1}(\sqrt{\lambda}\pi)}{\sqrt{\lambda}^{d-1}} + \frac{\sin^{d-2}(\sqrt{\lambda}\pi)}{\sqrt{\lambda}^{d}}\right. \\ &\quad \times \sum_{j\neq l\in\{1,2,\dots,d\}}(b_l - \cos(\sqrt{\lambda}\pi)K_l(\pi,\pi)) + \frac{\sin^{d-3}(\sqrt{\lambda}\pi)}{\sqrt{\lambda}^{d+1}}\end{aligned}$$

$$\times \sum_{\substack{l<l_1\\ j\neq l\\ l_1\in\{1,2,\cdots,d\}}} (b_l-\cos(\sqrt{\lambda}\pi)K_l(\pi,\pi))(b_{l_1}-\cos(\sqrt{\lambda}\pi)K_{l_1}(\pi,\pi))$$

$$+\cdots+\frac{1}{\sqrt{\lambda}^{2d-2}}\prod_{j\neq l\in\{1,2,\cdots,d\}}(b_l-\cos(\sqrt{\lambda}\pi)K_l(\pi,\pi))\Bigg]$$

$$=\sum_{j=1}^{d}\Bigg[\frac{\cos(\sqrt{\lambda}\pi)\sin^{d-1}(\sqrt{\lambda}\pi)}{\sqrt{\lambda}^{d-1}}+\frac{\cos(\sqrt{\lambda}\pi)\sin^{d-2}(\sqrt{\lambda}\pi)}{\sqrt{\lambda}^{d}}$$

$$\times\sum_{j\neq l\in\{1,2,\cdots,d\}} b_l-\frac{\cos^2(\sqrt{\lambda}\pi)\sin^{d-2}(\sqrt{\lambda}\pi)}{\sqrt{\lambda}^{d}}$$

$$\times\sum_{j\neq l\in\{1,2,\cdots,d\}} K_l(\pi,\pi)$$

$$+\cos(\sqrt{\lambda}\pi)\sum_{m=1}^{d-2}\frac{\alpha_m^j}{\sqrt{\lambda}^{d+m}}+\frac{\sin^d(\sqrt{\lambda}\pi)}{\sqrt{\lambda}^{d}}K_j(\pi,\pi)$$

$$+K_j(\pi,\pi)\sin(\sqrt{\lambda}\pi)\sum_{m=1}^{d-1}\frac{\beta_m^j}{\sqrt{\lambda}^{d+m}}+\frac{\sin^{d-1}(\sqrt{\lambda}\pi)a_j}{\sqrt{\lambda}^{d}}$$

$$+a_j\sum_{m=1}^{d-1}\frac{\beta_m^j}{\sqrt{\lambda}^{d+m}}\Bigg].$$

Therefore, equation (2.7) is obvious and we finish the proof. □

If $q_j(x)\equiv 0$, then, by substituting $s_j(\lambda,\pi)=(\sin(\sqrt{\lambda}\pi))/\sqrt{\lambda}$ and $s_j'(\lambda,\pi)=\cos(\sqrt{\lambda}\pi)$ into (2.3), we get

$$\varphi_1(\lambda)=\cos(\sqrt{\lambda}\pi)\frac{\sin^{d-1}(\sqrt{\lambda}\pi)}{\sqrt{\lambda}^{d-1}}.$$

The set of zeros of this characteristic function, which consists of eigenvalues of the operator $A_1$ as $q_j(x)\equiv 0$.

The geometric multiplicity of an eigenvalue is the dimension of its eigenspace, that is, the number of its linearly independent eigenfunctions. For the operator $A_1$ this number is 1 to $d$. The algebraic multiplicity is defined in terms of a characteristic function. This is a function whose zeros are precisely the eigenvalues of the problem. The order of a zero is the algebraic multiplicity of the corresponding eigenvalue. In particular, the geometric multiplicity of each eigenvalue of the self-adjoint operator $A_1$ is equal to its algebraic multiplicity. The set of zeros of the function $\varphi_1(\lambda)$ is $\{n^2\}_{n=1}^{\infty} \cup \{(n-(1/2))^2\}_{n=1}^{\infty}$. Since

$$\frac{d}{d\lambda}\varphi_1(\lambda)|_{\lambda=(n-(1/2))^2} \neq 0$$

and

$$\frac{d}{d\lambda}\varphi_1(\lambda)|_{\lambda=n^2} = \cdots = \frac{d^{d-2}}{d\lambda^{d-2}}\varphi_1(\lambda)|_{\lambda=n^2} = 0,$$
$$\frac{d^{d-1}}{d\lambda^{d-1}}\varphi_1(\lambda)|_{\lambda=n^2} \neq 0,$$

the algebraic multiplicity of each eigenvalue in $\{(n-(1/2))^2\}_{n=1}^{\infty}$ is 1 and the algebraic multiplicity of each eigenvalue in $\{n^2\}_{n=1}^{\infty}$ is $d-1$. Thus the following lemma is obtained.

**Lemma 2.2.** *If $q_j(x) \equiv 0$, $j = 1, 2, \ldots, d$, then*

$$\sigma(A_1) = \{n^2\}_{n=1}^{\infty} \bigcup \left\{ \left(n - \frac{1}{2}\right)^2 \right\}_{n=1}^{\infty}.$$

*Moreover, the (geometric) multiplicity of each eigenvalue in $\{(n-(1/2))^2\}_{n=1}^{\infty}$ is simple and the (geometric) multiplicity of each eigenvalue in $\{n^2\}_{n=1}^{\infty}$ is equal to $d-1$.*

Now we investigate the equation for eigenvalues of the problem $(1.1_j)$, $(1.2_j)_2$, (1.3) and (1.4), and denote by $\widetilde{s}_j(\lambda, x)$, $j = 1, 2, \ldots, d$, the solution of $(1.1_j)$ that satisfies the conditions

$$(2.8_j) \qquad \widetilde{s}_j(\lambda, 0) = 1, \qquad \widetilde{s}_j'(\lambda, 0) = 0.$$

Then the solutions of equations $(1.1_j)$ that satisfy the conditions $(1.2_j)_2$ are

$$(2.9_j) \qquad y_j(\lambda, x) = \widetilde{c}_j \widetilde{s}_j(\lambda, x),$$

where $\widetilde{c}_j$ are constants. Substituting $(2.9_j)$ into (1.3) and (1.4), we obtain the following equation for eigenvalues of the operator $A_2$: $\lambda$ is an eigenvalue of the operator $A_2$ if and only if

$$\varphi_2(\lambda) = \sum_{j=1}^{d} \widetilde{s}'_j(\lambda, \pi) \prod_{j \neq l \in \{1,2,\cdots,d\}} \widetilde{s}_l(\lambda, \pi) = 0. \tag{2.10}$$

Using the formulas in [**7, 10**], we have

$$\begin{aligned} \widetilde{s}_j(\lambda, x) &= \cos(\sqrt{\lambda}x) + \frac{\sin(\sqrt{\lambda}x)}{\sqrt{\lambda}} \widetilde{K}_j(x, x) \\ &\quad - \frac{1}{\sqrt{\lambda}} \int_0^x \widetilde{K}_{j,t}(x, t) \sin(\sqrt{\lambda}t)\, dt; \\ \widetilde{s}'_j(\lambda, x) &= -\sqrt{\lambda} \sin(\sqrt{\lambda}x) + \widetilde{K}_j(x, x) \cos(\sqrt{\lambda}x) \\ &\quad + \int_0^x \widetilde{K}_{j,x}(x, t) \cos(\sqrt{\lambda}t)\, dt, \end{aligned} \tag{$2.11_j$}$$

where $\widetilde{K}_j(x, t)$, $j = 1, \ldots, d$, have first partial derivatives $\widetilde{K}_{j,x}(x, t)$ and $\widetilde{K}_{j,t}(x, t)$ belonging to $L^2[0, \pi]$ as functions of $t$ for the fixed $x$, and

$$\widetilde{K}_j(\pi, \pi) = \frac{1}{2} \int_0^\pi q_j(x)\, dx. \tag{2.12}$$

For brevity, we set

$$c_j = -\int_0^\pi \widetilde{K}_{j,t}(\pi, t) \sin(\sqrt{\lambda}t)\, dt, \quad d_j = \int_0^\pi \widetilde{K}_{j,x}(\pi, t) \cos(\sqrt{\lambda}t)\, dt;$$

then the Riemann-Lebesgue lemma implies

$$c_j \longrightarrow 0, \qquad d_j \longrightarrow 0 \quad (\text{as } \lambda \to \infty). \tag{2.13}$$

**Lemma 2.3.** *The function $\varphi_2(\lambda)$ defined by* (2.10) *corresponding to the operator $A_2$ can be presented as follows*:

$$\begin{aligned}\varphi_2(\lambda) = & -d\sqrt{\lambda}\sin(\sqrt{\lambda}\pi)\cos^{d-1}(\sqrt{\lambda}\pi) - (d-1)\\ & \times \sin(\sqrt{\lambda}\pi)\cos^{d-2}(\sqrt{\lambda}\pi)\sum_{j=1}^{d} c_j - (d-1)\\ & \times \sin^2(\sqrt{\lambda}\pi)\cos^{d-2}(\sqrt{\lambda}\pi)\\ & \times \sum_{j=1}^{d}\widetilde{K}_j(\pi,\pi) + \cos^{d-1}(\sqrt{\lambda}\pi)\sum_{j=1}^{d} d_j\\ & + \cos^{d}(\sqrt{\lambda}\pi)\sum_{j=1}^{d}\widetilde{K}_j(\pi,\pi) + \sin(\sqrt{\lambda}\pi)\sum_{j=1}^{d}\sum_{m=1}^{d-1}\frac{f_m^j}{\sqrt{\lambda}^m}\\ & + \cos(\sqrt{\lambda}\pi) \times \sum_{j=1}^{d}\sum_{m=1}^{d-1}\frac{\widetilde{K}_j(\pi,\pi)g_m^j}{\sqrt{\lambda}^m} + \sum_{j=1}^{d}\sum_{m=1}^{d-1}\frac{d_j h_m^j}{\sqrt{\lambda}^m},\end{aligned} \tag{2.14}$$

*where $f_m^j$, $g_m^j$, $h_m^j$, $1 \le m \le d-1$, $1 \le j \le d$, are entire functions in $\lambda$, which are the finite summation of the finite products with $c_j$,* $\sin(\sqrt{\lambda}\pi)$ *and* $\cos(\sqrt{\lambda}\pi)$.

*Proof.* From (2.10) and $(2.11_j)$ we have

$$\begin{aligned}\varphi_2(\lambda) = & \sum_{j=1}^{d}\Big[-\sqrt{\lambda}\sin(\sqrt{\lambda}\pi) + \widetilde{K}_j(\pi,\pi)\cos(\sqrt{\lambda}\pi) + d_j\Big]\\ & \times \prod_{j\neq l\in\{1,2,\dots,d\}}\Big[\cos(\sqrt{\lambda}\pi) + \frac{\sin(\sqrt{\lambda}\pi)}{\sqrt{\lambda}}\widetilde{K}_l(\pi,\pi) + \frac{c_l}{\sqrt{\lambda}}\Big]\\ = & \sum_{j=1}^{d}\Big[-\sqrt{\lambda}\sin(\sqrt{\lambda}\pi) + \widetilde{K}_j(\pi,\pi)\cos(\sqrt{\lambda}\pi) + d_j\Big]\\ & \times \Big[\cos^{d-1}(\sqrt{\lambda}\pi) + \frac{\cos^{d-2}(\sqrt{\lambda}\pi)\sin(\sqrt{\lambda}\pi)}{\sqrt{\lambda}}\\ & \times \sum_{j\neq l\in\{1,2,\dots,d\}}\widetilde{K}_l(\pi,\pi) + \frac{\cos^{d-2}(\sqrt{\lambda}\pi)}{\sqrt{\lambda}}\end{aligned}$$

$$
\begin{aligned}
&\times \sum_{j\neq l\in\{1,2,\dots,d\}} c_l + \frac{\cos^{d-3}(\sqrt{\lambda}\pi)}{\lambda} \\
&\times \sum_{\substack{l<l_1 \\ j\neq l \\ l_1\in\{1,2,\dots,d\}}} (c_l + \sin(\sqrt{\lambda}\pi)\widetilde{K}_l(\pi,\pi))(c_{l_1} + \sin(\sqrt{\lambda}\pi)\widetilde{K}_{l_1}(\pi,\pi)) \\
&\qquad + \cdots + \frac{1}{\sqrt{\lambda}^{d-1}} \prod_{j\neq l\in\{1,2,\dots,d\}} (c_l + \sin(\sqrt{\lambda}\pi)\widetilde{K}_l(\pi,\pi))\Bigg] \\
= &\sum_{j=1}^{d}\Bigg[ -\sqrt{\lambda}\sin(\sqrt{\lambda}\pi)\cos^{d-1}(\sqrt{\lambda}\pi) - \sin(\sqrt{\lambda}\pi)\cos^{d-2}(\sqrt{\lambda}\pi) \\
&\quad \times \sum_{j\neq l\in\{1,2,\dots,d\}} c_l - \sin^2(\sqrt{\lambda}\pi)\cos^{d-2}(\sqrt{\lambda}\pi) \\
&\quad \times \sum_{j\neq l\in\{1,2,\dots,d\}} \widetilde{K}_l(\pi,\pi) \\
&\quad + \sin(\sqrt{\lambda}\pi)\sum_{m=1}^{d-1}\frac{f_m^j}{\sqrt{\lambda}^m} + \cos^d(\sqrt{\lambda}\pi)\widetilde{K}_j(\pi,\pi) \\
&\quad + \widetilde{K}_j(\pi,\pi)\cos(\sqrt{\lambda}\pi)\sum_{m=1}^{d-1}\frac{g_m^j}{\sqrt{\lambda}^m} \\
&\quad + d_j\cos^{d-1}(\sqrt{\lambda}\pi) + d_j\sum_{m=1}^{d-1}\frac{h_m^j}{\sqrt{\lambda}^m}\Bigg].
\end{aligned}
$$

The proof of the lemma is complete. □

If $q_j(x) \equiv 0$, then, by substituting $\widetilde{s}_j(\lambda,\pi) = \cos(\sqrt{\lambda}\pi)$, $\widetilde{s}'_j(\lambda,\pi) = -\sqrt{\lambda}\sin(\sqrt{\lambda}\pi)$ into (2.10), we get

$$\varphi_2(\lambda) = -d\sqrt{\lambda}\sin(\sqrt{\lambda}\pi)\cos^{d-1}(\sqrt{\lambda}\pi).$$

The set of zeros of this characteristic function, which consists of eigenvalues of the operator $A_2$ as $q_j(x)\equiv 0$. The arguments concerning the multiplicity of each eigenvalue refer to Lemma 2.2. From this, the following lemma is true.

**Lemma 2.4.** *If* $q_j(x) \equiv 0$, $j = 1, 2, \dots, d$, *then*

$$\sigma(A_2) = \{0\} \bigcup \{n^2\}_{n=1}^{\infty} \bigcup \left\{ \left(n - \frac{1}{2}\right)^2 \right\}_{n=1}^{\infty}.$$

*Moreover, the (geometric) multiplicity of each eigenvalue in* $\{n^2\}_{n=0}^{\infty}$ *is simple and the (geometric) multiplicity of each eigenvalue in* $\{(n-(1/2))^2\}_{n=1}^{\infty}$ *is equal to* $d-1$.

**3. The proof.** Now we can prove theorems in this paper through a series of lemmata established in Section 2.

*Proof of Theorem* 1.1. Since the equation for eigenvalues of the operator $A_2$ is that $\varphi_2(\lambda) = 0$ and $\{m_k^2 : k = 1, 2, \dots\} \subset \sigma(A_2)$, for all $k = 1, 2, \dots$, we obtain $\varphi_2(m_k^2) = 0$. From equation (2.14) in Lemma 2.3 and $\{m_k\}_{k=1}^{\infty} \subset \mathbf{N}$, $\varphi_2(m_k^2) = 0$ implies

$$\begin{aligned} \varphi_2(m_k^2) = (-1)^{dm_k} \sum_{j=1}^{d} \widetilde{K}_j(\pi,\pi) + (-1)^{m_k} \sum_{j=1}^{d} \sum_{m=1}^{d-1} \frac{\widetilde{K}_j(\pi,\pi) g_m^j}{m_k^m} \\ + (-1)^{(d-1)m_k} \sum_{j=1}^{d} d_j + \sum_{j=1}^{d} \sum_{m=1}^{d-1} \frac{d_j h_m^j}{m_k^m} = 0. \end{aligned} \tag{3.1}$$

Since $m_k$ is a strictly ascending infinite sequence of positive integers, letting $m_k \to \infty$ in (3.1), together with (2.12) and (2.13), it follows that

$$\frac{1}{2} \int_0^{\pi} \sum_{j=1}^{d} q_j(x)\, dx = \sum_{j=1}^{d} \widetilde{K}_j(\pi,\pi) = 0. \tag{3.2}$$

Next we show that

$$Y_0 = \Bigg( \underbrace{\frac{1}{\sqrt{d\pi}}, \cdots, \frac{1}{\sqrt{d\pi}}}_{d} \Bigg)^T$$

is the first eigenfunction of $A_2$. By the variational principle, we obtain

$$\inf_{Y \in D(A_2), \sum_{j=1}^{d} ||y_j||^2 = 1} \left( - \int_0^{\pi} \sum_{j=1}^{d} y_j'' \overline{y_j}\, dx + \int_0^{\pi} \sum_{j=1}^{d} q_j(x) |y_j|^2\, dx \right) = 0, \tag{3.3}$$

where $Y = (y_1, y_2, \dots, y_d)^T$, $\|y_j\|^2 = \int_0^\pi |y_j|^2\,dx$. Now $||Y_0|| = 1$ and $Y_0 \in D(A_2)$ are obvious, and so

$$0 \le (A_2Y_0, Y_0) = \int_0^\pi \sum_{j=1}^d q_j(x)\left|\frac{1}{\sqrt{d\pi}}\right|^2 dx = \frac{1}{d\pi}\int_0^\pi \sum_{j=1}^d q_j(x)\,dx,$$

and by (3.2), the righthand side is exactly 0, the test function $Y_0$ makes the functional $(A_2Y, Y)/||Y||^2$ achieve its minimum value and is thus the first eigenfunction. Substituting $Y_0$ which is the eigenfunction of eigenvalue 0 into the equation $(1.1_j)$, we obtain $q_j(x) = 0$, $j = 1, 2, \dots, d$, almost everywhere in $L^2[0, \pi]$. The proof is finished. □

*Proof of Theorem* 1.2. Since the equation for eigenvalues of the operator $A_2$ is that $\varphi_2(\lambda) = 0$ and $\{(m_k - (1/2))^2 : k = 1, 2, \dots\} \subset \sigma(A_2)$, for all $k = 1, 2, \dots$, we obtain $\varphi_2((m_k - (1/2))^2) = 0$. Since the multiplicity of each eigenvalue $(m_k - (1/2))^2$ is $d - 1$, we have

$$\lim_{\lambda\to(m_k-(1/2))^2} \frac{\varphi_2(\lambda)}{\cos^{d-2}(\sqrt{\lambda}\pi)} \equiv 0 \tag{3.4}$$

identically in $k \in \mathbf{N}$.

From equation (2.14) in Lemma 2.3 and $\{m_k\}_{k=1}^\infty \subset \mathbf{N}$, using the identities $\cos((m_k - (1/2))\pi) = 0$ and $\sin((m_k - (1/2))\pi) = (-1)^{m_k+1}$, by (3.4) we get

$$\begin{aligned} 0 &\equiv \lim_{\lambda\to(m_k-(1/2))^2} \frac{\varphi_2(\lambda)}{\cos^{d-2}(\sqrt{\lambda}\pi)} \\ &= (-1)^{m_k}(d-1)\sum_{j=1}^d c_j - (d-1)\sum_{j=1}^d \widetilde{K}_j(\pi,\pi) \\ &\quad + \lim_{\lambda\to(m_k-(1/2))^2} \frac{A(\lambda)}{\sqrt{\lambda}^{d-1}\cos^{d-2}(\sqrt{\lambda}\pi)}, \end{aligned} \tag{3.5}$$

where the entire function in $\lambda$

$$
\begin{aligned}
A(\lambda) &= \sin(\sqrt{\lambda}\pi)\sum_{j=1}^{d}\sum_{m=1}^{d-1} f_m^j \sqrt{\lambda}^{d-m-1} \\
&\quad + \cos(\sqrt{\lambda}\pi)\sum_{j=1}^{d}\sum_{m=1}^{d-1} \widetilde{K}_j(\pi,\pi) g_m^j \sqrt{\lambda}^{d-m-1} \\
&\quad + \sum_{j=1}^{d}\sum_{m=1}^{d-1} d_j h_m^j \sqrt{\lambda}^{d-m-1}.
\end{aligned} \tag{3.6}
$$

Thus, $\lim_{\lambda\to(m_k-(1/2))^2}(A(\lambda)/\sqrt{\lambda}^{d-1}\cos^{d-2}(\sqrt{\lambda}\pi))$ exist for all $k \in \mathbf{N}$, but $(m_k-(1/2))^2$ is a zero of $\cos^{d-2}(\sqrt{\lambda}\pi)$ with multiplicity $d-2$, which implies that $(m_k-(1/2))^2$ is a zero of $A(\lambda)$ with multiplicity $d-2$ at least. (1) If $(m_k-(1/2))^2$ is a zero of $A(\lambda)$ with multiplicity $m > d-2$, then

$$
\lim_{\lambda\to(m_k-(1/2))^2} \frac{A(\lambda)}{\sqrt{\lambda}^{d-1}\cos^{d-2}(\sqrt{\lambda}\pi)} = 0; \tag{3.7}
$$

(2) If $(m_k-(1/2))^2$ is a zero of $A(\lambda)$ with multiplicity $d-2$ exactly, by (3.6) then

$$
\lim_{\lambda\to(m_k-(1/2))^2} \frac{A(\lambda)}{\sqrt{\lambda}^{d-1}\cos^{d-2}(\sqrt{\lambda}\pi)} = O(1/m_k). \tag{3.8}
$$

Combining (3.5), (3.7) and (3.8), one can readily see that

$$
(-1)^{m_k}(d-1)\sum_{j=1}^{d} c_j - (d-1)\sum_{j=1}^{d}\widetilde{K}_j(\pi,\pi) + O(1/m_k) = 0 \tag{3.9}
$$

or

$$
(-1)^{m_k}(d-1)\sum_{j=1}^{d} c_j - (d-1)\sum_{j=1}^{d}\widetilde{K}_j(\pi,\pi) = 0. \tag{3.10}
$$

Since $m_k$ is a strictly ascending infinite sequence of positive integers, letting $m_k \to \infty$ in (3.9) or (3.10), together with (2.12) and (2.13), it follows that

$$
\frac{1}{2}\int_0^{\pi}\sum_{j=1}^{d} q_j(x)\,dx = \sum_{j=1}^{d}\widetilde{K}_j(\pi,\pi) = 0.
$$

Next, using methods in the proof of Theorem 1.1, the proof is finished. □

*Proof of Theorem* 1.3. The sufficient part is obvious by Lemma 2.2. To prove the converse, we employ Lemma 2.1. Since the equation for eigenvalues of the operator $A_1$ is that $\varphi_1(\lambda) = 0$ and $\{(m_k - (1/2))^2 : k = 1, 2, \dots\} \subset \sigma(A_1)$, for all $k = 1, 2, \dots$, we obtain $\varphi_1((m_k - (1/2))^2) = 0$. From equation (2.7) in Lemma 2.1 and $\{m_k\}_{k=1}^{\infty} \subset \mathbf{N}$, $\varphi_1((m_k - (1/2))^2) = 0$ implies

$$\begin{aligned}\varphi_1\left(\left(m_k - \frac{1}{2}\right)^2\right) &= \frac{(-1)^{(m_k+1)d} \sum_{j=1}^{d} K_j(\pi,\pi)}{(m_k - (1/2))^d} \\ &\quad + (-1)^{(m_k+1)} \sum_{j=1}^{d} \sum_{m=1}^{d-1} \frac{K_j(\pi,\pi)\beta_m^j}{(m_k - (1/2))^{d+m}} \\ &\quad + \frac{(-1)^{(m_k+1)(d-1)} \sum_{j=1}^{d} a_j}{(m_k - (1/2))^d} \\ &\quad + \sum_{j=1}^{d} \sum_{m=1}^{d-1} \frac{a_j \gamma_m^j}{(m_k - (1/2))^{d+m}} \\ &= 0,\end{aligned}$$

which implies

(3.11)
$$\begin{aligned}(-1)^{(m_k+1)d} \sum_{j=1}^{d} K_j(\pi,\pi) &+ (-1)^{(m_k+1)} \sum_{j=1}^{d} \sum_{m=1}^{d-1} \frac{\beta_m^j K_j(\pi,\pi)}{(m_k - (1/2))^m} \\ &+ (-1)^{(m_k+1)(d-1)} \sum_{j=1}^{d} a_j + \sum_{j=1}^{d} \sum_{m=1}^{d-1} \frac{\gamma_m^j}{(m_k - (1/2))^m} = 0.\end{aligned}$$

Since $m_k$ is a strictly ascending infinite sequence of positive integers, letting $m_k \to \infty$ in (3.11), together with (2.5) and (2.6), it follows that

$$\frac{1}{2} \int_0^{\pi} \sum_{j=1}^{d} q_j(x)\, dx = \sum_{j=1}^{d} K_j(\pi,\pi) = 0. \tag{3.12}$$

Next we show that $Y_{1/4} = (\underbrace{\sin(x/2), \dots, \sin(x/2)}_{d})^T$ is the first eigenfunction of $A_1$. By the variational principle, we obtain

(3.13)

$$\begin{aligned}\frac{1}{4} &= \inf_{Y\in D(A_1), ||Y||=1} (A_1Y, Y)\\ &= \inf_{Y\in D(A_1), \sum_{j=1}^d ||y_j||^2=1} \bigg(-\int_0^\pi \sum_{j=1}^d y_j''\overline{y_j}\, dx + \int_0^\pi \sum_{j=1}^d q_j(x)|y_j|^2\, dx\bigg),\end{aligned}$$

where $Y = (y_1, y_2, \dots, y_d)^T$, $\|y_j\|^2 = \int_0^\pi |y_j|^2\, dx$. Now $Y_{1/4} \in D(A_1)$ is obvious, and so

$$\begin{aligned}\frac{1}{4} &\le \frac{(A_1Y_{1/4}, Y_{1/4})}{||Y_{1/4}||^2}\\ &= \frac{1/4\sum_{j=1}^d \int_0^\pi \sin^2(x/2)\, dx + \int_0^\pi \sum_{j=1}^d q_j(x)\sin^2(x/2)\, dx}{\sum_{j=1}^d \int_0^\pi \sin^2(x/2)\, dx}\\ &= \frac{1}{4} + \frac{\int_0^\pi \sum_{j=1}^d q_j(x)(1-\cos x)\, dx}{d\pi}\\ &= \frac{1}{4} + \frac{\int_0^\pi \sum_{j=1}^d q_j(x)\, dx}{d\pi} - \frac{\int_0^\pi \cos x \sum_{j=1}^d q_j(x)\, dx}{d\pi},\end{aligned}$$

by (3.12) and assumption, the righthand side is exactly $1/4$, the test function $Y_{1/4}$ makes the functional $(A_1Y, Y)/||Y||^2$ attain its minimum value and is thus the first eigenfunction. Substituting $Y_{1/4}$, which is the eigenfunction of eigenvalue $1/4$, into equation $(1.1_j)$, we obtain

$$\frac{1}{4}\sin\frac{x}{2} + q_j(x)\sin\frac{x}{2} = \frac{1}{4}\sin\frac{x}{2};$$

therefore, $q_j(x) = 0$, $j = 1, 2, \dots, d$, almost everywhere in $L^2[0, \pi]$. □

*Proof of Theorem* 1.4. The sufficient part is obvious by Lemma 2.2. To prove the necessity, we use Lemma 2.1. Since the equation for eigenvalues of the operator $A_1$ is that $\varphi_1(\lambda) = 0$ and $\{m_k^2 : k = 1, 2, \dots\} \subset \sigma(A_1)$, for all $k = 1, 2, \dots$, we obtain $\varphi_1(m_k^2) = 0$. Since the multiplicity of each eigenvalue $m_k^2$ is $d-1$, we have

$$\lim_{\lambda\to m_k^2}\frac{\varphi_1(\lambda)}{\sin^{d-2}\sqrt{\lambda}\pi}\equiv 0 \tag{3.14}$$

for all $k\in\mathbf{N}$.

From equation (2.7) in Lemma 2.1 and $\{m_k\}_{k=1}^{\infty}\subset\mathbf{N}$, using the identities $\cos(m_k\pi)=(-1)^{m_k}$ and $\sin(m_k\pi)=0$, by (3.14) we get

$$\begin{aligned}0&\equiv\lim_{\lambda\to m_k^2}\frac{\varphi_1(\lambda)}{\sin^{d-2}\sqrt{\lambda}\pi}\\&=\frac{(-1)^{m_k}(d-1)\sum_{j=1}^{d}b_j}{m_k^d}-\frac{(d-1)\sum_{j=1}^{d}K_j(\pi,\pi)}{m_k^d}\\&\quad+\lim_{\lambda\to m_k^2}\frac{B(\lambda)}{\sqrt{\lambda}^{2d-1}\sin^{d-2}(\sqrt{\lambda}\pi)},\end{aligned}\tag{3.15}$$

where the entire function in $\lambda$

$$\begin{aligned}B(\lambda)=\cos(\sqrt{\lambda}\pi)&\sum_{j=1}^{d}\sum_{m=1}^{d-2}\alpha_m^j\sqrt{\lambda}^{d-m-1}\\&+\sin(\sqrt{\lambda}\pi)\sum_{j=1}^{d}\sum_{m=1}^{d-1}K_j(\pi,\pi)\beta_m^j\sqrt{\lambda}^{d-m-1}\\&+\sum_{j=1}^{d}\sum_{m=1}^{d-1}a_j\gamma_m^j\sqrt{\lambda}^{d-m-1}.\end{aligned}\tag{3.16}$$

Thus, $\lim_{\lambda\to m_k^2}(B(\lambda)/\sqrt{\lambda}^{2d-1}\sin^{d-2}(\sqrt{\lambda}\pi))$ exist for all $k\in\mathbf{N}$, but $m_k^2$ is a zero of $\sin^{d-2}(\sqrt{\lambda}\pi)$ with multiplicity $d-2$, which implies that $m_k^2$ is a zero of $B(\lambda)$ with multiplicity $d-2$ at least. (1) If $m_k^2$ is a zero of $B(\lambda)$ with multiplicity $m>d-2$, then

$$\lim_{\lambda\to m_k^2}\frac{B(\lambda)}{\sqrt{\lambda}^{2d-1}\sin^{d-2}(\sqrt{\lambda}\pi)}=0; \tag{3.17}$$

(2) If $m_k^2$ is a zero of $B(\lambda)$ with multiplicity $d-2$ exactly, by (3.16) then

$$\lim_{\lambda\to m_k^2}\frac{B(\lambda)}{\sqrt{\lambda}^{2d-1}\sin^{d-2}(\sqrt{\lambda}\pi)}=O(1/m_k^{d+1}). \tag{3.18}$$

Combining (3.15), (3.17) and (3.18), we obtain

$$(-1)^{m_k}\sum_{j=1}^{d} b_j - \sum_{j=1}^{d} K_j(\pi,\pi) + O(1/m_k^{d+1}) = 0. \tag{3.19}$$

Since $m_k$ is a strictly ascending infinite sequence of positive integers, letting $m_k \to \infty$ in (3.19), together with (2.5) and (2.6), it follows that

$$\frac{1}{2}\int_0^{\pi}\sum_{j=1}^{d} q_j(x)\,dx = \sum_{j=1}^{d} K_j(\pi,\pi) = 0.$$

Now it is sufficient to use the arguments in the proof of Theorem 1.3 to finish the proof. □

*Remark* 3.1. The potential $q_j = c$, $j = 1, 2, \dots, d$, with a real constant $c$ in equation $(1.1_j)$, can be reconstructed from its spectrum. We need only apply Theorems 1.1–1.4 to equation $(1.1_j)$ with potential $q_j - c$.

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