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SOME REMARKS ON FACTORIAL QUOTIENT RINGS

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ABSTRACT. Let D be a Weil divisor with rational coefficients on an integral, normal, projective scheme X defined over a field K. Assume that ND is an ample Cartier divisor for some $N > 0$. Then $A(X, D) = \oplus_{n\geq 0} H^0(X, \mathcal{O}_X(nD))T^n \subseteq K(X)[T]$ is a finitely generated, integrally closed, graded K-algebra. Since factorial domains are integrally closed, it is natural to ask for criteria which imply the factoriality of $A(X, D)$. In 1984 Robbiano found the shape of the divisor D such that $A(X, D)$ is factorial, in the case $\mathrm{Cl}\,(X) = \mathbf{Z}$. The main result in this paper is Theorem 29 where we give a characterization of such factorial rings valid over a field of any characteristic.

In the last part of the paper we study how the task of factorizing an element of a UFD, given as a quotient R/I, can be achieved by simply calculating inside the ring R.

1. Introduction. In 1979 Demazure developed the theory of Weil divisors with rational coefficients on normal schemes. He showed that if D is such a divisor on an integral, normal, projective scheme X defined over a field K, and if ND is an ample Cartier divisor for some $N > 0$, then $A(X, D) = \oplus_{n\geq 0} H^0(X, \mathcal{O}_X(nD))T^n \subseteq K(X)[T]$ is a finitely generated, integrally closed, graded K-algebra.

In [**12**] Robbiano studied the connection between weighted projective spaces, rational coefficient Weil divisors and unique factorization domains. In particular, he showed that if $\dim(X) \geq 1$, the ring $A(X, D)$ is almost factorial if and only if $\mathrm{rank}\,(\mathrm{Cl}\,(X)) = 1$. Moreover, he described the shape that a Weil divisor with rational coefficients must have to give rise to a factorial ring. This result is described in our Theorem 8.

The main result in this paper is Theorem 29 where we give a characterization of such factorial rings. Let X be an integral normal projective scheme defined over a field K. Assume that $\dim(X) \geq 1$, $\mathrm{Cl}\,(X) = \mathbf{Z}$ generated by $[D_0]$, where D_0 is very ample, and let $D =$

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$\sum_V (p_V/q_V)V$ be a rational coefficient Weil divisor such that $A(X,D)$ is a factorial ring. Then there exists an equivariant isomorphism

$$A(X,D) \cong A(X,D_0)[X_0, \dots X_s]/(X_0^{\delta_0} - f_0, \dots , X_s^{\delta_s} - f_s)$$

where $f_0, \dots , f_s \in A(X,D_0)$ are homogeneous generators of the prime ideals of $A(X,D_0)$ corresponding to the prime divisors in the support of D.

A more general structure theorem was proved in a recent paper by Tomari and Watanabe [**15**], where they deal with integral normal projective schemes defined over a field K with $\mathrm{char}\,(K) = 0$. Their result is also valid under the hypothesis that $\mathrm{char}\,(K) = p > 0$, but in this case they need the extra assumptions $\mathrm{GCD}\,(\mathrm{LCM}\,(q_V), p) = 1$ and $\mathrm{GCD}\,(p, p_V) = 1$. Our result is obtained using a more direct approach which does not require any restrictions on the characteristic of the field K.

The aim of the last part of the paper is to understand how the task of factorizing an element of a UFD, given as a quotient R/I, can be achieved by simply calculating inside the ring R. In fact, it is natural to ask for a method to compute the factorization of an element in a factorial quotient ring R/I where $R = K[X_1, \dots , X_n]$ is a polynomial ring over a field K, and I is a homogeneous ideal of R with respect to a positive graduation of R. Therefore, we provide an algorithm to compute the factorization of the residue class of F in $K[X_1, \dots , X_n]/I$ by the computation of the minimal primes of the ideal $(F) + I$ in $K[X_1, \dots , X_n]$.

In the last section we give an alternative algorithm to compute the factorization of an element in R/I in the case where I is a principal ideal. The approach we use is similar to the one for factoring polynomials in several variables over an algebraic number field (Trager, [**16**]).

2. Rings associated to Weil divisors with rational coefficients. In this section we recall some basic definitions, and we study the rings associated to Weil divisors with rational coefficients over a normal projective scheme. In the following, K is a field and $(X, \mathcal{O}_X)$ is an integral projective normal scheme defined over K. We denote by W–div(X) the set of Weil divisors on X, by W–div$(X, \mathbf{Q}) =$

W–div$(X)\otimes_{\mathbf{Z}}\mathcal{Q}$, the set of Weil divisors on X with rational coefficients, and by $\mathcal{D}_X$ the set of divisors $D \in$ W–div$(X, \mathbf{Q})$ such that ND is an ample Cartier divisor for some $N > 0$. If $D \in \mathcal{D}_X$, then $\lfloor D \rfloor$ denotes the integral part of the D, i.e., $\lfloor D \rfloor$ is the divisor defined by $\sup\{\Delta \in \text{W–div}(X) : \Delta \leq D\}$. If $\mathcal{O}_X(D)$ is the sheaf defined by $\mathcal{O}_X(D)(U) = \{f \in K(X) : ((f)+D)_{|U} \geq 0\}$, then $\mathcal{O}_X(D) = \mathcal{O}_X(\lfloor D \rfloor)$. Moreover, when we write $D = \sum_{i=1}^{s} \eta_i/\delta_i \ V_i \in \mathcal{D}_X$, we assume that $\delta_i > 0$, GCD $(\eta_i, \delta_i) = 1$ and V_i are distinct prime divisors.

Theorem 1 ([**4**, Proposition 3.3]). *Let $(X, \mathcal{O}_X)$ be an integral projective normal scheme. Let $D \in \mathcal{D}_X$, let T be a new indeterminate of degree 1, and let $A(X, D)$ be the graded subring of $K(X)[T]$ defined by the equality $A(X, D) = \oplus_{n\geq 0} H^0(X, \mathcal{O}_X(nD))T^n$. Then*:

(1) *$A(X, D)$ is an integral domain and a finitely generated graded K-algebra such that $A(X, D)_0 = K$.*

(2) *$A(X, D)$ is integrally closed.*

(3) *There exists an isomorphism of schemes $j : X \to \text{Proj}\,(A(X, D))$ such that $j_*\mathcal{O}_X(nD) = \mathcal{O}_{\text{Proj}\,(A(X,D))}(n)$ and $j^*\mathcal{O}_{\text{Proj}\,(A(X,D))}(n) = \mathcal{O}_X(nD)$.*

(4) *$K(A(X, D)) = K(X)(T)$.*

(5) *Let f be a rational function over X. Then there exists an equivariant isomorphism $\psi : A(X, D) \to A(X, D + (f))$.*

Conversely, suppose that A is an integrally closed domain and a finitely generated graded K-algebra such that $A_0 = K$. Then we want to find a divisor $D \in$ W–div(Proj$\,(A), \mathbf{Q})$ such that $A = A(\text{Proj}\,(A), D)$.

Let $p : \text{Spec}\,(A) \setminus \{\mathfrak{m}\} \to \text{Proj}\,(A)$ be the canonical projection. We recall that if V is a prime divisor of Proj$\,(A)$, $\mathfrak{p}$ the prime ideal of height 1 such that $V = \text{Proj}\,(A/\mathfrak{p})$, and $W = \text{Spec}\,(A/\mathfrak{p}) \setminus \{\mathfrak{m}\}$ the unique prime divisor of $\text{Spec}\,(A) \setminus \{\mathfrak{m}\}$ such that $p(W) = V$, then the *pull-back* of V is the divisor $p^*(V) = e_{W|V}W$ where $e_{W|V} = l(A_{\mathfrak{p}}/\mathfrak{m}_{A_{(\mathfrak{p})}}A_{\mathfrak{p}})$ is the length of the $A_{\mathfrak{p}}$-module $A_{\mathfrak{p}}/\mathfrak{m}_{A_{(\mathfrak{p})}}A_{\mathfrak{p}}$.

Theorem 2 ([**4**], Theorem 3.5). *Let A be an integrally closed domain and a finitely generated graded K-algebra such that $A_0 = K$. Then let $T \in K(A)$ be a homogeneous element of degree one, and* *let* $A = A(\mathrm{Proj}\,(A), D)$. *Then there exists a unique divisor* $D = \sum_{i=1}^{s} \eta_i/\delta_i\ V_i \in \mathcal{D}_X$ *with the property that* $A_n = H^0(X, \mathcal{O}_X(nD))T^n$, $\mathcal{O}_X(n) = \mathcal{O}_X(nD)T^n$, *and* $p^*(D) = \mathrm{div}\,(T)$. *In particular,* $p^*(V_i) = \delta_i W_{V_i}$ *for every* $i = 1, \dots, s$.

Let $(X, \mathcal{O}_X)$ be an integral projective normal scheme such that $\dim\,(X) \geq 1$, and let $D \in \mathcal{D}_X$. The following result states the existence of an exact sequence which connects the class groups of $A(X, D)$ and X.

Theorem 3 ([**17**, Theorem 1.6]). *Let* $(X, \mathcal{O}_X)$ *be an integral projective normal scheme such that* $\dim\,(X) \geq 1$, *let* $D = \sum_{i=1}^{s} \eta_i/\delta_i\ V_i \in \mathcal{D}_X$ *and let* $L_D = \mathrm{LCM}\,(\delta_1, \dots, \delta_s)$. *Then let* $\phi : \mathbf{Z} \to \mathrm{Cl}\,(X)$ *be the map defined by* $\phi(1) = [L_D \cdot D]$, *let* $\alpha : \mathbf{Z} \to \oplus_{i=1}^{s} \mathbf{Z}/\delta_i\mathbf{Z}$ *be the homomorphism defined by* $\alpha(1) = (\eta_1 \mod \delta_1, \dots, \eta_s \mod \delta_s)$, *and let* $\psi : \mathrm{Cl}\,(X) \to \mathrm{Cl}\,(A(X, D))$ *be the map induced by* p^*. *The following sequence*

$$0 \longrightarrow \mathbf{Z} \xrightarrow{\phi} \mathrm{Cl}\,(X) \xrightarrow{\psi} \mathrm{Cl}\,(A(X, D)) \longrightarrow \mathrm{Coker}\,(\alpha) \longrightarrow 0$$

is exact.

Definition 4. A normal ring A is said to be *factorial* if $\mathrm{Cl}\,(A) = 0$. A normal ring A is said to be *almost factorial* if $\mathrm{Cl}\,(A)$ is torsion.

Corollary 5. *Let* $(X, \mathcal{O}_X)$ *be an integral projective normal scheme such that* $\dim\,(X) \geq 1$, *let* $D = \sum_{i=1}^{s} \eta_i/\delta_i\ V_i \in \mathcal{D}_X$, *and denote by* L_D *the integer* $\mathrm{LCM}\,(\delta_1, \dots, \delta_s)$. *Then*

(1) $A(X, D)$ *is almost factorial if and only if* $\mathrm{rank}\,(\mathrm{Cl}\,(X)) = 1$.

(2) $A(X, D)$ *is factorial if and only if* $\mathrm{Cl}\,(X) = \mathbf{Z}$ *generated by* $[L_D \cdot D]$ *and the* δ_i*s are pairwise coprime.*

Definition 6. A divisor $D = \sum_{i=1}^{s} \eta_i/\delta_i\ V_i \in \mathcal{D}_X$ is said to be a *pairwise coprime divisor* if $\mathrm{GCD}\,(\delta_i, \delta_j) = 1$, $i \neq j$.

Remark 7. Suppose that $\mathrm{Cl}\,(X) = \mathbf{Z}$, and let $[D_0]$ be a generator. Then either D_0 or $-D_0$ is in $\mathcal{D}_X$. In fact, if we consider the embedding of X in some $\mathbf{P}^N$, and denote by H a hyperplane section of X, then $[nD_0] = [H]$ for some $n \in \mathbf{Z}$. This relation implies that either D_0 or $-D_0$ is an ample Cartier divisor. Therefore, we can assume that $D_0 \in \mathcal{D}_X \cap \text{W–div}(X)$ is such that $[D_0]$ generates $\text{Cl}\,(X)$, and for every $D \in \text{W–div}(X)$ we denote by $\deg\,(D)$ the integer defined by the relation $D \sim \deg\,(D) D_0$.

Theorem 8 [**12**, Theorem 3.5]. *Let* $(X, \mathcal{O}_X)$ *be an integral projective normal scheme such that* $\dim\,(X) \geq 1$, *let* $D = \sum_{i=1}^{s} \eta_i/\delta_i \; V_i \in \mathcal{D}_X$, *and assume that* $\text{Cl}\,(X) = \mathbf{Z}$ *is generated by* $[D_0]$ *and that* $D_0 \in \mathcal{D}_X$. *The following conditions are equivalent*:

(1) $A(X, D)$ *is factorial.*

(2) D *is a pairwise coprime divisor such that* $\sum_{i=1}^{s} (\eta_i \cdot \deg\,(V_i))/\delta_i = 1/(\prod_{i=1}^{s} \delta_i)$.

2.1. Normalized grading. Let $Q = (q_0 \ldots, q_r)$ be an $r+1$-tuple of positive integers, $q = \text{GCD}\,(q_0 \ldots, q_r)$, $d_i = \text{GCD}\,(q_0, \ldots, q_{i-1}, q_{i+1}, \ldots, q_r)$, $a_i = \text{LCM}\,(d_0, \ldots, d_{i-1}, d_{i+1}, \ldots, d_r)$, and $a = \text{LCM}\,(d_0, \ldots, d_r)$. The $r+1$-tuple Q is said to be *reduced* if $q = 1$.

In the following remark we collect some easy facts about reduced tuples of positive integers.

Remark 9. Suppose that Q is reduced. The following statements hold:

(1) $a_i \mid q_i$ for every $i = 0, \ldots, r$.

(2) $\text{GCD}\,(q_i, d_i) = 1$ for every $i = 0, \ldots, r$.

(3) $\text{GCD}\,(d_j, d_i) = 1$ for every $j \neq i$.

(4) $\text{GCD}\,(a_i, d_i) = 1$ for every $i = 0, \ldots, r$.

(5) $a_i d_i = a$ for every $i = 0, \ldots, r$.

(6) $d_j | a_i$ for every $j \neq i$.

Definition 10. We say that Q is *normalized* if $d_i = 1$ for every $i = 0, \ldots, r$. The *normalization* of Q is the $r+1$-tuple $\overline{Q} = ((q_0/a_0), \ldots, (q_r/a_r))$. Clearly, $\overline{Q}$ is normalized.

Now, we study some properties related to an $\mathbf{N}$-grading of a K-algebra A and the induced map $p^* : \operatorname{Spec}(A) \setminus \{\mathfrak{m}\} \to \operatorname{Proj}(A)$. Let A be an integrally closed domain and a finitely generated $\mathbf{N}$ graded K-algebra, assume that $A_0 = K$, and let $\{t_0, \dots , t_r\}$ be a minimal set of homogeneous generators of A as a K-algebra. Let $q_i = \deg(t_i) > 0$ for $i = 1, \dots , r$, and let $Q = (q_0 \dots , q_r)$. The next lemma is easy to prove, so we omit the proof.

Lemma 11. *The following conditions are equivalent*:

(1) $\operatorname{GCD}\{n \in \mathbf{N}^* : A_n \neq (0)\} = q$.

(2) $\operatorname{GCD}(q_0, \dots , q_r) = q$.

(3) $A_{qn} = (0)$ for $n > 0$, and there exists a homogeneous element $T \in K(A)$ such that $\deg(T) = q$.

The grading on A is said to be *reduced* if the $r+1$-tuple Q is reduced, or equivalently if there exists a homogeneous element $T \in K(A)$ of degree one.

It is known that for every integer $n > 0$ there exists an isomorphism of schemes $\psi : \operatorname{Proj}(A) \to \operatorname{Proj}(A^n)$, where $A^{(n)}$ is the n-*Veronese* of A, and it is the graded ring $A^{(n)} = \oplus_{m\geq 0}[A^{(n)}]_m$ such that $[A^{(n)}]_m = A_{mn}$.

In particular, $\operatorname{Proj}(A) \cong \operatorname{Proj}(A^{(q)})$, where $q = \operatorname{GCD}(q_0, \dots , q_r)$. Moreover, $A^{(q)} = K[t_0, \dots , t_r]$ where $\deg(t_i) = q_i/q$. Therefore, in the following we can assume that the grading on A is reduced, and it is said to be *normalized* if Q is normalized.

Proposition 12. *Let A be an integrally closed domain and a finitely generated $\mathbf{N}$-graded K-algebra. Assume that $A_0 = K$. With the above notation, we have $A^{(a)} = K[t_0^{d_0}, \dots , t_r^{d_r}]$ and $\deg(t_i^{d_i}) = q_i/a_i$.*

Proof. In the ring A we have $\deg(t_i^{d_i}) = q_i d_i = a_i d_i q_i/a_i = a q_i/a_i$ and $q_i/a_i \in \mathbf{N}^*$, hence $t_i^{d_i} \in A^{(a)}$.

Conversely, if $t_0^{s_0} \cdots t_r^{s_r} \in A^{(a)}$, then $s_0 q_0 + \cdots + s_r q_r = \lambda a$, for some $\lambda \in \mathbf{N}$; therefore,

$$s_i q_i = -\sum_{j\neq i} s_j q_j + \lambda a_i d_i.$$

Since $d_i \mid q_j$ where $j \neq i$, we obtain $s_i q_i \in (d_i)$. Moreover, q_i and d_i are coprime, so $s_i \in (d_i)$ for every $i = 0, \dots, r$. □

We have seen that $\overline{Q}$ is normalized but in general $\{t_0^{d_0}, \dots, t_r^{d_r}\}$ is not a minimal system of homogeneous generators of $A^{(a)}$, so the grading on $A^{(a)}$ is not necessarily normalized.

Proposition 13. *Let A be an integrally closed domain and a finitely generated $\mathbf{N}$ graded K-algebra. Assume that $A_0 = K$. Let V be a prime divisor of* $\operatorname{Proj}(A)$*, let $\mathfrak{p}$ be the corresponding prime ideal of height* 1*, and let W be the unique prime divisor of* $\operatorname{Spec}(A) \setminus \{\mathfrak{m}\}$ *such that $p(W) = V$. Then $e_{W|V} = \operatorname{GCD}\{n \in \mathbf{N}^* : (A/\mathfrak{p})_n \neq (0)\}$.*

Proof. Let $d_{\mathfrak{p}} = \operatorname{GCD}\{n \in \mathbf{N}^* : (A/\mathfrak{p})_n \neq (0)\}$. Since $A_{\mathfrak{p}}$ is a regular ring of dimension 1, then $\mathfrak{p}A_{\mathfrak{p}}$ is principle. We can suppose that $\mathfrak{p}A_{\mathfrak{p}} = (f)$, where $f \in \mathfrak{p}$ is homogeneous. The claim is

$$\mathfrak{m}_{A_{(\mathfrak{p})}} A_{\mathfrak{p}} = (f^{d_{\mathfrak{p}}}),$$

where $\mathfrak{m}_{A_{(\mathfrak{p})}}$ is the maximal ideal of $A_{(\mathfrak{p})}$.

First, we observe that for every homogeneous element $u \notin \mathfrak{p}$, $d_{\mathfrak{p}} \mid \deg(u)$. Moreover, by Lemma 11, there exists an element $u' = u_1/u_2 \in K(A)$, with u_1 and u_2 homogeneous elements in A not in $\mathfrak{p}$ and $\deg(u') = \deg(u_1) - \deg(u_2) = d_{\mathfrak{p}}$.

Since $u' \notin \mathfrak{p}A_{\mathfrak{p}}$, if we prove that $\operatorname{GCD}(\deg(f), d_{\mathfrak{p}}) = 1$, then

$$\mathfrak{m}_{A_{(\mathfrak{p})}} = \left(\frac{f^{d_{\mathfrak{p}}}}{u'^{\deg(f)}}\right),$$

and the conclusion follows. Let d' be the $\operatorname{GCD}(\deg(f), d_{\mathfrak{p}})$. If $g \in A$ is a homogeneous element not in $\mathfrak{p}$, then $d_{\mathfrak{p}} \mid \deg(g)$, so $d' \mid \deg,(g)$. On the other hand, if $g \neq 0$ is a homogeneous element in $\mathfrak{p}$, then in $A_{\mathfrak{p}}$ we have $g = (l_1/l_2) f^s$, where l_1, l_2 are homogeneous elements in A not in $\mathfrak{p}$.

Then $d' \mid d_{\mathfrak{p}} \mid \deg(l_1)$ and $d' \mid d_{\mathfrak{p}} | \deg(l_2)$. Moreover $d' \mid \deg(f)$, so $d' \mid \deg(g)$. Therefore, the degree of every homogeneous element in A is a multiple of d'. Since the grading on A is reduced, then $d' = 1$. □

Proposition 14. *Let A be an integrally closed domain and a finitely generated $\mathbf{N}$ graded K-algebra. Assume that $A_0 = K$. With the above notation, suppose that* $\operatorname{div}(t_i) = \sum_{j=1}^{m_i} s_{i,j} W_{i,j}$*, with $W_{i,j}$ prime divisors of* $\operatorname{Spec}(A) \setminus \{\mathfrak{m}\}$*, and let $V_{i,j} = p(W_{i,j})$ be the corresponding prime divisors of* $\operatorname{Proj}(A)$*. Then*

(1) *$d_i | e_{W_{i,j}|V_{i,j}}$ for every $j = 1, \dots, m_i$.*

(2) *If $\operatorname{div}(t_i)$ is prime for some $i = 0, \dots, r$, then $p^*(V_i) = d_i \operatorname{div}(t_i)$.*

Proof. Let $\mathfrak{p}_{i,j}$ be the homogeneous prime ideal of height 1 corresponding to $V_{i,j}$. By Proposition 13, $e_{W_{i,j}|V_{i,j}} = \operatorname{GCD}\{n \in \mathbf{N}* : [A/\mathfrak{p}_{i,j}]_n \neq (0)\}$. Since $t_i \in \mathfrak{p}_{i,j}$, then $A/\mathfrak{p}_{i,j}$ is a quotient algebra of $A/(t_i)$. Hence, $[A/\mathfrak{p}_{i,j}]_n \neq 0$ implies $[A/(t_i)]_n \neq 0$. Therefore $\operatorname{GCD}\{n \in \mathbf{N}^* : [A/(t_i)]_n \neq (0)\}$ divides $\operatorname{GCD}\{n \in \mathbf{N}^* : [A/\mathfrak{p}_{i,j}]_n \neq (0)\} = e_{W_{i,j}|V_{i,j}}$.

By Proposition 11 and the observation that $t_0, \dots, t_{i-1}, t_{i+1}, \dots, t_r$ is a minimal system of homogeneous generators of the ring $A/(t_i)$, we have the equality $\operatorname{GCD}\{n \in \mathbf{N}^* : [A/(t_i)]_n \neq (0)\} = d_i$. $\square$

2.2. Normal embeddings. In this section we study the case where the grading of the ring associated to a Weil divisor with rational coefficients over a projective scheme is normalized. Let $(X, \mathcal{O}_X)$ be an integral projective normal scheme such that $\dim(X) \geq 1$, and let $D \in \mathcal{D}_X$. Let $\{t_0, \dots, t_r\}$ be a minimal system of homogeneous generators of $A(X, D)$ as a K-algebra, and denote by Q the tuple $(q_0, \dots, q_r)$, where $q_i = \deg(t_i)$ for every $i = 0, \dots, r$. Then there exists a homogeneous isomorphism $K[T_0, \dots, T_r]/I \cong A(X, D)$ such that $\overline{T_i} \mapsto t_i$ gives a closed embedding of X in the weighted projective space $\mathbf{P}(Q)$.

Proposition 15. *Let $(X, \mathcal{O}_X)$ be an integral projective normal scheme such that $\dim(X) \geq 1$, and let $D \in \mathcal{D}_X$. With the above notation, if D is a Weil divisor and t_i is prime in $A(X, D)$ for every $i = 0, \dots, r$, then Q is normalized.*

Proof. Let $T = \prod_{i=0}^r t_i^{n_i}$ be such that $p^*(D) = \operatorname{div}(T) = \sum_{i=0}^r n_i \operatorname{div}(t_i)$. We have $\sum_{i=0}^r n_i q_i = 1$. If $n_i = 0$, then $d_i = 1$

since $1 = \sum_{j \neq i} n_i q_i$. If $n_i \neq 0$, then $D = \sum_{i=0}^{r} n_i/d_i p\,(\mathrm{div}\,(t_i))$ by Theorem 2 and by Proposition 14, so $d_i = 1$ since D is a Weil divisor. □

Definition 16. We say that X is *normally embedded* by $D \in \mathcal{D}_X$ if the grading of $A(X, D)$ is normalized. The embedding is said to be factorial if $A(X, D)$ is factorial and it is said to be almost factorial if $A(X, D)$ is almost factorial.

Corollary 17. *Let $(X, \mathcal{O}_X)$ be an integral projective normal scheme such that* $\dim\,(X) \geq 1$*. If $D \in \mathcal{D}_X \cap$ W–div(X) and $A(X, D)$ is factorial, then X is normally embedded by D.*

Proof. Since $t_0, \ldots, t_r$ are minimal generators of $A(X, D)$, then they are irreducible. Moreover, $A(X, D)$ is factorial, hence the element t_i is prime for every $i = 0, \ldots, r$. Then Q is normalized by Proposition 15. □

Proposition 18. *Let $(X, \mathcal{O}_X)$ be an integral projective normal scheme such that* $\dim\,(X) \geq 1$*, and let $D \in \mathcal{D}_X$. If $A(X, D)$ is factorial and its grading is normalized, then D is a Weil divisor.*

Proof. Let $T = \prod_{i=0}^{r} {t_i}^{n_i}$ be such that $p^*(D) = \mathrm{div}\,(T) = \sum_{i=0}^{r} n_i \mathrm{div}\,(t_i)$. Since $A(X, D)$ is factorial and t_i is irreducible, $\mathrm{div}\,(t_i)$ is a prime divisor for every i. By Theorem 2 and Proposition 14, we have $D = \sum_{i=0}^{r} n_i/d_i\; p(\mathrm{div}\,(t_i))$. Therefore, D is a Weil divisor. □

Theorem 19. *Let $(X, \mathcal{O}_X)$ be an integral projective normal scheme such that* $\dim\,(X) \geq 1$*. Then the following conditions are equivalent*:

(1) *There exists $D \in \mathcal{D}_X$ such that $A(X, D)$ is factorial.*

(2) $\mathrm{Cl}\,(X) = \mathbf{Z}$.

(3) *There exists $D_0 \in \mathcal{D}_X \cap$W–div$(X)$ such that $A(X, D_0)$ is factorial, and its grading is normalized.*

Proof. (1) $\Longrightarrow$ (2) is a consequence of Theorem 3.

We prove (2) $\Longrightarrow$ (3). Let $D_0 \in$ W–div(X) be such that $[D_0]$ generates $\mathrm{Cl}\,(X)$. After Remark 7 we can assume that $D_0 \in \mathcal{D}_X$.

Moreover, $A(X, D_0)$ is factorial by Corollary 5, and its grading is normalized by Corollary 17.

Finally, (3) $\Longrightarrow$ (1) is obvious. □

Remark 20. Let $(X, \mathcal{O}_X)$ be an integral projective normal scheme such that $\dim(X) \geq 1$, and assume that $\mathrm{Cl}(X) = \mathbf{Z}$. We have seen that there exists a divisor $D_0 \in \mathcal{D}_X \cap \text{W–div}(X)$ such that $A(X, D_0)$ is factorial and its grading is normalized. Let $D \in \mathcal{D}_X$ be such that $A(X, D)$ is factorial and its grading is normalized. By Proposition 18 we have that D is a Weil divisor. Assume that $D = \sum_{i=1}^{s} \eta_i\ V_i \in \mathcal{D}_X \cap \text{W–div}(X)$. Then by Theorem 8 we have $\sum_{i=1}^{s} \eta_i\ \deg(V_i) = 1$, i.e, D is linearly equivalent to D_0.

3. Factorial graded algebras. We have seen in Corollary 5 that if X is an integral normal projective scheme such that $\dim(X) \geq 1$ then either $\mathrm{rk}(\mathrm{Cl}(X)) > 1$, and in this case we cannot obtain almost factorial rings from X, or $\mathrm{rk}(\mathrm{Cl}(X)) = 1$, and for every $D \in \mathcal{D}_X$, $A(X, D)$ is almost factorial. Moreover, by Theorem 19, if $\mathrm{Cl}(X) = \mathbf{Z}$ then there exists a divisor $D_0 \in \mathcal{D}_X \cap \text{W–div}(X)$ such that $A(X, D_0)$ is factorial, and we obtain factorial rings from X using the "recipe" described in Theorem 8. In this section we are going to characterize such rings (Theorem 29).

In the following, by A we mean an integrally closed domain which is a finitely generated $\mathbf{N}$-graded K-algebra. Moreover, we assume that $A_0 = K$ and that the grading is reduced. If $A = \oplus_{n\geq 0} A_n$ we denote by $\mathfrak{m}_A$ the ideal $= \oplus_{n>0} A_n$.

In the following, by $\{t_0, \ldots, t_r\}$ we denote a minimal set of homogeneous generators of A as a K-algebra, and by $q_i = \deg(t_i) > 0$ for $i = 0, \ldots, r$.

Proposition 21. *Let $f \in A_m$, $m > 0$, be a homogeneous element, and let $n > 0$ be such that* $\mathrm{GCD}(n, m) = 1$. *Then $X^n - f$ is prime in $A[X]$.*

Proof. Let $\deg(X) = m$, and let $\deg(t_i) = q_i \cdot n$ for every $i = 0, \ldots, r$. Then $A[X]$ is a graded K-algebra such that $X^n - f$ is homogeneous of degree nm. Let $B = A[x]/(x^n - f)$. If B is not an integral domain, then there exists a homogeneous zero divisor. But a homogeneous element of B is of the form ax^i with $a \in A$ homogeneous, $0 \le i < n$ and $x = \overline{X}$. □

Proposition 22. *Let m be a positive integer, let $f \in A_m$ be a homogeneous prime element, let $n > 1$ be an integer such that* GCD $(n, m) = 1$, *and let $B = A[X]/(X^n - f)$.*

(1) *f is a minimal generator of $\mathfrak{m}_A$ if and only if there exists $i \in \{0, \dots, r\}$ such that $\{t_0, \dots, t_{i-1}, t_{i+1}, \dots, t_r, x\}$ is a minimal system of homogeneous generators of B as a K-algebra.*

(2) *Let $g \in A$ be a homogeneous prime element such that $(g) \ne (f)$. Then g is prime in B.*

Proof. Let $\mathfrak{m}_B = (t_0, \dots, t_r, x)$. We have to prove that $\overline{f} \ne 0$ in $\mathfrak{m}_A/\mathfrak{m}_A^2$ if and only if there exists an $i \in \{0, \dots, r\}$ such that $\overline{t_0}, \dots, \overline{t_{i-1}}, \overline{t_{i+1}}, \dots, \overline{t_r}, \overline{x}$ form a basis of $\mathfrak{m}_B/\mathfrak{m}_B^2$. Clearly $\mathfrak{m}_B^2 \cap A = \mathfrak{m}_A^2 + (f)$.

Assume that f is a minimal generator of $\mathfrak{m}_A$, and let $t_0 = f$. Since $t_0 = x^n$ in B, $\overline{t_1}, \dots, \overline{t_r}, \overline{x}$ generate $\mathfrak{m}_B/\mathfrak{m}_B^2$. They are linearly independent. In fact, assume that $\sum_{j=1}^r \lambda_j t_j + \lambda x \in \mathfrak{m}_B^2$, where $\lambda_1, \dots, \lambda_r, \lambda \in K$. Since $\mathfrak{m}_B^2$ is a homogeneous ideal and $\deg(x) \ne \deg(t_i)$ for every i, we have $\lambda x \in \mathfrak{m}_B^2$, hence $\lambda = 0$. Therefore, $\sum_{j=1}^r \lambda_j t_j \in \mathfrak{m}_B^2 \cap A = \mathfrak{m}_A^2 + (t_0)$, so $\sum_{j=1}^r \lambda_j \overline{t_j} - \lambda_0 \overline{t_0} = 0$ in $\mathfrak{m}_A/\mathfrak{m}_A^2$, for a suitable $\lambda_0 \in K$. But $\overline{t_0}, \dots, \overline{t_r}$ are linearly independent, therefore $\lambda_i = 0$ for every $i = 1, \dots, r$.

Conversely, let $i \in \{0, \dots, r\}$ be such that $\{\overline{t_0}, \dots, \overline{t_{i-1}}, \overline{t_{i+1}}, \dots, \overline{t_r}, \overline{x}\}$ is a basis of $\mathfrak{m}_B/\mathfrak{m}_B^2$. Then $\overline{t_i} = \sum_{j \ne i} \lambda_j \overline{t_j} + \lambda \overline{x}$, i.e., $t_i - \sum_{j \ne i} \lambda_j t_j + \lambda x \in \mathfrak{m}_B^2$. This implies $\lambda = 0$ and $t_i - \sum_{j \ne i} \lambda_j t_j \in \mathfrak{m}_B^2 \cap A$. Therefore, $t_i - \sum_{j \ne i} \lambda_j t_j - af \in \mathfrak{m}_A^2$ for some $a \in A$, i.e., $\overline{t_i} - \sum_{j \ne i} \lambda_j \overline{t_j} - \overline{a}\overline{f} = 0$ in $\mathfrak{m}_A/\mathfrak{m}_A^2$. But $t_0, \dots, t_r$ are linearly independent, so $\overline{a}\,\overline{f} \ne 0$, i.e., f is a minimal generator.

Now we prove (2). Since g is a homogeneous prime element, $A/(g)$ is a graded domain. Then $f \in [A/(g)]_m$ and $f \ne 0$ in $A/(g)$. By Proposition 21 we have that $A/(g)[X]/(X^n - f) \cong A[X]/(X^n - f)/(g)$ is a domain, i.e., g is prime. □

Proposition 23. *Suppose that A is factorial and that its grading is normalized. Let $T \in K(A)$ be a homogeneous element of degree one, and let D be the unique divisor in $\mathcal{D}_{\mathrm{Proj}\,(A)}$ with the property that $A = A(\mathrm{Proj}\,(A), D)$ and $p^*(D) = \mathrm{div}\,(T)$. Then*:

(1) *D is a Weil divisor.*

(2) *$[D]$ generates $\mathrm{Cl}\,(\mathrm{Proj}\,(A)) = \mathbf{Z}$.*

(3) *If V is a prime divisor of $\mathrm{Proj}\,(A)$, then there exists a homogeneous prime element $f_V \in A$ such that $V \cong \mathrm{Proj}\,(A/(f_V))$.*

(4) *If $\deg(V)$ is defined by $[V] = \deg(V)[D]$, then $\deg(V) = \deg(f_V)$.*

Proof. (1) follows from Proposition 18 and (2) follows from Theorem 3. Let $\mathfrak{p}_V \subset A$ be the unique homogeneous prime ideal of height 1 such that V is isomorphic to $\mathrm{Proj}\,(A/\mathfrak{p}_V)$. Since A is factorial, there exists a homogeneous prime element $f_V \in A$ such that $\mathfrak{p}_V = (f_V)$.

Since $(T^{\deg(f_V)}/f_V)$ defines a rational function over $\mathrm{Proj}\,(A)$, we have that $\deg(f_V)D \sim V$. □

With the same notation as in Section 2, let $d_i = \mathrm{GCD}\,(q_0, \dots, q_{i-1}, q_{i+1}, \dots, q_r)$, $a_i = \mathrm{LCM}\,(d_0, \dots, d_{i-1}, d_{i+1}, \dots, d_r)$ and $a = \mathrm{LCM}\,(d_0, \dots, d_r)$.

Lemma 24. *Suppose that $A^{(a)}$ is factorial and that its grading is normalized. If t_i is a prime element in A, then $\deg(V_i) = q_i/a_i$, where $V_i = p(\mathrm{div}\,(t_i))$.*

Proof. By Proposition 12 we have $A^{(a)} = K[t_0^{d_0}, \dots, t_r^{d_r}]$. Moreover, $t_i^{d_i}$ is irreducible in $A^{(a)}$ because t_i is prime in A. Since $A^{(a)}$ is factorial, $t_i^{d_i}$ is prime in $A^{(a)}$. Moreover,

$$\mathrm{Proj}\,(A^{(a)}/(t_i^{d_i})) = \mathrm{Proj}\,((A/(t_i))^{(a)}) \cong \mathrm{Proj}\,(A/(t_i)) = V_i.$$

Therefore, by Proposition 23, $\deg(V_i)$ is equal to the degree of $t_i^{d_i}$ in $A^{(a)}$, i.e., q_i/a_i. □

Lemma 25. *If A is factorial, then $A^{(a)}$ is factorial and its grading is normalized.*

Proof. Let $X = \mathrm{Proj}\,(A)$ and let $T = \prod_{i=0}^{r} t_i^{n_i} \in K(A)$ be such that $\deg\,(T) = 1$. Since A is factorial and t_i is irreducible, t_i is a prime element for every $i = 0, \dots, r$. Therefore, by Theorem 2 and by Proposition 14, the divisor D such that $A = A(X, D)$ and $p^*(D) = \mathrm{div}\,(T)$ is $D = \sum_{i=0}^{r}(n_i/d_i)\; p(\mathrm{div}\,(t_i))$. By Theorem 3, $[aD]$ generates $\mathrm{Cl}\,(X)$. Moreover, by the definition of $A(X, D)$, we have $A^{(a)} = A(X, aD)$, hence $A^{(a)}$ is factorial by Corollary 5. Since aD is a Weil divisor, the grading on $A^{(a)}$ is normalized by Corollary 17. □

Here we show how the methods explained before can be used to prove the following fact. For another proof, see [**13, 14**].

Theorem 26. *Let A be a finitely generated factorial K-algebra. Assume that A is $\mathbf{N}$-graded, $A_0 = K$, and the grading is reduced. Let m be a positive integer, let $f \in A_m$ be a prime element and let $n > 0$ be such that* $\mathrm{GCD}\,(n, m) = 1$. *Then $A[X]/(X^n - f)$ is factorial.*

Proof. If $n = 1$, then $A[X]/(X - f) \cong A$. Therefore, we suppose that $n > 1$ and we denote $A[X]/(X^n - f)$ by B.

First, suppose that f is not a minimal generator of $\mathfrak{m}_A$. By Proposition 22, B is minimally generated by $t_0, \dots, t_r, x$, and $Q' = (nq_0, \dots, nq_r, m)$ is the vector of degrees. Assume that $f = \sum \alpha_j t_0^{w_{0,j}} \cdots t_r^{w_{r,j}}$, where $\alpha_j \in K$. Since f is prime and is not a minimal generator of $\mathfrak{m}_A$, in $\sum \alpha_j t_0^{w_{0,j}} \cdots t_r^{w_{r,j}}$ there is an addendum without t_i, for every $i = 0, \dots, r$. Therefore, m is a multiple of $d_i = \mathrm{GCD}\,(q_0, \dots, q_{i-1}, q_{i+1}, \dots, q_r)$ for every $i = 0, \dots, r$. In order to normalize Q', we consider

$$d_i' = \mathrm{GCD}\,(q_0 n, \dots, q_{i-1} n, q_{i+1} n, \dots, q_r n, m) = \mathrm{GCD}\,(d_i n, m) = d_i$$

for $i = 0, \dots, r$ and

$$d_{r+1}' = \mathrm{GCD}\,(q_0 n, \dots, q_r n) = n.$$

Since $d_i \mid m$ and $\mathrm{GCD}\,(n, m) = 1$, we have $\mathrm{GCD}\,(n, d_i) = 1$. Then

$$a_i' = \mathrm{LCM}\,(d_0, \dots, d_{i-1}, d_{i+1}, \dots, d_r, n) = a_i n,$$

and $a'_{r+1} = \mathrm{LCM}\,(d_0, \dots, d_r, n) = a \cdot n$. Therefore, $\overline{Q'} = ((q_0/a_0), \dots, (q_r/a_r), (m/a))$ and

$$B^{(na)} = K[t_0^{d_0}, \dots, t_r^{d_r}, X^n]/(X^n - f) \cong K[t_0^{d_0}, \dots, t_r^{d_r}] = A^{(a)}.$$

Since A is factorial and $\mathrm{Proj}\,(A) \cong \mathrm{Proj}\,(A^{(a)}) \cong \mathrm{Proj}\,(B^{(na)}) \cong \mathrm{Proj}\,(B)$, we obtain that B is almost factorial. Moreover, by Lemma 25, $A^{(a)}$ is factorial and its grading is normalized, so $B^{(na)}$ has the same properties.

Let $T = \prod_{i=0}^r t_i^{s_i} \in K(A)$ be a homogeneous element of degree one. Let α and β be two integers such that $\alpha m + \beta n = 1$. Then we have $\alpha m + \beta \sum_{i=0}^r (nq_i)s_i = 1$, i.e., $T' = x^\alpha \prod_{i=0}^r t_i^{\beta s_i} \in K(B)$ is a homogeneous element of degree one. But $(t_i) \neq (f)$ for every $i = 0, \dots, r$, so, by Proposition 22, t_i is a homogeneous prime element in B for every i. Moreover x is prime in B since $B/(x) \cong A/(f)$. By Theorem 2 and by Proposition 14, the divisor D on $\mathrm{Proj}\,(B)$ such that $B = A(\mathrm{Proj}\,(B), D)$ and $p^*(D) = \mathrm{div}\,(T')$ is

$$D = \frac{\alpha}{n} V_{r+1} + \sum_{i=0}^r s_i \frac{\beta}{d_i} V_i,$$

where $p^*(V_{r+1}) = n\mathrm{div}\,(x)$ and $p^*(V_i) = d_i \mathrm{div}\,(t_i)$ for every $i = 0, \dots, r$. By Lemma 24, we have the equalities $\deg\,(V_i) = q_i/a_i$ for every $i = 0, \dots, r$, and $\deg\,(V_{r+1}) = m/a$. Since

$$\begin{aligned}\left(\frac{\alpha}{n}\right)\frac{m}{a} + \sum_{i=0}^r \left(s_i \frac{\beta}{d_i}\right)\left(\frac{q_i}{a_i}\right) &= \left(\frac{\alpha}{n}\right)\frac{m}{a} + \beta \sum_{i=0}^r s_i \frac{q_i}{a} = \left(\frac{\alpha}{n}\right)\frac{m}{a} + \frac{\beta}{a} \\ &= \frac{\alpha m + \beta n}{na} = \frac{1}{na} = \frac{1}{n \prod_{i=0}^r d_i},\end{aligned}$$

we conclude that B is factorial using Theorem 8.

Now, assume that f is a minimal generator of $\mathfrak{m}_A$, so let $t_0 = f$. By Proposition 22, B is minimally generated by $t_1, \dots, t_r, x$, and $Q' = (nq_1, \dots, nq_r, q_0)$ is the vector of degrees. We normalize Q'. Since $m = q_0$ and $\mathrm{GCD}\,(n, m) = 1$, we have $\mathrm{GCD}\,(q_1 n, \dots, q_{i-1} n, q_{i+1} n, \dots, q_r n, q_0) = d_i$ for $i = 1, \dots, r$. Moreover, $\mathrm{GCD}\,(q_1 n, \dots, q_r n) = nd_0$.

Since $d_i \mid q_0$ for every $i \neq 0$, and $\mathrm{GCD}\,(n, q_0) = 1$, we have $\mathrm{GCD}\,(n, d_i) = 1$ for every $i = 1, \dots, r$. Therefore, $\mathrm{LCM}\,(d_1, \dots, d_{i-1},$

$d_{i+1}, \ldots, d_r, nd_0) = a_i n$, LCM $(d_1, \ldots, d_r) = a_0$, and LCM $(d_1, \ldots, d_r, d_0 \cdot n) = a \cdot n$. So this implies that $\overline{Q'} = ((q_1/a_1), \ldots, (q_r/a_r), (q_0/a_0))$. By Proposition 12, we have

$$B^{(na)} = K[t_1^{d_1}, \ldots, t_r^{d_r}, X_0^{nd_0}]/(X_0^n - t_0) \cong K[t_1^{d_0}, \ldots, t_r^{d_r}, t_0^{d_0}] = A^{(a)},$$

hence $B^{(na)}$ is factorial and its grading is normalized. Let $T = \prod_{i=0}^r t_i^{s_i}$ be a homogeneous element of degree one in $K(A)$. Let α and β be such that $\alpha q_0 + \beta n = 1$. Then $\alpha q_0 + \beta \sum_{i=0}^r (nq_i)s_i = 1$, i.e., $T' = x^\alpha \prod_{i=0}^r t_i^{\beta s_i}$ is a homogeneous element of degree one in $K(B)$. By Proposition 22, $t_1, \ldots, t_r$ are homogeneous prime elements in B. Moreover $t_0 = x^n$, and x is prime in B, since $B/(x_i) \cong A/(t_0)$. Then

$$\begin{aligned} \operatorname{div}\left(x^\alpha \prod_{i=0}^r t_i^{\beta s_i}\right) &= \alpha \operatorname{div}(x) + s_0\beta \operatorname{div}(t_0) + \sum_{i=1}^r \beta s_i \operatorname{div}(t_i) = \\ &= \alpha \operatorname{div}(x) + s_0\beta \operatorname{div}(x^n) + \sum_{i=1}^r \beta s_i \operatorname{div}(t_i) \\ &= (\alpha + ns_0\beta) \operatorname{div}(x) + \sum_{i=1}^r \beta s_i \operatorname{div}(t_i). \end{aligned}$$

By Theorem 2 and by Proposition 14, the divisor D such that $B = A(X, D)$ and $p^*(D) = \operatorname{div}(T')$ is

$$D = \frac{\alpha + ns_0\beta}{nd_0} V_{r+1} + \sum_{i=1}^r s_i \frac{\beta}{d_i} V_i,$$

where V_{r+1} is such that $p^*(V_{r+1}) = nd_0 \operatorname{div}(x)$, and V_i is such that $p^*(V_i) = d_i \operatorname{div}(t_i)$ for every $i = 1, \ldots, r$. Moreover, by Proposition 14, $\deg(V_i) = q_i/a_i$ for every $i = 1, \ldots, r$, and $\deg(V_{r+1}) = q_0/a_0$. Since

$$\begin{aligned} \frac{\alpha + ns_0\beta}{nd_0}\frac{q_0}{a_0} + \sum_{i=0}^r s_i \frac{\beta}{d_i}\frac{q_i}{a_i} &= \frac{\alpha q_0}{na} + \beta\frac{s_0 q_0}{a} + \beta \sum_{i=1}^r \frac{s_i q_i}{a} \\ &= \frac{\alpha q_0}{na} + \beta \sum_{i=0}^r \frac{s_i q_i}{a} = \alpha \frac{q_0}{n} + \frac{\beta}{a} \\ &= \frac{\alpha q_0 + \beta n}{na} = \frac{1}{na} = \frac{1}{nd_0 \prod_{i=1}^r d_i}, \end{aligned}$$

we conclude that B is factorial by Theorem 8. □

Remark 27. Using Proposition 7.1 of [**13**], it is possible to show that a graded ring is factorial if and only every irreducible homogeneous element is prime. Then Theorem 26 easily follows by Proposition 22.

Lemma 28. *Let A be a domain and a finitely generated* $\mathbf{N}$ *graded K-algebra such that $A_0 = K$. Assume that the grading on A is normalized. Then for each homogeneous prime $f \in A$ the grading on $A/(f)$ is reduced.*

Proof. Let $t_0, \dots, t_r$ be a minimal system of homogeneous generators of A as a K-algebra. Then $\overline{t_0}, \dots, \overline{t_r}$ is a system of generators of $A/(f)$. If $\overline{t_0}, \dots, \overline{t_r}$ is a minimal system of generators of $A/(f)$ as a K-algebra, we have the assumption. Suppose that for some $i \in \{0, \dots, r\}$ we have $\overline{t_i} = \overline{g}$, where $g \in K[t_0, \dots, t_{i-1}, t_{i+1}, \dots, t_r]_{\deg(t_i)}$. Then $t_i - g \in (f)$, and $t_i - g$ is irreducible since t_i is irreducible. Hence $t_i - g$ is equal to cf, where c is invertible. Therefore $t_0, \dots, t_{i-1}, f, t_{i+1}, \dots, t_r$ is a minimal system of homogeneous generators of A. Then $A/(f) \cong K[t_0, \dots, t_{i-1}, t_{i+1}, \dots, t_r]$, and it is minimally generated by $t_0, \dots, t_{i-1}, t_{i+1}, \dots, t_r$. But the grading on A is normalized, so the grading on $A/(f)$ is reduced. $\square$

Let $(X, \mathcal{O}_X)$ be an integral projective normal scheme. Suppose that $\mathrm{Cl}(X) = \mathbf{Z}$ is generated by $[D_0]$. After Remark 7, we can assume that $D_0 \in \mathcal{D}_X$. Then $A(X, D_0)$ is factorial by Corollary 5 and his grading is normalized by Proposition 23. Moreover, for every prime divisor V on X, there exists a homogeneous prime element $f_V \in A(X, D_0)$ such that $V \cong \mathrm{Proj}\,(A(X, D_0)/(f_V))$ and $\deg(V) = \deg(f_V)$.

The next theorem shows that for every $D \in \mathcal{D}_X$ satisfying the hypotheses of Theorem 19, i.e., such that $A(X, D)$ is factorial, we have a complete description of $A(X, D)$ if we know $A(X, D_0)$.

Theorem 29. *Let $(X, \mathcal{O}_X)$ be an integral projective normal scheme. Suppose that $\mathrm{Cl}(X) = \mathbf{Z}$ is generated by $[D_0]$, where $D_0 \in \mathcal{D}_X$. Let $\{t_0, \dots, t_r\}$ be a minimal system of homogeneous generators of $A(X, D_0)$ as a K-algebra, and let $Q = (q_0, \dots, q_r)$, where $q_i = \deg(t_i)$ for $i = 0, \dots, r$. Let $D = \sum_{i=0}^{s} (\eta_i/\delta_i)\ V_i$ be a pairwise coprime divisor in $\mathcal{D}_X$ such that $\sum_{i=0}^{s} (\eta_i \cdot \deg(V_i))/\delta_i = 1/\prod_{i=0}^{s} \delta_i$. Let $\delta =$*

$\prod_{i=0}^{s} \delta_i = \mathrm{LCM}\,(\delta_1, \dots, \delta_s)$, *and denote by* $f_0, \dots, f_s$ *the homogeneous elements in* $A(X, D_0)$ *corresponding to* $V_0, \dots, V_s$*. Then there exists an equivariant isomorphism*

$$A(X, D) \cong K[t_0, \dots, t_r][X_0, \dots X_s]/(X_0^{\delta_0} - f_0, \dots, X_s^{\delta_s} - f_s)$$

such that $\deg(t_i) = \delta \cdot q_i$ *and* $\deg(X_j) = \deg(f_j) \cdot \delta/\delta_j$ *for every* $i = 0, \dots, r$ *and for every* $j = 0, \dots, s$.

Proof. We denote by A the ring $A(X, D_0)$ and by A' the quotient ring $K[t_0, \dots, t_r][X_0, \dots X_s]/(X_0^{\delta_0} - f_0, \dots, X_s^{\delta_s} - f_s)$ graded by $\deg(t_i) = \delta \cdot q_i$ for every $i = 0, \dots, r$, and $\deg\ (X_j) = \deg(f_j) \cdot \delta/\delta_j$ for every $j = 0, \dots, s$. We want to prove that $\mathrm{Proj}\,(A') \cong X$, and $A(X, D) = A'$. First, we determine a minimal system of homogeneous generators of A' as a K-algebra, using recursively Proposition 22. We observe that $\sum_{i=0}^{s} (\eta_i \cdot \deg(f_i) \cdot \delta/\delta_i) = 1$ and $\delta_i \mid (\delta/\delta_j)$ if $i \neq j$, so:

I) $\mathrm{GCD}\,(\deg(f_0) \cdot \delta/\delta_1, \dots, \deg(f_s) \cdot \delta/\delta_s) = 1$,

II) $\mathrm{GCD}\,(\delta_i, \delta/\delta_i) = 1$,

III) $\mathrm{GCD}\,(\delta_i, \deg(f_i)) = 1$.

Now, for every $i = 0, \dots, s$, consider

$$B_i = K[t_0, \dots, t_r][X_0, \dots, X_i]/(X_0^{\delta_0} - f_0, \dots, X_i^{\delta_i} - f_i),$$

where $\deg\ (t_h) = \delta_0 \cdots \delta_i \cdot q_h$ for $h = 0, \dots, r$, and $\deg(X_j) = \deg(f_j) \cdot \delta_0 \cdots \delta_i/\delta_j$ for $j = 0, \dots, i$.

Since δ_0 and $\deg(f_0)$ are coprime, Theorem 26 implies that B_0 is factorial, and that f_i is a prime element in B_0 of degree $\delta_0 \deg(f_i)$, for $i = 1, \dots, s$. Moreover, x_0 is prime in B_0 where x_0 is the residue class of X_0 in B_0. Suppose that B_i is factorial and that $x_0, \dots, x_i, f_{i+1}, \dots, f_s$ are homogeneous prime elements in B_i. Since $\mathrm{GCD}\,(\delta_{i+1}, \deg(f_{i+1})\delta_0 \cdots \delta_i) = 1$, Theorem 26 implies that B_{i+1} is factorial and that $x_0, \dots, x_i, f_{i+2}, \dots, f_s, x_{i+1}$ are a prime element in B_{i+1}. In particular, $A' = \mathcal{B}_s$, hence it is factorial, and $x_0, \dots, x_s$ are prime in A'.

Now we look for a minimal system of homogeneous generators of A'. Suppose that $\delta_{l+1} = \cdots = \delta_s = 1$ where $-1 \le l \le s$, and $\delta_i > 1$ for every $0 \le i \le l$. Then $A' = K[t_0, \dots, t_r, x_0, \dots, x_l]$. Moreover, assume that $\{f_0, \dots, f_i, t_{i+1}, \dots, t_r\}$ is a minimal set of homogeneous generators of A for some $-1 \leq i \leq l$, and that $f_h \in (f_0, \dots, f_i) + \mathfrak{m}_A^2$, for every $i < h \leq l$. We can choose $t_0 = f_0, \dots, t_i = f_i$, and then $\{x_0, \dots, x_i, t_{i+1}, \dots, t_r\}$ is a minimal system of generators of B_i by Proposition 22. Moreover, f_{h+1} is not minimal in B_h for every $h \geq i$, since $f_{h+1} \in \mathfrak{m}_A^2 + (f_0, \dots, f_i) = \mathfrak{m}_{B_h}^2 \cap A$. Then, using Proposition 22 recursively, we conclude that $\{x_0, \dots, x_i, t_{i+1}, \dots, t_r, x_{i+1}, \dots, x_l\}$ is a minimal system of generators of A'.

Since $T' = x_0^{\eta_0} \cdots x_s^{\eta_s}$ is an element of degree one in $K(A')$, the grading on A' is reduced. Let

$$\begin{aligned} Q' &= (q'_0, \dots, q'_i, q'_{i+1}, \dots, q'_r, q'_{r+1}, \dots, q'_{r+l-i}) \\ &= \left(q_0 \cdot \frac{\delta}{\delta_0}, \dots, q_i \cdot \frac{\delta}{\delta_i}, \delta \cdot q_{i+1}, \dots, \delta \cdot q_r, \deg(f_{i+1}) \cdot \frac{\delta}{\delta_{i+1}}, \dots, \right. \\ &\qquad \left. \deg(f_l) \cdot \frac{\delta}{\delta_l} \right). \end{aligned}$$

We normalize Q'. Let $d'_h = \operatorname{GCD}(q'_0, \dots, q'_{h-1}, q'_{h+1}, \dots q'_{r+l-i})$. We observe that $d'_h = 1$ for every $h = i+1, \dots, r$. In fact, if p is prime and $p \mid d'_h$, then $p \mid \delta \cdot q_j$ for every $j = 0, \dots, r$, $j \neq h$. But Q is normalized, so $p \mid \delta$, i.e., there exists $k \in \{0, \dots, l\}$ such that $p \mid \delta_k$. Moreover, p divides $\deg(f_k) \cdot \delta/\delta_k$ which is coprime with δ_k, hence $p = 1$. Now we prove that $d'_{r-i+h} = \delta_h$ for every $h = i+1, \dots, l$. Let $h \in \{i+1, \dots, l\}$. Then $\delta_h \mid d'_{r-i+h}$, since $\delta_h \mid (\delta/\delta_h)$ if $i \neq h$. Let p be a prime factor of d'_{r-i+h}. Since Q is normalized, $p \mid \delta$. Then $p \mid \delta_j$ for some j. If $j \neq h$, then $p \mid (\deg(f_j) \cdot \delta/\delta_j)$. By II) and III), $\deg(f_j) \cdot \delta/\delta_j$ is coprime with δ_j, so $p \mid \delta_h$. In the same way we prove that $d'_h = \delta_h$ for $h \in \{0, \dots, i\}$. Therefore,

$$\begin{aligned} &(d'_0, \dots, d'_i, d'_{i+1}, \dots, d'_r, d'_{r+1}, \dots, d'_{r+l-i}) \\ &\qquad = (\delta_0, \dots, \delta_i, 1, \dots, 1, \delta_{i+1}, \dots, \delta_l). \end{aligned}$$

Let $a'_h = \operatorname{LCM}(d'_0, \dots, d'_{h-1}, d'_{h+1}, \dots, d'_{r+l-i})$ for $h = 0, \dots, r+l-i$. Then

$$\begin{aligned} &(a'_0, \dots, a'_i, a'_{i+1}, \dots, a'_r, a'_{r+1}, \dots, a'_{r+l-i}) \\ &\qquad = \left(\frac{\delta}{\delta_0}, \dots, \frac{\delta}{\delta_i}, \delta, \dots, \delta, \frac{\delta}{\delta_{i+1}}, \dots, \frac{\delta}{\delta_l} \right). \end{aligned}$$

Therefore, we have that $\overline{Q'} = (q_0, \dots, q_i, q_{i+1}, \dots, q_r, \deg(f_{i+1}), \dots, \deg(f_l))$. Since $a' = \mathrm{LCM}(d'_0, \dots, d'_{r+l-i}) = \delta$, and

$$\begin{aligned} A'^{(\delta)} &= K[x_0^{\delta_0}, \dots, x_i^{\delta_i}, t_{i+1}, \dots, t_r, x_{i+1}^{\delta_{i+1}}, \dots, x_l^{\delta_l}] \\ &= K[t_0, \dots, t_i, t_{i+1}, \dots, t_r, f_{i+1}, \dots, f_l] = A, \end{aligned}$$

we conclude that $\mathrm{Proj}(A') \cong X$.

Let now $D' \in \mathcal{D}_X$ be the unique divisor such that $A(X, D') = A'$ and $p^*(D') = \mathrm{div}(T')$. On $\mathrm{Spec}(A') - \{\mathfrak{m}\}$ we have $\mathrm{div}(T') = \sum_{j=0}^{s} \eta_j \,\mathrm{div}(x_j)$, and $p(\mathrm{div}(x_j)) = V_j$, since

$$\begin{aligned} &\mathrm{Proj}(A'/(x_j)) \\ &\quad\cong \mathrm{Proj}((A'/(x_j))^{(\delta)}) \\ &\quad= \mathrm{Proj}(K[x_0^{\delta_0}, \dots, x_i^{\delta_i}, t_{i+1}, \dots, t_r, x_{i+1}^{\delta_{i+1}}, \dots, x_l^{\delta_l}]/(x_j^{\delta_j})) \\ &\quad= \mathrm{Proj}(K[t_0, \dots, t_i, t_{i+1}, \dots, t_r, f_{i+1}, \dots, f_l]/(f_j)) \\ &\quad= \mathrm{Proj}(K[t_0, \dots, t_i, t_{i+1}, \dots, t_r]/(f_j)) \\ &\quad= \mathrm{Proj}(A/(f_j)) = V_j. \end{aligned}$$

Next, we determine the integers $e_{\mathrm{div}(x_j)|V_j}$ such that $p^*(V_j) = e_{\mathrm{div}(x_j)|V_j} \mathrm{div}(x_j)$ for every $j = 0, \dots, s$. Proposition 14 implies that $e_{\mathrm{div}(x_j)|V_j} = \delta_j$ for every $j = 0, \dots, l$. Let us prove that $e_{\mathrm{div}(x_j)|V_j} = 1$ for every $j > l$. By Proposition 13, $e_{\mathrm{div}(x_j)|V_j} = \mathrm{GCD}\{n \in \mathbf{N}^* : (A'/(x_j))_n \neq (0)\}$. By Lemma 28, the claim is that there exists a homogeneous element of degree one in the field of fraction of $A'/(x_j) = A'/(f_j) = A/(f_j)[x_0, \dots, x_l]$. By Lemma 28, there exists an element of degree one in $K(A/(f_j))$, and its degree is δ in $K(A'/(f_j))$. Moreover, in $K(A'/(f_j))$, $\deg(x_i) = \deg(f_i)\delta/\delta_i$ for every $i = 0, \dots, l$. Since $\mathrm{GCD}(\delta, \deg(f_1)\delta/\delta_1, \dots, \deg(f_l)\delta/\delta_l) = 1$, there exists an element of degree one in $K(A'/(f_j))$. Therefore we have $e_{\mathrm{div}(x_j)|V_j} = \delta_j = 1$ for every $j > l$. By Theorem 2 and Proposition 14, we conclude that $D' = \sum_{j=0}^{s} \eta_j/\delta_j \, V_j = D$. □

4. Factorization in a graded quotient ring. The aim of this section is to understand how the task of factorizing an element of a UFD, given as a quotient R/I, can be achieved by simply calculating inside the ring R.

Classical examples of varieties whose coordinate ring is a graded factorial domain include generic surfaces of $\mathbf{P}_K^3$ with order $m \geq 4$, nonsingular quadrics of $\mathbf{P}_K^n$, $n \geq 4$, Grassmannians, nonsingular complete intersections with dimension ≥ 3 in $\mathbf{P}_{\mathbf{C}}^n$, where $n \geq 4$. Moreover, as we have seen, it is possible to construct factorial domains taking $A(X, D)$ where X is an integral, normal, projective scheme defined over a field K whose divisor class group is $\mathrm{Cl}(X) = \mathbf{Z}$, and D is a well-defined Weil divisor with rational coefficients.

Since all these examples are finitely generated K-algebras, it is natural to ask for a method to compute the factorization of an element in a factorial quotient ring R/I where $R = K[X_1, \dots, X_n]$ is a polynomial ring over a field K, and I is a homogeneous ideal of R to respect to a positive graduation of R.

In order to have the factorization (Proposition 39), we need an algorithm to compute the greatest common divisor of two elements in a quotient ring. First, we show that we can assume the elements are homogeneous.

Let K be a field. Let I be a homogeneous ideal in $R = K[X_1, \dots, X_n]$ where $\deg(X_i) = q_i > 0$ for every $i = 1, \dots, r$. Assume that R/I is a unique factorization domain, and let W be a new indeterminate. Then the ring $R[W]/IR[W] \cong R/I[W]$ is a UFD, and it admits an induced positive grading with $\deg(W) = 1$. Moreover, $(R/I)_0 = (R[W]/IR[W])_0 = K$, and we recall that in a positively graded domain the degree of an invertible element is zero. In the following, by S we denote $R[W]$, by IS the extension of the ideal I in S, and by w the residue class of W in S/IS.

Definition 30. Let $f \in R/I$. Let $f = f_0 + \cdots + f_{\deg(f)}$ where $f_i \in (R/I)_i$ for every $i = 0, \dots, \deg(f)$. Then ${}^h f = w^{\deg(f)} f_0 + w^{\deg(f)-1} f_1 + \cdots + f_{\deg(f)}$ is a homogeneous element in S/IS of degree $\deg(f)$, and it is called the *homogenization* of f.

Definition 31. Let $g \in S/IS$. Then ${}^d g = g|_{w=1} \in R/I$ is called the *dehomogenization* of g.

Remark 32. Given $f \in R/I$, let $F \in R$ be such that f is the residue class of F in R/I. Then $\overline{{}^hF} = {}^hf \cdot w^{\deg(F)-\deg(f)}$ where ${}^hF$ is the usual homogenization of a polynomial in R.

Given $g \in S/IS$, let $G \in S$ be such that g is the residue class of G in S/I. Then $\overline{{}^dG} = {}^dg$ where ${}^dG$ is the usual dehomogenization.

The next lemma is easy to prove, so we omit the proof.

Lemma 33. *The following statements hold*:

(1) *Let* $f', f'' \in R/I$. *Then* ${}^h(f'f'') = {}^hf'\,{}^hf''$.

(2) *Let* $g', g'' \in S/IS$. *Then* ${}^d(g'g'') = {}^dg'\,{}^dg''$.

(3) *Let* $f \in R/I$. *Then* ${}^d({}^hf) = f$.

(4) *Let* $g \in S/IS$ *be such that* w *does not divide* g. *Then* ${}^h({}^dg) = g$.

Remark 34. Let g be a homogeneous element in S/IS. Then ${}^dg \in K$ if and only if $g = kw^{\deg(g)}$ for some $k \in K$. Indeed, let $g = a_0 + a_1w + \cdots + a_sw^s$ where $a_i \in (R/I)_{\deg(g)-i}$, and $a_s \neq 0$. Then ${}^dg = a_0 + a_1 + \cdots + a_s \in K$ if and only if $s = \deg(g)$, and $a_0 = \cdots = a_{s-1} = 0$.

Proposition 35. *If* $f \in R/I$ *is an irreducible element, then* ${}^hf$ *is irreducible in* S/IS. *Conversely, if* $g \in S/IS$ *is irreducible, then* ${}^dg$ *is irreducible in* R/I.

Proof. Suppose f irreducible and ${}^hf = g_1 \cdot g_2$ where g_1 and g_2 are homogeneous elements in S/IS. Then $f = {}^dg_1 \cdot {}^dg_2$, so either ${}^dg_1 \in K$ or ${}^dg_2 \in K$. Assume that ${}^dg_1 \in K$. By Remark 34, we have $g_1 = kw^{\deg(g_1)}$ for some $k \in K$. Since w does not divide ${}^hf$, then $\deg(g_1) = 0$, so ${}^hf$ is irreducible.

Conversely, suppose that ${}^dg = f_1 \cdot f_2$. Since g is irreducible, w cannot divide g, so ${}^h({}^dg) = g = {}^hf_1 \cdot {}^hf_2$. Then either ${}^hf_1 \in K$ or ${}^hf_2 \in K$, so either f_1 or f_2 is in K. □

Proposition 36. *Let* $f, g \in R/I$.

(1) $\mathrm{GCD}(f, g) = {}^d\mathrm{GCD}({}^hf, {}^hg)$.

(2) *Let* $F, G \in R$ *be such that* $f = \overline{F}$ *and* $g = \overline{G}$. *Then* $\mathrm{GCD}\,(f,g) = {}^d\mathrm{GCD}\,(\overline{{}^hF}, \overline{{}^hG})$.

Proof. (1) If either $f = 0$ or $g = 0$, the conclusion follows. So, assume that $f \neq 0$, $g \neq 0$, and let $q = \mathrm{GCD}\,(f,g)$. Then we have to prove that ${}^hq = \mathrm{GCD}\,({}^hf,\, {}^hg)$. Since $q \mid f$ and $q \mid g$, then ${}^hq \mid {}^hf$ and ${}^hq \mid {}^hg$. Let $p \in S/IS$ be such that $p \mid {}^hf$ and $p \mid {}^hg$. Therefore, ${}^dp$ divides f and g, so ${}^dp \mid q$. Moreover w cannot divide p, hence $p = {}^h({}^dp)$. Then we conclude that $p =\mid {}^hq$, i.e., ${}^hq = \mathrm{GCD}\,({}^hf,\, {}^hg)$.

(2) follows by $\mathrm{GCD}\,(f,g) = {}^d\mathrm{GCD}\,({}^hf,\, {}^hg) = {}^d\mathrm{GCD}\,({}^hf w^{\deg(F)-\deg(f)}, {}^hg w^{\deg(G)-\deg(g)}) = {}^d\mathrm{GCD}\,(\overline{{}^hF}, \overline{{}^hG})$ □

Therefore, if we want to calculate $\mathrm{GCD}\,(f,g)$ where $f, g \in R/I$, we may assume that f and g are homogeneous elements.

Proposition 37. *Let* f, g *be homogeneous elements in* R/I. *Let* f_1, g_1 *be such that* $f = f_1\mathrm{GCD}\,(f,g)$ *and* $g = g_1\mathrm{GCD}\,(f,g)$. *Let* F, G *be homogeneous elements in* R *such that* f, g *are the residue class of* F *and* G, *respectively, in* R/I. *Let* $I_1, \dots, I_r \in R$ *be a minimal system of homogeneous generators of* I. *Let* $\mathbf{V_1}, \dots, \mathbf{V_s}$ *be a minimal system of homogeneous generators of* $\mathrm{Syz}\,(F, -G, I_1, \dots, I_r)$. *Let* $m \in \{1, \dots, s\}$ *be such that* $\mathbf{V_m}$ *has minimum degree between the vectors whose first component is not in* I. *Then* $\mathbf{V_m} = k(G_1, F_1, J_1, \dots, J_r)$ *where* $\overline{G_1} = g_1$, $\overline{F_1} = f_1$, *and* $k \in K^*$.

Proof. Let $\mathbf{V_h} = (V_{1,h}, \dots, V_{r+2,h}) \in R^{r+2}$ for every $h = 1, \dots, s$, and let $v_{l,h}$ be the residue class of $V_{l,h}$ in R/I. Let $\mathbf{v_h} = (v_{1,h}, v_{2,h})$ for every $h = 1, \dots, s$. Then, $v_1, \dots, v_s$ is a minimal system of homogeneous generators of $\mathrm{Syz}\,(f, -g)$. Since R/I is a unique factorization domain, $\mathrm{Syz}\,(f, -g)$ is minimally generated by (g_1, f_1). Therefore, $\mathbf{v_l} = k(g_1, f_1)$ for some $l \in \{1, \dots, s\}$, and for some $k \in K^*$. This implies that $V_{1,l} = kG_1$ and $V_{2,l} = kF_1$ where $\overline{F_1} = f_1$ and $\overline{G_1} = g_1$. We prove that $l = m$. Since $\mathbf{v_h} = t\mathbf{v_l}$ for every $h = 1, \dots, s$ and for some $t \in R/I$, $V_{1,h} - TV_{1,l} \in I$ where $\overline{T} = t$. If $T \in I$, then $V_{1,h} \in I$; if $T \notin I$, then $\deg(V_{1,h}) \geq \deg(V_{1,l})$. Therefore, $l = m$. □

Corollary 38. *Let a, f, g be homogeneous elements in R/I for which $a = f/g$. Let F, G be homogeneous elements in R such that $f = \overline{F}$ and $g = \overline{G}$. Let $I_1, \ldots, I_r \in R$ be a minimal system of homogeneous generators of I. Finally, let $\mathbf{V_1}, \ldots, \mathbf{V_s}$ be a minimal system of homogeneous generators of* $\operatorname{Syz}(F, -G, I_1, \ldots, I_r)$. *Then there exists an index $h \in \{1, \ldots, s\}$ such that $\mathbf{V_h} = (k, kA, J_1, \ldots, J_r)$ where $k \in K^*$ and $\overline{A} = a$.*

Algorithm DivQuotient

input: F, G two polynomials, $I = (I_1, \ldots, I_r)$ an ideal, W a new variable

output: either a polynomial H such that the residue class of F is equal to the residue class of the product of H and G in R/I, or the string "error"

$F1 = \text{Homogenized}(W, F)$;
$G1 = \text{Homogenized}(W, G)$;
$[V_1, \ldots, V_s] = \text{Syzygies}([F1, -G1, I_1, \ldots, I_r])$;
$\mathit{UsefulSyz} = [V_j : V_j[1]$ **IsNot** 0 **And**
$\quad \mathit{Deg}(V_j[1]) = 0$ **For** $j = 1, \ldots, s]$;
If Length ($\mathit{UsefulSyz}$) **IsNot** 0 **Then**
$\quad VH = \mathit{UsefulSyz}[1]$;
$\quad H = \text{Dehomogenized}(W, VH[2]/VH[1])$;
$\quad$ **Return** H;
Else Return "error";
EndIf;
End

Algorithm GCDQuotient

input: F, G two polynomials, $I = (I_1, \ldots, I_r)$ an ideal, W a new variable

output: a polynomial H such that the residue class of H is equal to the GCD of the residue class of F and G in R/I

$F1 = \text{Homogenized}\,(W, F)$;
$G1 = \text{Homogenized}\,(W, G)$;
$[V_1, \ldots, V_s] = \text{Syzygies}\,([F1, -G1, I_1, \ldots, I_r])$;
$\mathit{FirstCompOfV} = [\text{NormalForm}\,(V_j[1], I) :$
$j = 1, \ldots, s]$;
$\mathit{CandG1overH} = [L$ **In** $\mathit{FirstCompOfV} : L$ **IsNot** $0]$;
$\mathit{MinDeg} = \text{Min}\,([\mathit{Deg}(M) : M$ **In** $\mathit{CandG1overH}])$;
$\mathit{G1overH} = [L$ **In** $\mathit{CandG1overH} : \text{Deg}\,(L) = M]$;
$H = \text{DivQuotient}\,(G1, \mathit{G1overH}[1], I, W)$;
Return Dehomogeneized (W, H);
End

By the uniqueness of the minimal primary decomposition we have the following proposition.

Proposition 39. *Let R be a Noetherian ring, and let I be an ideal in R such that R/I is a* UFD. *Let $f \in R/I$, and let $f = p_1^{n_1} \cdots p_r^{n_r}$ be the unique factorization of f in R/I. Let $F \in R$ be such that f is the residue class of F in R/I, and let $\mathfrak{q}_1 \cap \cdots \cap \mathfrak{q}_s$ be a minimal primary decomposition of $(F) + I$ in R. Then $r = s$ and, after possibly reindexing, $\mathfrak{q}_i/I = (p_i^{n_i})$ for each $i = 1, \ldots, r$.*

In particular if $\sqrt{\mathfrak{q}_i} = \mathfrak{p}_i = (P_{i,1}, \ldots, P_{i,r_i})$, for every $i = 1, \ldots, r$, and if $P_i \in R$ is such that $\overline{P_i} = \text{GCD}\,(\overline{P_{i,1}}, \ldots, \overline{P_{i,r_i}})$, then $p_i = \overline{P_i}$ for every $i = 1, \ldots, r$.

Proof. Clearly $(p_1^{n_1}) \cap \cdots \cap (p_r^{n_r})$ and $\mathfrak{q}_1/I \cap \cdots \cap \mathfrak{q}_s/I$ are minimal primary decompositions of (f). Then $r = s$ and, after possibly reindexing, $\mathfrak{q}_i/I = (p_i^{n_i})$ for each $i = 1, \ldots, r$. □

Algorithm FactorQuotient

input: F a polynomial, I an ideal, W a new variable

output: a list $[[p_1, n_1], \ldots, [p_r, n_r]]$ of the irreducible factors of the residue class of F with their respective multiplicities

```
  [P_1, ..., P_r] = MinimalPrimes (I + (F));
  For i = 1 To r Do
       n_i = 1;
       [(P_i)_1, ..., (P_i)_t] = Generators (P_i);
       p_i = (P_i)_1;
       For k = 2 To t Do
              p_i = GCDQuotient ( p_i, (P_i)_k, I, W);
       EndFor;
       While DivQuotient(F, p_i, I, W) IsNot "error" Do
              n_i = n_i + 1;
              F = DivQuotient(F, p_i, I, W);
       EndWhile;
       ListFactorization [i] = [p_i, n_i];
  EndFor;
  Return ListFactorization;
End
```

4.1. A special case. In this section we give an alternative algorithm to compute the factorization of an element in R/I in the case where I is a principal ideal. Then we suppose that $I = (F)$ is a homogeneous ideal in $R = K[X_1, \dots, X_n]$ where $\deg(X_i) = q_i > 0$ for every $i = 1, \dots, r$. Assume that $R/(F)$ is a UFD. Let $\mathrm{LT}_{X_1}(F)$ be the leading coefficient of F to respect the variable X_1, and assume that $\mathrm{LT}_{X_1}(F) \in K$.

Proposition 40. *Let $G \in R$. If $\overline{G}$ is irreducible in $R/(F)$, then the resultant of F and G with respect to X_1 is the power of an irreducible polynomial in $K[X_2, \dots, X_n]$.*

Proof. Assume $\mathrm{Resultant}(G, F, X_1) = C \cdot D$, where $\mathrm{GCD}(C, D) = 1$. We observe that $R/(F)$ is an algebraic extension of $K[X_2, \dots, X_n]$. Let $X_{1,1}, X_{1,p}$ be the distinct roots of F, and let $G_i = G(X_{i,1})$. It is well known that $\mathrm{Resultant}(G, F, X_1) = \mathrm{Norm}(G) = \prod_i G_i$. Then $\overline{G} = G_1$ divides the residue class of $\mathrm{Resultant}(G, F, X_1)$ in $R/(F)$; therefore, either $\overline{G} \mid \overline{C}$ or $\overline{G} \mid \overline{D}$. Assume for concreteness that $\overline{G} \mid \overline{C}$, i.e., $\overline{C} = G_1 \cdot H_1$. The rings $K[X_2, \dots, X_n][X_{1,1}]$ and $K[X_2, \dots, X_n][X_{1,j}]$ are canonically isomorphic under a mapping ϕ_j which sends $X_{1,1}$ to $X_{1,j}$, and it is the identity on $K[X_2, \dots, X_n]$. So $\overline{C}$ is invariant under ϕ_j, and the equation $\overline{C} = G_1 \cdot H_1$ becomes

$\overline{C} = G_j \cdot H_j$. Therefore, $G_j \mid \overline{C}$ for every j. But GCD $(C, D) = 1$ implies that GCD $(G_j, \overline{D}) = 1$ for all j, so GCD $(\prod G_j, \overline{D}) = 1$. But $D \mid$ Resultant $(F, G, X_1) =$ Norm $(\overline{G}) = \prod G_j$. So we have $D = 1$. ◻

Algorithm FactorPrincQuotient

input: F a polynomial, $\mathcal{I}$ a polynomial, W a new variable

output: a list $[[p_1, n_1], \ldots, [p_r, n_r]]$ of the irreducible factors of the residue class of F with their respective multiplicities

```
[[F_1, m_1], ..., [F_r, m_r]] = Factorize (Resultant (F, I, X_1));
For i = 1 To r Do
      p_i = GCDQuotient ( F, F_i, (I), W);
      If Resultant (p_i, F, X_1) Is Squarefree Then
                n_i = 0;
                p_i = GCDQuotient (F, F_i, (I), W);
                F = DivQuotient (F, p_i, (I), W);
                While F IsNot 'error' Do
                      n_i = n_i + 1;
                      F = DivQuotient(F, p_i, (I), W);
                EndWhile;
      Endif;
      ListFactor [i] = [p_i, n_i];
EndFor;
ListFactor = [ListFactor,FactorQuotient (F, Ideal(I), W)];
Return ListFactor;
End
```

Theorem 41. *Let $G \in R$ be such that* Resultant (F, G, X_1) *is square free, and let $\prod G_i(X_2, \ldots, X_n)$ be a complete factorization of* Resultant (F, G, X_1) *in $K[X_2, \ldots, X_n]$. Then $\prod(\text{GCD}\,(\overline{G}, \overline{G_i}))$ is a complete factorization of $\overline{G}$ in $R/(F)$.*

Proof. Let $g_1 =$ GCD $(\overline{G}, \overline{G_i})$. Then we must show that each g_i is irreducible and that all the irreducible factors of $g = \overline{G}$ are among the g_i. Let $v = \overline{V}$ be an irreducible factor of g. Then Resultant (V, F, X_1) is the power of an irreducible polynomial in $K[X_2, \ldots, X_n]$, but $v \mid g$, so Norm $(v) \mid$ Norm (g) and Norm (g) is square free. Therefore the

Norm (v) is irreducible and must be one of the G_i. Since the Norm (g) is equal to the product of the norms of each of the irreducible factors of g, each G_i must be the norm of some irreducible factor of g.

Then assume that both v_1 and v_2 divide GCD $(g, \overline{G_i})$, where v_1 and v_2 are distinct irreducible factors of g. Since $v_1 \mid \overline{G_i}$, we have Norm $(v_1) \mid$ Norm (G_i), but $G_i \in K[X_2, \dots, X_n]$, so Norm $(G_i) = (G_i)^p$. The norm of v_1 is irreducible in $K[X_2, \dots, X_n]$ and divides the irreducible polynomial G_i, so Norm $(v_1) = G_i$. Similarly, Norm $(v_2) = G_i$. But $v_1 v_2 | g$, so Norm $(v_1 v_2) = G_i^2 \mid$ Norm (g), and this contradicts the assumption that Norm (g) is square free. Therefore, GCD (g, G_i) must be irreducible for all i. $\square$

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