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THE CONVERGENCE OF q-BERNSTEIN POLYNOMIALS $(0 < q < 1)$ AND LIMIT q-BERNSTEIN OPERATORS IN COMPLEX DOMAINS

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ABSTRACT. Due to the fact that the convergence properties of q-Bernstein polynomials are not similar to those in the classical case $q = 1$, their study has become an area of intensive research with a wide scope of open problems and unexpected results. The present paper is focused on the convergence of q-Bernstein polynomials, $0 < q < 1$, and related linear operators in complex domains. An analogue of the classical result on the simultaneous approximation is presented. The approximation of analytic functions with the help of the limit q-Bernstein operator is studied.

1. Introduction. After q-Bernstein polynomials were introduced by Phillips [**11**] in 1997, these polynomials have been brought to the spotlight and studied by a number of authors from different perspectives. While, for $q = 1$, the q-Bernstein polynomials coincide with classical Bernstein polynomials, for $q \neq 1$ we obtain a new class of polynomials with rather unexpected properties. Reviews of the results on q-Bernstein polynomials along with arrays of references on this matter are given in [**12**, Chapter 7] (results obtained before 2000) and in [**7**] (results obtained in 2000–2004). The subject remains under intensive study, and there are new papers constantly emerging, see, for example, papers [**8, 9, 10, 17, 18**] appeared after [**7**]. A two-parametric generalization of q-Bernstein polynomials and a version of the Bernstein-Durrmeyer operator related to q-Bernstein polynomials have been considered in [**2, 4**], respectively.

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For the sequel we need the following definitions, see [**12**, Chapter 8, subsection 8.1].

Let $q > 0$. For any nonnegative integer k, the *q-integer* $[k]_q$ is defined by

$$[k]_q := 1 + q + \cdots + q^{k-1} \ (k = 1, 2, \dots), \quad [0]_q := 0;$$

and the *q-factorial* $[k]_q!$ by

$$[k]_q! := [1]_q[2]_q \dots [k]_q \ (k = 1, 2, \dots), \quad [0]_q! := 1.$$

For integers k, n with $0 \le k \le n$, the *q-binomial* coefficient is defined by

$$\begin{bmatrix} n \\ k \end{bmatrix}_q := \frac{[n]_q!}{[k]_q![n-k]_q!}.$$

We use the following standard notation, cf., e.g., [**1**, Chapter 10, subsection 10.2]:

$$(z;q)_0 := 1; \quad (z;q)_k := \prod_{j=0}^{k-1}(1 - zq^j); \quad (z;q)_\infty := \prod_{j=0}^{\infty}(1 - zq^j).$$

Definition 1.1 ([**11**]). Let $f : [0,1] \to \mathbf{C}$. The *q-Bernstein polynomials* of f are:

$$B_{n,q}(f;z) := \sum_{k=0}^{n} f\left(\frac{[k]_q}{[n]_q}\right) p_{nk}(q;z), \quad n = 1, 2, \dots, \tag{1.1}$$

where

$$p_{nk}(q;z) := \begin{bmatrix} n \\ k \end{bmatrix}_q z^k (z;q)_{n-k}, \quad k = 0, 1, \dots n. \tag{1.2}$$

When $q = 1$, we recover the classical Bernstein polynomials:

$$B_{n,1}(f;z) = B_n(f;z) = \sum_{k=0}^{n} f\left(\frac{k}{n}\right) \binom{n}{k} z^k (1-z)^{n-k}.$$

It has been shown by Phillips et al. that the q-Bernstein polynomials take after some of the properties of the classical Bernstein polynomials, cf. [**12**, Chapter 7]. Among those properties taken after are the *end-point interpolation* property, the *shape-preserving* properties in the case $0 < q < 1$, and the representation via q-differences. Like the classical Bernstein polynomials, the q-Bernstein polynomials reproduce linear functions and are degree reducing on the set of polynomials, that is, if T is a polynomial of degree m, then $B_{n,q}(T; z)$ is a polynomial of degree at most $\min\{m, n\}$.

On the other hand, the examination of the convergence properties of the q-Bernstein polynomials reveals that these properties are essentially *different* from those of the classical ones. What is more, the cases $0 < q < 1$ and $q > 1$ are not similar to each other. The approximation by q-Bernstein polynomials was first considered in [**11**] and later on in [**3, 5, 8, 14, 16, 17, 19**]. Mostly, these papers deal with the case $0 < q < 1$, when q-Bernstein polynomials are *positive* linear operators on $C[0, 1]$. It should be emphasized that, although in the case $0 < q < 1$, q-Bernstein polynomials are positive linear operators, they do not satisfy the conditions of Korovkin's theorem, because

$$B_{n,q}(t^2; x) = x^2 + \frac{x(1-x)}{[n]_q} \longrightarrow x^2 + (1-q)x(1-x) \neq x^2, \quad n \to \infty.$$

However, they satisfy the conditions of Wang's Korovkin type theorem ([**16**, Theorem 2]) and may be regarded as an exemplary model for this theorem. Wang's theorem guarantees the existence of the limit operator $B_{\infty,q}$ for the sequence $\{B_{n,q}\}$, which, contrary to the situation in the classical case, is not the identity operator. An explicit form of $B_{\infty,q}(f; x)$ for $x \in [0, 1]$ has been found in [**3**], see Theorem A below. The approximation by means of $B_{\infty,q}$ and various properties of this operator have been studied in [**3, 6, 9, 14, 15, 17, 19**]. In [**18**], Wang observed that the same operator arises as the limit for a sequence of q-Meyer-König and Zeller operators.

This paper focuses on some problems of the approximation with q-Bernstein polynomials and the limit q-Bernstein operator in complex domains. From here on, we assume that $q \in (0, 1)$ is fixed. The following theorem on the convergence of q-Bernstein polynomials in this case has been proved in [**3**].

Theorem A ([**3**]). *Given* $0 < q < 1$, $f \in C[0,1]$. *The sequence* $\{B_{n,q}(f;x)\}$ *converges uniformly on* $[0,1]$ *to the limit function* $B_{\infty,q}(f;x)$, *where*

$$B_{\infty,q}(f;x) := \begin{cases} \sum_{k=0}^{\infty} f(1-q^k)p_{\infty k}(q;x) & \text{if } x \in [0,1), \\ f(1) & \text{if } x = 1, \end{cases} \tag{1.3}$$

and

$$p_{\infty k}(q;z) := \frac{z^k}{(q;q)_k}(z;q)_\infty, \quad k = 0,1,2,\dots . \tag{1.4}$$

We note that

$$p_{\infty k}(q;z) = \lim_{n\to\infty} p_{nk}(q;z), \tag{1.5}$$

with $p_{nk}(z)$ *given by* (1.2), *and the convergence being uniform on any compact set in* $\mathbf{C}$. *Clearly, each* $p_{\infty k}(z)$ *is an entire function.*

Definition 1.2. Let $0 < q < 1$. The linear operator on $C[0,1]$ given by

$$B_{\infty,q} : f \longmapsto B_{\infty,q}(f;.)$$

is called the *limit q-Bernstein operator*.

Theorem A shows that this operator comes out naturally as we consider the limit of the q-Bernstein polynomials in the case $0 < q < 1$.

Since, for $f \in C[0,1]$, the sequence $\{f(1-q^k)/(q;q)_k\}$ is bounded, it follows that $B_{\infty,q}(f;x)$ admits an analytic continuation from $[0,1]$ into the open unit disc $\{z : |z| < 1\}$. In general, $B_{\infty,q}(f;x)$ may not admit an analytic continuation into a disc $\{z : |z| < r\}$ with $r > 1$. The possibility of an analytic continuation for $B_{\infty,q}(f;x)$ has been studied in [**6, 9**]. Whenever $B_{\infty,q}(f;x)$ admits an analytic continuation into a domain $\Omega \subset \mathbf{C}$, we denote a continued function by $B_{\infty,q}(f;z)$, $z \in \Omega$. It should be pointed out that representation (1.3) may not be true for a continued function if $z \notin [0,1]$.

It has been proved in [**3**] that, for any $f \in C[0,1]$, we have: $B_{\infty,q}(f;x) \to f(x)$ uniformly on $[0,1]$ as $q \to 1^-$. The generalizations of this assertion have been obtained by Videnskii in [**14, 15**]. His results imply, in particular, that for $f \in C^2[0,1]$,

$$|B_{\infty,q}(f;x) - f(x)| \le C_f(1-q) \quad \text{for } x \in [0,1]. \tag{1.6}$$

In this paper, we consider extensions of both Theorem A and estimate (1.6) to complex domains. As our main results, we obtain an analogue of the classical result on the simultaneous approximation for the Bernstein polynomials and study the approximation of analytic functions with the help of $B_{\infty,q}(f;z)$.

2. Statement of results. In the sequel, for $r > 0$, we denote $D(r) := \{z : |z| < r\}$ and $\overline{D(r)} := \{z : |z| \leq r\}$. By $[x]^*$ we denote the greatest integer strictly less than x. For $K \subset \mathbf{C}$, the expression $g_n(z) \rightrightarrows g(z)$, $n \to \infty$, $z \in K$ means uniform convergence as $n \to \infty$ for $z \in K$, and similarly, $B_{\infty,q}(f;z) \rightrightarrows f(z)$, $q \to 1^-$, $z \in K$ means uniform convergence as $q \to 1^-$ for $z \in K$.

For $m \in \mathbf{Z}_+$, $\alpha \in [0,1]$, $\beta \geq 0$ and $f \in C[0,1]$, we write $f \in \mathrm{Lip}\,(m;\alpha;\beta)$ if f possesses m derivatives in a left neighborhood of 1 and, in addition, $f^{(m)}$ satisfies the condition:

$$\left|f^{(m)}(x) - f^{(m)}(1)\right| \leq M(1-x)^{\alpha}(\ln(1/(1-x)))^{-\beta} \quad \text{for some } M > 0. \tag{2.1}$$

We take $C[0,1] = \mathrm{Lip}\,(0;0;0)$. Clearly, $\mathrm{Lip}\,(m;\alpha;0)$ denotes the set of continuous functions whose mth derivative satisfies the Lipschitz condition of order α at 1:

$$\left|f^{(m)}(x) - f^{(m)}(1)\right| \leq M(1-x)^{\alpha} \quad \text{for some } M > 0. \tag{2.2}$$

Obviously, $\mathrm{Lip}\,(m;\alpha;\beta) \subset \mathrm{Lip}\,(m;\alpha;0)$ for all $\beta > 0$ and $C^{m+1}[0,1] \subset \mathrm{Lip}\,(m;1;0) \subset \mathrm{Lip}\,(m;\alpha;0)$ for all $\alpha \in [0,1]$.

It has been proved in [**6**] that if $f \in \mathrm{Lip}\,(m;\alpha;0)$, then $B_{\infty,q}(f;x)$ admits an analytic continuation from $[0,1]$ into the disc $D(q^{-(m+\alpha)})$. This result is sharp, that is, in general, $B_{\infty,q}(f;x)$ does not admit an analytic continuation into a wider disc. Apart from that, if $f \in \mathrm{Lip}\,(m;\alpha;\beta)$ and $\beta > 1$, then $B_{\infty,q}(f;z)$ possesses $[\beta-1]^*$ continuous derivatives in $\overline{D(q^{-(m+\alpha)})}$. These results have been developed in [**10**], where it has been shown that $f \in \mathrm{Lip}\,(m;\alpha;0)$ implies

$$B_{n,q}(f;z) \rightrightarrows B_{\infty,q}(f;z), \quad n \to \infty, \quad z \in K$$

for any compact set $K \subset D(q^{-(m+\alpha)})$.

Our first theorem provides an analogue of the well-known result on simultaneous approximation by the Bernstein polynomials.

Theorem 2.1. *Given* $0 < q < 1$, $\alpha \in [0,1]$, $\beta > 1$ *and* $f \in \mathrm{Lip}\,(m;\alpha;\beta)$, *we have for each* $j = 0, 1, \dots, [\beta-1]^*$:

$$B_{n,q}^{(j)}(f;z) \rightrightarrows B_{\infty,q}^{(j)}(f;z), \quad n \to \infty, \quad z \in \overline{D(q^{-(m+\alpha)})}.$$

Corollary 2.2. *Let* $q \in (0,1)$. *If a continuous function* f *on* $[0,1]$ *is infinitely differentiable* (*from the left*) *at* 1, *then for any compact set* $K \subset \mathbf{C}$, *we have*

$$B_{n,q}^{(j)}(f;z) \rightrightarrows B_{\infty,q}^{(j)}(f;z), \quad n \to \infty, \quad z \in K, \; j = 0, 1, \dots .$$

The theorems below are related to the approximation of analytic functions with the help of $B_{\infty,q}(f;z)$.

Theorem 2.3. *Let* $f \in C[0,1]$ *admit an analytic continuation from* $[0,1]$ *into* $\{z : |z-1| < 1+\varepsilon\}$. *Then, for any compact set* $K \subset D(\varepsilon)$,

$$B_{\infty,q}(f;z) \rightrightarrows f(z), \quad q \to 1^-, \quad z \in K.$$

Corollary 2.4. *If* f *is an entire function, then, for any compact set* $K \subset \mathbf{C}$,

$$B_{\infty,q}(f;z) \rightrightarrows f(z), \quad q \to 1^-, \quad z \in K.$$

Finally, we provide an estimate for the rate of approximation for functions analytic in $D(r)$, $r > 1$.

Theorem 2.5. *Let* $f(z)$ *be analytic in a closed disk* $\overline{D(r)}$ *with* $r > 1$. *Then, for* $z \in \overline{D(r)}$, *we have*:

$$|B_{\infty,q}(f;z) - f(z)| \le C_{f,r}(1-q).$$

Remark 2.1. Clearly, Corollary 2.4 can be derived from Theorem 2.5 as well. Moreover, we obtain that the order of approximation for analytic functions equals $(1-q)$. Using the growth estimates for f, we can estimate $C_{f,r}$ for $r>1$, see Remark 3.2.

3. Proofs of the theorems. We start with the following assertion.

Lemma 3.1. *For any $R>0$ and each $j=0,1,\dots$, there exists a constant $C_{j,q}(R)$ independent of n and k such that, for $|z|\le R$, the following estimates hold*:

$$\left|p_{nk}^{(j)}(q;z)\right| \le C_{j,q}(R)R^k k^j, \quad \left|p_{\infty k}^{(j)}(q;z)\right| \le C_{j,q}(R)R^k k^j. \tag{3.1}$$

Proof. We set

$$\psi_n(q;z) := (z;q)_n, \quad \psi(z) := (z;q)_\infty.$$

Clearly, $\psi_n(q;z)$ converges to $\psi(q;z)$ uniformly and absolutely on any compact set $K\subset \mathbf{C}$ and consequently (see, e.g., [**13**, Chapter II, subsection 2.8.1]):

$$\psi_n^{(j)}(q;z) \rightrightarrows \psi^{(j)}(q;z), \quad n\to\infty, \quad z\in K \text{ for each } j=0,1,\dots. \tag{3.2}$$

It follows from (3.2) that, for each $j=0,1,\dots$, the functions $\psi_n^{(j)}(q;z)$, $n=1,2,\dots$ and $\psi^{(j)}(q;z)$ are uniformly bounded on any compact set $K\subset\mathbf{C}$. Hence, for any $R>0$ and each $j=0,1,\dots$, there exists a constant $c_{j,q}(R)$ independent of n so that for $|z|\le R$,

$$\left|\psi_n^{(j)}(q;z)\right| \le c_{j,q}(R), \quad \left|\psi^{(j)}(q;z)\right| \le c_{j,q}(R).$$

Then, for $|z|\le R$,

$$\begin{aligned}\left|p_{nk}^{(j)}(q;z)\right| &\le \begin{bmatrix} n \\ k \end{bmatrix}_q \sum_{l=0}^{j} \binom{j}{l} |(z^k)^{(j-l)}| \cdot |\psi_{n-k}^{(l)}(q;z)| \\ &\le 1/(q;q)_\infty \sum_{l=0}^{j} c_{l,q}(R) \binom{j}{l} R^{k-j+l} k^{j-l} \\ &\le \frac{R^k k^j}{(q;q)_\infty} \sum_{l=0}^{j} c_{l,q}(R) \binom{j}{l} R^{l-j} := C_{j,q}(R) R^k k^j.\end{aligned}$$

Similarly, we can prove the second inequality in (3.1). □

Remark 3.1. For $j = 0$, we may take

$$C_{0,q}(R) := \frac{(-R;q)_\infty}{(q;q)_\infty}.$$

Proof of Theorem 2.1. Suppose that $f \in \mathrm{Lip}\,(m;\alpha;\beta)$. Then we have by Taylor's formula

$$f(x) = \sum_{j=0}^{m} \frac{f^{(j)}(1)}{j!}(x-1)^j + r_m(x) := T_m(x) + r_m(x),$$

where $T_m(x)$ is a polynomial of degree at most m, and the remainder $r_m(x)$ is estimated by:

$$|r_m(x)| \le M(1-x)^{m+\alpha} \left(\ln\left(1/(1-x)\right)\right)^{-\beta}, \quad x \in [0,1]. \tag{3.3}$$

Obviously, $B_{\infty,q}(f;z) = B_{\infty,q}(T_m;z) + B_{\infty,q}(r_m;z)$. Since $B_{n,q}$ are degree-reducing and T_m is a polynomial of degree at most m, Theorem A implies that $B_{\infty,q}(T_m;x)$ is a polynomial of degree at most m and $B_{n,q}(T_m;z) \rightrightarrows B_{\infty,q}(T_m;z)$, $n \to \infty$ on any compact set $K \subset \mathbf{C}$. It suffices, therefore, to prove that

$$B_{n,q}^{(j)}(r_m;z) \rightrightarrows B_{\infty,q}^{(j)}(r_m;z), \quad n \to \infty,$$
$$z \in \overline{D(q^{-(m+\alpha)})}, \quad j = 0, 1, \ldots, [\beta-1]^*.$$

First, we note that

$$(3.4)\qquad \begin{aligned} B_{\infty,q}(r_m;z) &= (z;q)_\infty \sum_{k=0}^{\infty} \frac{r_m(1-q^k)}{(q;q)_k} z^k \\ &= \sum_{k=0}^{\infty} r_m(1-q^k) p_{\infty k}(q;z) \quad \text{for } z \in \overline{D(q^{-(m+\alpha)})}. \end{aligned}$$

This is because, by virtue of estimate (3.3), the series on the right-hand side of (3.4) converges absolutely and uniformly in $\overline{D(q^{-(m+\alpha)})}$. Moreover, estimate (3.1) implies that the series may be differentiated $[\beta-1]^*$ times term by term.

Now, given $\varepsilon > 0$, we choose N_0 so that

$$M\,(\ln(1/q))^{-\beta}\, C_j \sum_{k=N_0}^{\infty} k^{j-\beta} < \varepsilon/3 \quad \text{for } j = 0, 1, \ldots, [\beta-1]^*,$$

where M is as in (3.3) and $C_j := C_{j,q}(q^{-(m+\alpha)})$ are given by Lemma 3.1. For $n > N_0$ and $z \in \overline{D(q^{-(m+\alpha)})}$, we have

$$\begin{aligned} \triangle_j :&= |B^{(j)}_{n,q}(r_m;z) - B^{(j)}_{\infty,q}(r_m;z)| \\ &= \left| \sum_{k=0}^{n} r_m([k]_q/[n]_q) p^{(j)}_{nk}(q;z) - \sum_{k=0}^{\infty} r_m(1-q^k) p^{(j)}_{\infty k}(q;z) \right| \\ &\le \sum_{k=0}^{N_0} \left| r_m([k]_q/[n]_q) p^{(j)}_{nk}(q;z) - r_m(1-q^k) p^{(j)}_{\infty k}(q;z) \right| \\ &\quad + \sum_{k=N_0+1}^{n} \left| r_m([k]_q/[n]_q) \right| |p^{(j)}_{nk}(q;z)| \\ &\quad + \sum_{k=N_0+1}^{\infty} \left| r_m(1-q^k) \right| |p^{(j)}_{\infty k}(q;x)| \\ &=: I_1 + I_2 + I_3. \end{aligned}$$

Using (1.5) and the fact that $r_m([k]_q/[n]_q) \to r_m(1-q^k)$ as $n \to \infty$, we conclude that $I_1 < \varepsilon/3$ for n sufficiently large.

To estimate I_2, we notice that by (3.3),

$$\begin{aligned}\left|r_m\left(\frac{[k]_q}{[n]_q}\right)\right| &\le M\left(1-\frac{[k]_q}{[n]_q}\right)^{m+\alpha}\left(\ln\frac{1}{1-[k]_q/[n]_q}\right)^{-\beta}\\ &\le M\left(\ln(1/q)\right)^{-\beta} q^{k(m+\alpha)}k^{-\beta}.\end{aligned}$$

Applying Lemma 3.1 with $R = q^{-(m+\alpha)}$, we obtain for $z \in \overline{D(q^{-(m+\alpha)})}$:

$$\begin{aligned}I_2 &= \sum_{k=N_0+1}^{n}\left|r_m([k]_q/[n]_q)\right| |p_{nk}^{(j)}(q;z)|\\ &\le M\left(\ln(1/q)\right)^{-\beta} C_j \sum_{k=N_0+1}^{n} k^{j-\beta} < \varepsilon/3.\end{aligned}$$

Similarly, (3.3) implies that

$$\left|r_m(1-q^k)\right| \le M\left(\ln(1/q)\right)^{-\beta} q^{k(m+\alpha)}k^{-\beta}$$

and

$$I_3 \le M\left(\ln(1/q)\right)^{-\beta} C_j \sum_{k=N_0+1}^{\infty} k^{j-\beta} < \varepsilon/3.$$

Thus, $\triangle_j < \varepsilon$. □

Proof of Theorem 2.3. From Theorem 5 of [**6**] we know that $B_{\infty,q}(f;z)$ is an entire function represented by

$$B_{\infty,q}(f;z) = \sum_{k=0}^{\infty}\frac{(-1)^k f^{(k)}(1)}{k\,!}\prod_{j=0}^{k-1}(1-q^j z).$$

Then, for any compact set $K \subset D(\varepsilon)$, we choose ρ, γ in such a way that $0 < \rho < \gamma < \varepsilon$, $K \subset \overline{D(\rho)}$. Using the Cauchy estimates for the derivatives $f^{(k)}(1)$, we get

$$|f^{(k)}(1)| \le \frac{Mk\,!}{(1+\gamma)^k}, \qquad M = \max_{|z-1|=1+\gamma}|f(z)|.$$

If $z \in K,\ q \in (0,1)$, then $|z| \le \rho < \gamma$ and $\prod_{j=0}^{k-1} |1 - q^j z| \le (1+\rho)^k$, so

$$|B_{\infty,q}(f;z)| \le \sum_{k=0}^{\infty} \frac{1}{k\,!} \frac{M k\,!}{(1+\gamma)^k} (1+\rho)^k = M \frac{1+\gamma}{\gamma - \rho} < \infty.$$

Hence, $B_{\infty,q}(f;z)$ is uniformly bounded for $q \in (0,1)$ in the disk $\overline{D(\rho)}$. Besides

$$B_{\infty,q}(f;x) \rightrightarrows f(x), \quad x \in [0,1], \quad q \to 1^-.$$

The statement now follows from the Vitali theorem, cf. e.g., [**13**, Chapter 5, subsection 5.2]. □

Proof of Theorem 2.5. Let

$$f(z) = \sum_{k=0}^{\infty} c_k z^k, \quad |z| \le r.$$

Then

$$B_{\infty,q}(f;z) = \sum_{k=0}^{\infty} c_k B_{\infty,q}(t^k;z), \quad |z| \le r. \tag{3.5}$$

Indeed, by Lemma 3 of [**5**],

$$B_{\infty,q}(t^k;z) = \sum_{j=1}^{k} \alpha_j z^j,$$

where $0 \le \alpha_j \le 1$, $\sum_{j=1}^{k} \alpha_j = 1$ and $\alpha_k = q^{k(k-1)/2}$.

Therefore,

$$\left|B_{\infty,q}(t^k;z)\right| \le |z|^k \quad \text{for } |z| \ge 1.$$

The latter estimate implies that the series in (3.5) converges absolutely and uniformly in $\overline{D(r)}$ and therefore represents $B_{\infty,q}(f;z)$.

Now, consider for $z \in \overline{D(r)}$, the difference:

$$|B_{\infty,q}(f;z) - f(z)| \le \sum_{k=0}^{\infty} |c_k| \cdot \left|B_{\infty,q}(t^k;z) - z^k\right|.$$

For $|z| \geq 1$, we have:

$$\left|B_{\infty,q}(t^k;z) - z^k\right| \leq (1-\alpha_k)|z|^k + \sum_{j=1}^{k-1} \alpha_j |z|^j \leq 2(1-\alpha_k)|z|^k$$
$$= 2\left(1 - q^{k(k-1)/2}\right)|z|^k \leq (1-q)k(k-1)|z|^k.$$

Hence, for $z \in \overline{D(r)}$, we obtain:

$$|B_{\infty,q}(f;z) - f(z)| \leq (1-q)\sum_{k=2}^{\infty} |c_k| k(k-1) r^k := C_{f,r}(1-q). \qquad \square$$

Remark 3.2. We can find an estimate for $C_{f,r}$ using $M(R;f)$, $R > r$, where f is analytic in $\overline{D(R)}$ and $M(R;f) := \max_{|z|=R} |f(z)|$. Indeed, by the Cauchy estimates,

$$|c_k| \leq \frac{M(R;f)}{R^k}$$

and we obtain:

$$C_{f,r} \leq M(R;f)\sum_{k=2}^{\infty} k(k-1)\left(\frac{r}{R}\right)^k = M(R;f)\left(\frac{r}{R}\right)^2 \cdot \frac{2R^3}{(R-r)^3}.$$

If f is entire, then we obtain the following estimate: for any $r > 1$, we have

$$|B_{\infty,q}(f;z) - f(z)| \leq 4M(2r;f)(1-q), \quad |z| \leq r.$$

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