ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 39, Number 4, 2009

TRIGONOMETRIC QUASI-GREEDY BASES FOR $L^p(\mathbf{T}; w)$

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ABSTRACT. We give a complete characterization of 2π-periodic weights w for which the usual trigonometric system forms a quasi-greedy basis for $L^p(\mathbf{T}; w)$, i.e., bases for which simple thresholding approximants converge in norm. The characterization implies that this can happen only for $p = 2$ and whenever the system forms a quasi-greedy basis, the basis must actually be a Riesz basis.

1. Introduction. Let $\mathcal{B} = \{e_n\}_{n\in\mathbf{N}}$ be a bounded Schauder basis for a Banach space X, i.e., a basis for which $0 < \inf_n \|e_n\|_X \leq \sup_n \|e_n\|_X < \infty$. An approximation algorithm associated with $\mathcal{B}$ is a sequence $\{A_n\}_{n=1}^{\infty}$ of (possibly nonlinear) maps $A_n : X \to X$ such that for $x \in X$, $A_n(x)$ is a linear combination of at most n elements from $\mathcal{B}$. We say that the algorithm is convergent if $\lim_{n\to\infty} \|x - A_n(x)\|_X = 0$ for every $x \in X$. For a Schauder basis there is a natural convergent approximation algorithm. Suppose the dual system to $\mathcal{B}$ is given by $\{e_k^*\}_{k\in\mathbf{N}}$. Then the linear approximation algorithm is given by the partial sums $S_n(x) = \sum_{k=1}^n e_k^*(x)e_k$.

Another quite natural approximation algorithm is the greedy approximation algorithm where the partial sums are obtained by thresholding the expansion coefficients. Greedy approximation algorithms are often applied successfully in applications such as denoising and compression using wavelets, see e.g., [**3, 4**]. The algorithm is defined as follows. For each element $x \in X$, we define the greedy ordering of the coefficients as the map $\rho : \mathbf{N} \to \mathbf{N}$ with $\rho(\mathbf{N}) \supseteq \{j : e_j^*(x) \neq 0\}$ such that, for $j < k$, we have either $|e_{\rho(k)}^*(x)| < |e_{\rho(j)}^*(x)|$ or $|e_{\rho(k)}^*(x)| = |e_{\rho(j)}^*(x)|$ and $\rho(k) > \rho(j)$. Then the greedy m-term approximant to x is given by $\mathcal{G}_m(x) = \sum_{j=1}^m e_{\rho(j)}^*(x)e_{\rho(j)}$. The question is whether the greedy algorithm is convergent. This is clearly the case for an *unconditional*

Keywords and phrases. Quasi-greedy basis, Schauder basis, trigonometric system.
Received by the editors on December 6, 2006, and in revised form on January 6, 2007.
DOI:10.1216/RMJ-2009-39-4-1267

basis where the expansion $x = \sum_{k=1}^{\infty} e_k^*(x)e_k$ converges regardless of the ordering. However, Temlyakov and Konyagin [**7**] showed that the greedy algorithm may also converge for certain conditional bases. This led them to define so-called quasi-greedy bases, see [**7**].

Definition 1.1. A bounded Schauder basis for a Banach space X is called *quasi-greedy* if there exists a constant C such that $\|\mathcal{G}_m(x)\|_X \leq C\|x\|_X$ for $x \in X$ and $m \in \mathbf{N}$.

It was proved by Wojtaszczyk that a Schauder basis is quasi-greedy exactly when the greedy approximation algorithm is convergent.

Theorem 1.2 [**13**]. *A bounded Schauder basis for a Banach space X is quasi-greedy if and only if* $\lim_{m\to\infty} \|x - \mathcal{G}_m(x)\|_X = 0$ *for every element $x \in X$.*

In this note we consider the standard trigonometric system $\mathcal{T} := \{(2\pi)^{-1/2}e^{ikx}\}_{k\in\mathbf{Z}}$ on $\mathbf{T} := [-\pi, \pi)$. As is very well known, $\mathcal{T}$ is an unconditional (orthonormal) basis for $L^2(\mathbf{T})$ and it is immediate that the greedy algorithm converges. However, we are not so fortunate when we consider $\mathcal{T}$ in $L^p(\mathbf{T})$, $p \neq 2$. It was proved by Temlyakov [**11**] that $\mathcal{T}$ fails to be a quasi-greedy basis for $L^p(\mathbf{T})$, $1 \leq p \leq \infty$, $p \neq 2$. This negative result was also proved independently by Córdoba and Fernández for $1 \leq p < 2$, see [**2**]. So we have to look for spaces other than $L^p(\mathbf{T})$ if we want to extend the positive result for $\mathcal{T}$ in $L^2(\mathbf{T})$. Konyagin and Temlyakov [**8, 9**] have considered sufficient conditions on individual $L^p(\mathbf{T})$-functions that guarantee norm-convergence of the greedy algorithm.

Another possible path forward is to consider the weighted space

$$L^p(\mathbf{T}; w) := \left\{ f : \mathbf{T} \longrightarrow \mathbf{C}; \|f\|_{p,w}^p = \int_{-\pi}^{\pi} |f(t)|^p w(t)\, dt < \infty \right\}, \quad 1 < p < \infty,$$

where w is a nonnegative 2π-periodic weight. For a suitable choice of weight, we can make $L^p(\mathbf{T}; w)$ larger or smaller than $L^p(\mathbf{T})$. The dual

system to $\mathcal{T}$ in $L^p(\mathbf{T};w)$ for a positive weight w is (at least formally)

$$\left\{\frac{1}{\sqrt{2\pi}}\frac{e^{ikx}}{w(t)}\right\}_{k=1}^{\infty}$$

and the expansion relative to this system is

$$f=\frac{1}{2\pi}\sum_{k\in\mathbf{Z}}\int_{-\pi}^{\pi}f(t)\overline{w(t)^{-1}e^{ikt}}w(t)\,dt\,e^{ikt}=\frac{1}{2\pi}\sum_{k\in\mathbf{Z}}\langle f,e^{ikt}\rangle e^{ikt},$$

where $\langle\cdot,\cdot\rangle$ is the standard inner product on $L^2(\mathbf{T})$. Thus, the greedy algorithm for $\mathcal{T}$ in $L^p(\mathbf{T};w)$ coincides with the usual greedy algorithm for the trigonometric system. Our main result in Section 3 gives a complete characterization of the nonnegative weights w on $\mathbf{T}:=[-\pi,\pi)$ such that $\mathcal{T}$ forms a quasi-greedy basis $L^p(\mathbf{T};w)$. The characterizing condition is rather restrictive: we must have $p=2$, and for $p=2$, $\mathcal{T}$ forms a quasi-greedy basis $L^2(\mathbf{T};w)$ if and only if there exists $C>0$ such that $C^{-1}\le w(t)\le C$. As a consequence, we can conclude that $\mathcal{T}$ is a quasi-greedy basis $L^2(\mathbf{T};w)$ if and only if $\mathcal{T}$ is a Riesz basis for $L^2(\mathbf{T};w)$. This is perhaps surprising since a priori, the Riesz basis property is much more restrictive than the quasi-greedy one. In Section 2 we characterize the weights w such that $\mathcal{T}$ is a Schauder basis for $L^2(\mathbf{T};w)$. This characterization, and our main result in Section 3, is given in terms of the so-called Muckenhoupt A_2-condition. Finally, we consider an application to polynomial weights in Section 4.

2. Trigonometric Schauder bases for $L^p(\mathbf{T};w)$. In this section we give a characterization of when the trigonometric system forms a Schauder basis for $L^p(\mathbf{T};w)$. We need to have a Schauder basis in order for thresholding to make sense. The result is a direct consequence of the celebrated result by Hunt, Muckenhoupt, and Wheeden [**6**].

Let us first fix the notation. Let $e_k(t):=(2\pi)^{-1/2}e^{ikt}$, and let $\mathcal{T}=\{e_{n_k}\}_{k=1}^{\infty}$ be the "natural" ordering of the trigonometric system given by the enumeration $\{n_k\}_{k=1}^{\infty}=\{0,-1,1,-2,2,\dots\}$. We wish to consider both the symmetric partial sum operator

$$T_N(f)=\sum_{k=-N}^{N}\langle f,e_k\rangle e_k,$$

where $\langle \cdot, \cdot \rangle$ is the standard inner product on $L^2(\mathcal{T})$, and the partial sum operator

$$S_N(f) = \sum_{k=1}^{N} \langle f, e_{n_k} \rangle e_{n_k}.$$

We need the Muckenhoupt A_p-condition. We use the convention that $0 \cdot \infty = 0$.

Definition 2.1. A nonnegative 2π-periodic function w is called an A_p-weight, $1 < p < \infty$, if there exists a constant $K < \infty$ such that for every interval $I \subset \mathbf{R}$,

$$\left(\frac{1}{|I|} \int_I w(t)\, dt \right) \left(\frac{1}{|I|} \int_I w(t)^{-1/(p-1)}\, dt \right)^{p-1} \leq K.$$

The family of all A_p-weights is denoted $\mathcal{A}_p(\mathbf{T})$.

The two trivial A_p-weights, $w \equiv 0$ and $w \equiv \infty$, are not interesting from our point of view since the associated $L^p(\mathbf{T}, w)$ is either trivial or far too large to be useful. We therefore exclude the trivial weights, and notice that all the remaining A_p-weights satisfy $0 < w(t) < \infty$ almost everywhere, and one easily verifies that $w, w^{-1/(p-1)} \in L^1(\mathbf{T})$. The following theorem is proved in [**6**].

Theorem 2.2 [**6**]. *Let w be a nonnegative 2π-periodic weight and consider formally $T_N : L^p(\mathbf{T}; w) \to L^p(\mathbf{T}; w)$, $1 < p < \infty$. Let $\|T_N\|_{p,w}$ denote the corresponding operator norm. Then $\sup_N \|T_N\|_{p,w} < \infty$ if and only if $w \in \mathcal{A}_p(\mathbf{T})$.*

We now consider the following equivalent version, which gives a nice characterization of when $\mathcal{T}$ forms a Schauder basis for $L^p(\mathbf{T}; w)$.

Proposition 2.3. *Let w be a nonnegative 2π-periodic weight on $\mathbf{T}$. Then $\mathcal{T}$ is a Schauder basis for $L^p(\mathbf{T}; w)$, $1 < p < \infty$, if and only if $w \in \mathcal{A}_p(\mathbf{T})$.*

Proof. First, suppose $w \in \mathcal{A}_p(\mathbf{T})$. Then $0 < w(t) < \infty$ almost everywhere and $\mathcal{T}$ spans a dense subset of $L^p(\mathbf{T}; w)$. The natural

biorthogonal system to $\mathbf{T}$ is given by $\{w(t)^{-1}e_{n_k}\}_{k=1}^{\infty}$ where we notice that $w(t)^{-1}e_{n_k} \in L^q(\mathbf{T};w)$, $1/p + 1/q = 1$. The partial sum operator is given by

$$S_N(f) = \sum_{k=1}^{N} \int_{-\pi}^{\pi} f(t)\overline{w(t)^{-1}e_{n_k}(t)}w(t)\,dt\,e_{n_k} = \sum_{k=1}^{N} \langle f, e_{n_k}\rangle e_{n_k},$$

so, in particular, $S_{2N+1} = T_N$ for $N \geq 1$. Also,

$$S_{2N+2} = T_N + \langle f, e_{n_{2N+2}}\rangle e_{n_{2N+2}},$$

with

$$\|\langle f, e_{n_{2N+2}}\rangle e_{n_{2N+2}}\|_{p,w} \leq C|\langle fw^{1/p}, w^{-1/p}e_{n_{2N+2}}\rangle| \leq C'\|f\|_{p,w},$$

where we used that $w, w^{-q/p} \in L^1(\mathbf{T})$. Hence, by this observation and Theorem 2.2, we obtain $\sup_N \|S_N\|_{p,w} < \infty$, and it follows that $\mathcal{T}$ is a Schauder basis for $L^p(\mathbf{T};w)$. Next, suppose $\mathcal{T}$ is a Schauder basis for $L^p(\mathbf{T};w)$. Let $\{d_k\}_{k=1}^{\infty} \subset L^q(\mathbf{T};w)$ denote the unique dual (biorthogonal) system. We claim that $d_k = w^{-1}e_{n_k}$. To verify the claim, notice that

$$c_{j,k} := \int_{-\pi}^{\pi} d_k(t)\overline{e_{n_j}(t)}w(t)\,dt = \delta_{j,k},$$

where $(c_{j,k})_j$ are nothing but the Fourier coefficients of $d_k(t)w(t) \in L^1(\mathbf{T})$. Thus, $d_k(t)w(t) = e_{n_k}(t)$ almost everywhere. In particular, since $|d_k(t)| < \infty$ almost everywhere, $0 < w(t) < \infty$ almost everywhere, and $d_k(t) = w(t)^{-1}e_{n_k}(t)$. We have $S_N(f) = \sum_{k=1}^{N}\langle f, e_{n_k}\rangle e_{n_k}$. The fact that $\mathcal{T}$ is a Schauder basis now gives

$$\sup_N \|T_N\|_{p,w} \leq \sup_N \|S_N\|_{p,w} < \infty,$$

and we use Theorem 2.2 to conclude that $w \in \mathcal{A}_p(\mathbf{T})$. □

Remark 2.4. We can move the trigonometric Schauder basis in $L^p(\mathbf{T};w)$ to $L^p(\mathbf{T})$ using the isometric isomorphism $U : L^p(\mathbf{T};w) \to L^p(\mathbf{T})$ defined by $U(f) = w^{1/p}f$. Thus,

$$\left\{w(t)^{1/p}e_{n_k}(t)\right\}_{k\in\mathbf{N}} \text{ and } \left\{\frac{e_{n_k}(t)}{w(t)^{1/p}}\right\}_{k\in\mathbf{N}}$$

form a biorthogonal Schauder basis system in $L^p(\mathbf{T})$ whenever $w \in \mathcal{A}_p(\mathbf{T})$.

3. Trigonometric quasi-greedy bases for $L^p(\mathbf{T};w)$. Proposition 2.3 tells us that $\mathcal{T}$ is a Schauder basis for $L^p(\mathbf{T};w)$ if and only if $w \in \mathcal{A}_p(\mathbf{T})$. In this section we prove the main result of this note: $\mathcal{T}$ can be quasi-greedy in $L^p(\mathbf{T};w)$ only for $p = 2$, and we characterize the weights $w \in \mathcal{A}_2(\mathbf{T})$ for which $\mathcal{T}$ is quasi-greedy in $L^2(\mathbf{T};w)$. First, we need to recall some basic property of quasi-greedy bases.

The first result we state is due to Wojtaszczyk [**13**], see also [**5**]. It shows that quasi-greedy bases are unconditional for constant coefficients.

Lemma 3.1. *Suppose $\{b_k\}_{k\in\mathbf{N}}$ is a quasi-greedy basis in a Banach space X. Then there exist constants $0 < c_1 \le c_2 < \infty$ such that for every choice of signs $\varepsilon_k = \pm 1$ and any finite subset $A \subset \mathbf{N}$, we have*

$$c_1 \Big\| \sum_{k\in A} b_k \Big\|_X \le \Big\| \sum_{k\in A} \varepsilon_k b_k \Big\|_X \le c_2 \Big\| \sum_{k\in A} b_k \Big\|_X. \tag{3.1}$$

We can use Lemma 3.1 together with some basic facts about the geometry of $L^p(\mathbf{T};w)$ to prove the following result.

Proposition 3.2. *Suppose that the trigonometric system $\mathcal{T} = \{e_{n_k}\}_{k\in\mathbf{N}}$ is quasi-greedy in $L^p(\mathbf{T};w)$ for some $1 < p < \infty$. Then there exist constants $0 < c_1 \le c_2 < \infty$ such that for any $\varepsilon = \{\varepsilon_k\}_{k\in\mathbf{N}} \in \{-1,1\}^{\mathbf{N}}$ and any finite subset $A \subset \mathbf{N}$,*

$$c_1 |A|^{1/2} \le \Big\| \sum_{k\in A} \varepsilon_k e_{n_k} \Big\|_{L^p(\mathbf{T};w)} \le c_2 |A|^{1/2}. \tag{3.2}$$

Proof. First we consider the case $1 < p \le 2$. Let $r_1, r_2, \dots$ be the Rademacher functions on $[0,1]$ defined by $r_k(t) = \operatorname{sign}(\sin(2^k \pi t))$, and take any finite subset of integers $A = \{k_1, k_2, \dots, k_N\} \subset \mathbf{N}$. Put $D_N = \sum_{l=1}^N \varepsilon_{k_l} e_{n_{k_l}}$. Using Lemma 3.1, and the fact that $L^p(\mathbf{T};w)$ has

cotype 2, see, e.g., [**12**, Chapter 3], we obtain

$$\begin{aligned}\big\|D_N\big\|_{L^p(\mathbf{T};w)} &\asymp \int_0^1 \Big\| \sum_{n=1}^N r_n(u)\, e_{k_n} \Big\|_{L^p(\mathbf{T};w)}\, du \\ &\geq C\bigg(\sum_{n=1}^N \|e_{n_{k_l}}\|^2_{L^p(\mathbf{T};w)} \bigg)^{1/2} \asymp N^{1/2}.\end{aligned}$$

Now suppose $2 \geq p < \infty$. Then $L^p(\mathbf{T};w)$ has type 2 ([**12**, Chapter 3]), and using Lemma 3.1, we get the estimate

$$\begin{aligned}\big\|D_N\big\|_{L^p(\mathbf{T};w)} &\asymp \int_0^1 \Big\| \sum_{n=1}^N r_n(u)\, e_{n_{k_l}} \Big\|_{L^p(\mathbf{T};w)}\, du \\ &\leq C\bigg(\sum_{n=1}^N \|e_{n_{k_l}}\|^2_{L^p(\mathbf{T};w)} \bigg)^{1/2} \asymp N^{1/2}.\end{aligned}$$

The above estimates give $\|D_N\|_{L^2(\mathbf{T};w)} \asymp N^{1/2}$. For $1 < p < 2$, we notice that

$$\|D_N\|_{L^p(\mathbf{T};w)} \leq \|D_N\|_{L^2(\mathbf{T};w)} \asymp N^{1/2},$$

and (3.2) holds in the range $1 < p \leq 2$. For $2 < p < \infty$, we use

$$N^{1/2} \asymp \|D_N\|_{L^2(\mathbf{T};w)} \leq \|D_N\|_{L^p(\mathbf{T};w)}$$

to reach the conclusion. □

A sequence $\{b_n\}_{n\in\mathbf{N}}$ in a Banach space X is called *democratic* if there exists a D such that for any finite subsets $A, B \subset \mathbf{N}$ with the same cardinality $|A| = |B|$, we have

$$\Big\| \sum_{k\in A} e_k \Big\|_X \leq D \Big\| \sum_{k\in B} e_k \Big\|_X.$$

For any democratic sequence, we can define the fundamental function

$$\phi(n) := \sup_{A\subset\mathbf{N}:|A|\leq n} \Big\| \sum_{k\in A} e_k \Big\|_X. \tag{3.3}$$

Proposition 3.2 shows that whenever $\mathcal{T}$ is a quasi-greedy basis for $L^p(\mathbf{T};w)$, $\mathcal{T}$ is democratic with fundamental function $\phi(n) \asymp n^{1/2}$. For such bases, it is possible to prove a strong version of the Hausdorff-Young inequality. Let us introduce some notation.

For a sequence $\{a_n\}_{n=1}^{\infty}$ we denote by $\{a_n^*\}$ a nonincreasing rearrangement of the sequence $\{|a_n|\}$. Then we define the Lorentz norms

$$\|\{a_n\}\|_{2,\infty} := \sup_n n^{1/2} a_n^* \text{ and } \|\{a_n\}\|_{2,1} := \sum_{n=1}^{\infty} n^{-1/2} a_n^*.$$

The following important theorem was proved in [**13**].

Theorem 3.3 [**13**]. *Let $\mathcal{B} = \{b_k\}_{k\in\mathbf{N}}$ be a democratic quasi-greedy basis for a Banach space X. Suppose that the fundamental function* (3.3) *associated with $\mathcal{B}$ satisfies $\phi(n) \asymp n^{1/2}$. Then there exist constants $0 < c_1 \le c_2 < \infty$ such that for any coefficients $\{a_k\}$*

$$c_1\|\{a_k\}\|_{2,\infty} \le \left\| \sum_{k\in\mathbf{N}} a_k b_k \right\|_X \le c_2\|\{a_k\}\|_{2,1}.$$

Remark 3.4. Of special interest to us is the fact that $\|\cdot\|_{2,1}$ and $\|\cdot\|_{2,\infty}$ assign (approximately) the same norm to flat sequence. More precisely, for $\mathcal{B} = \{b_k\}_{k\in\mathbf{N}}$ a quasi-greedy basis satisfying the hypothesis of Theorem 3.3, there exist $c_1, c_2 > 0$ such that for any unimodular sequence $\{a_k\}_{k\in\Lambda}$, $\Lambda \subset \mathbf{N}$ (i.e., $|a_k| = 1$ for $k \in \Lambda$), we have

$$c_1|\Lambda|^{1/2} \le \left\| \sum_{k\in\Lambda} a_k b_k \right\|_X \le c_2|\Lambda|^{1/2}, \tag{3.4}$$

since $\|\{a_k\}_{k\in\Lambda}\|_{2,1} \asymp \|\{a_k\}_{k\in\Lambda}\|_{2,\infty} \asymp |\Lambda|^{1/2}$. The estimate (3.4) will be used below to prove our main result, Theorem 3.5.

Theorem 3.5. *Let w be a nonnegative 2π-periodic weight. Suppose $\mathcal{T}$ is a quasi-greedy basis for $L^p(\mathbf{T};w)$, $1 < p < \infty$. Then $p = 2$, $w \in A_2$, and there exists a positive constant C such that $C^{-1} \le w(t) \le C$ almost everywhere.*

Proof. Suppose $\mathcal{T}$ is a quasi-greedy basis for $L^p(\mathbf{T};w)$. Then, in particular, $\mathcal{T}$ is a Schauder basis for $L^p(\mathbf{T};w)$ and $w \in A_p$ by Proposition 2.3. Now we use the Dirichlet kernel $D_N := \sum_{k=1}^N e_{n_k}$ to study $w(t)$. For each $u \in \mathbf{T}$, we have $e_{n_k}(t-u) = e_{n_k}(t)e_{n_k}(-u)$ with $|e_{n_k}(-u)| = 1$, and we obtain

$$D_N(t-u) = \sum_{k=1}^N e_{n_k}(-u)e_{n_k}(t).$$

Now the estimate (3.4) gives uniformly in u,

$$c_1^2 N \le \int_{-\pi}^{\pi} \left| \sum_{k=1}^N e_{n_k}(t-u) \right|^2 w(t)\,dt \le c_2^2 N,$$

so

$$c_1^2 \le \int_{-\pi}^{\pi} \frac{1}{N} \left| \sum_{k=1}^N e_{n_k}(t-u) \right|^2 w(t)\,dt \le c_2^2. \tag{3.5}$$

Notice that $1/N|\sum_{k=1}^N e_{n_k}(t-u)|^2$ is an approximation to the identity at the point u. Thus, whenever $u \in \mathbf{T}$ is a Lebesgue point of $w \in L^1(\mathbf{T})$, we obtain

$$c_1^2 \le w(u) = \lim_{N\to\infty} \int_{-\pi}^{\pi} \frac{1}{N} \left| \sum_{k=1}^N e_{n_k}(t-u) \right|^2 w(t)\,dt \le c_2^2.$$

We conclude that $c_1^2 \le w(t) \le c_2^2$ almost everywhere. Now suppose $p \ne 2$. By Proposition 3.2, $\|D_N\|_{L^p(\mathbf{T};w)} \asymp N^{1/2}$, and it follows from Hölder's inequality that

$$N^{1/2} \asymp \|D_N\|_p \le \|D_N\|_1^{\theta} \|D_N\|_2^{1-\theta}, \quad 1 < p < 2, \qquad \theta = \frac{2}{p} - 1,$$

or

$$N^{1/2} \asymp \|D_N\|_2 \le \|D_N\|_1^{\theta} \|D_N\|_p^{1-\theta}, \quad 2 < p < \infty, \qquad \theta = \frac{p-2}{2p-2}.$$

In both cases we can conclude that $\|D_N\|_{L^1(\mathbf{T};w)} \asymp N^{1/2}$ since $\|D_N\|_{L^2(\mathbf{T};w)} \asymp N^{1/2}$. However, this is a contradiction since we have the well-known estimate of the Lebesgue constant for $\mathcal{T}$,

$$\|D_N\|_{L^1(\mathbf{T};w)} \asymp \|D_N\|_{L^1(\mathbf{T})} \le C\log(N),$$

where we used $c_1^2 \le w(t) \le c_2^2$ almost everywhere. Thus, $\mathcal{T}$ quasi-greedy implies that $p = 2$, $w \in \mathcal{A}_2(\mathbf{T})$ and $c_1^2 \le w(t) \le c_2^2$ almost everywhere. □

Theorem 3.5 shows that the class of weights $w \in \mathcal{A}_2(\mathbf{T})$ such that $\mathcal{T}$ is a quasi-greedy basis for $L^2(\mathbf{T};w)$ is very restrictive. In fact, the are no *conditional* quasi-greedy bases for $L^2(\mathbf{T};w)$ as the following corollary shows.

Corollary 3.6. *Let w be a positive 2π-periodic weight for which $\mathcal{T}$ is a quasi-greedy basis for $L^2(\mathbf{T};w)$. Then $\mathcal{T}$ is a Riesz basis for $L^2(\mathbf{T};w)$.*

Proof. Suppose $\mathcal{T}$ is a quasi-greedy basis for $L^2(\mathbf{T};w)$. According to Theorem 3.5, there exists a $C > 0$ such that $C^{-1} \le w(t) \le C$ almost everywere. Hence, for any finite sequence $\{a_k\}_k$,

$$\begin{aligned} C^{-1}\int_{-\pi}^{\pi}\left|\sum_k a_k e_{n_k}(t)\right|^2 dt &\le \int_{-\pi}^{\pi}\left|\sum_k a_k e_{n_k}(t)\right|^2 w(t)\,dt \\ &\le C\int_{-\pi}^{\pi}\left|\sum_k a_k e_{n_k}(t)\right|^2 dt. \end{aligned}$$

In particular, $\|\sum_k a_k e_{n_k}\|_{2,w}^2 \asymp \|\{a_k\}_k\|_{\ell^2}^2$, which shows that $\mathcal{T}$ is a Riesz basis for $L^2(\mathbf{T};w)$. □

4. An application. Here we consider an application for general polynomial weights of the results obtained in the previous two sections.

Proposition 4.1. *Let P be a polynomial of degree n with $|P(-\pi)| = |P(\pi)|$. For $-1/n < \mu < 1/n$, $\mathcal{T}$ is a Schauder basis for $L^2(\mathbf{T};|P|^\mu)$. For such a weight $|P|^\mu$, $\mathcal{T}$ is a quasi-greedy (and thus Riesz) basis for $L^2(\mathbf{T};|P|^\mu)$ if and only if P has no zeros on $\mathbf{T}$.*

Proof. Stein and Ricci [**10**] proved that for $n \in \mathbf{N}$ and $0 < \mu < 1/n$ there exists a uniform constant $c := c(n, \mu)$ such that

$$\int_{-1}^{1} |P(t)|^{-\mu}\, dt \le c\bigg(\int_{-1}^{1} |P(t)|\, dt \bigg)^{-\mu},$$

where P is any polynomial of degree n. It follows by Hölder's inequality that

$$\int_{-1}^{1} |P(t)|^{\mu}\, dt \le c' \bigg(\int_{-1}^{1} |P(t)|\, dt \bigg)^{\mu} \le c' c \bigg(\int_{-1}^{1} |P(t)|^{-\mu}\, dt \bigg)^{-1},$$

which together with the fact that the class of polynomials of degree n is invariant under any dilation and translation, proves that $|P|^{\mu}$ is in $\mathcal{A}_2(\mathbf{T})$ for $-1/n < \mu < 1/n$, provided $|P(-\pi)| = |P(\pi)|$. Thus, for $-1/n < \mu < 1/n$, $\mathcal{T}$ is a Schauder basis for $L^2(\mathbf{T}; |P|^{\mu})$. Obviously $|P|^{\mu}$ is bounded on $[-\pi, \pi]$ so $\mathcal{T}$ is a quasi-greedy (and thus a Riesz) basis for $L^2(\mathbf{T}; |P|^{\mu})$ if and only if P has no zeros on $\mathbf{T}$. □

Example 4.2. This is the famous example by Babenko of a conditional Schauder basis for $L^2(\mathbf{T})$ [**1**]. Using Remark 2.4 and Proposition 4.1, we see that the system $\{|t|^{\alpha} e_{n_k}\}_{k=1}^{\infty}$ forms a Schauder basis for $L^2(\mathbf{T})$ for $0 < \alpha < 1/2$ since, according to Proposition 4.1, $|t|^{\mu} \in \mathcal{A}_2$ for $-1 < \mu < 1$. The basis is conditional since t has a zero on $\mathbf{T}$.

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