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FOCK SPACE TECHNIQUES IN TENSOR ALGEBRAS OF DIRECTED GRAPHS

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ABSTRACT. In [**16**], Muhly and Solel developed a theory of tensor algebras over C^*-correspondences that extends the model theory of contractions in $B(H)$. The main examples are generated by Fock spaces, directed graphs and analytic cross products. In this paper we show that many results of tensor algebras of directed graphs, including dilations and commutant lifting theorems for $C_{\cdot 0}$ completely contractive representations, can be deduced from results on Fock spaces. One of the main tools we use is that of Poisson kernels, which we define for arbitrary C^*-correspondences. The Fock space approach allows us to consider "weighted" graphs, where the dilation and commutant lifting theorems hold. Additionally, we prove a rigidity result for submodules of induced representations of directed graphs, and we obtain projective resolutions of graph deformations.

1. Introduction. In the last 30 years there have been many attempts to generalize model theory for contractions in $B(H)$, particularly the Nagy-Foias dilation theory and the commutant lifting theorem. For example, Douglas and Paulsen [**7**] proposed Hilbert module language to extend these results to multivariate function theory. Popescu [**21, 22**] extended them to a noncommutative multivariate setting. And Muhly and Solel [**16, 17**] extended them to tensor algebras over C^*-correspondences (Hilbert bimodules over a C^*-algebra A). The language of Muhly and Solel is very general and it includes as special cases all of the previous examples. Additionally, it includes the tensor algebras generated by directed graphs, analytic cross products, and others.

Many aspects of the Nagy-Foias theory are reduced to study of the unilateral shift in the Hardy space H^2, which has orthonormal basis $\{z^n : n \geq 0\}$. Likewise, many aspects of Popescu's noncommutative

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theory are reduced to the study of left creation operators on the full Fock space, which are isometries with orthogonal ranges. The full Fock space of l_2^N is

$$\mathcal{F}\left(l_2^N\right) = \mathbf{C}1 \oplus l_2^N \oplus \left(l_2^N\right)^{\otimes 2} \oplus \left(l_2^N\right)^{\otimes 3} \oplus \left(l_2^N\right)^{\otimes 4} \oplus \cdots,$$

but we identify it with $l_2(\mathbf{F}_N^+)$, the Hilbert space with orthonormal basis $\{\delta_\alpha : \alpha \in \mathbf{F}_N^+\}$ indexed by the free semi-group on N-generators $g_1, g_2, \dots, g_N$. The left creation operators $L_i : l_2(\mathbf{F}_N^+) \to l_2(\mathbf{F}_N^+)$ are defined by $L_i(\delta_\alpha) = \delta_{g_i\alpha}$, for $i \leq N$.

One interesting feature of Popescu's approach is that many results, including dilation and commutant lifting theorems, pass from $l_2(\mathbf{F}_N^+)$ with $L_1, \dots, L_N$, to subspaces $\mathcal{M}$ that are invariant under adjoints of the left creation operators. We call these spaces $*$-invariant. For example, compression of the left creation operators to the symmetric Fock space gives rise to Arveson's d-shifts, and the dilation result of [**4**, Theorem 4.5] follows immediately. In this paper we will show that an induced representation of a tensor algebra of a directed graph is associated with a subspace of $l_2(\mathbf{F}_N^+) \otimes H$ invariant under $(L_1 \otimes I_H)^*, (L_2 \otimes I_H)^*, \dots, (L_N \otimes I_H)^*$. This allows us to apply Fock space techniques to these algebras.

Non self-adjoint algebras of directed graphs have been studied recently by Muhly and Solel [**17, 18**], Kribs and Power[**13**], Jury and Kribs [**9, 10**] and Katsoulis and Kribs [**11, 12**]. Kribs and Power [**13**]] introduced a generalized Fock space to study these algebras, but we will use the C^*-correspondence language of Muhly and Solel [**16, 17**]. The full Fock space techniques seem to adapt better to this language.

In Section 2 we review background material and define the Poisson kernels for C^*-correspondences. These are explicit dilations that are modeled after Popescu's Poisson kernel of the full Fock space.

In Section 3 we study non self-adjoint algebras generated by finite directed graphs. We assume that the directed graph G has N edges $g_1, g_2, \dots, g_N$ and n vertices. The formal span of the edges is the Cuntz-Krieger bimodule $E = X(G)$ over l_∞^n. And the Fock module

$$F\left(E\right) = l_\infty^n \oplus E \oplus E^{\otimes 2} \oplus E^{\otimes 3} \oplus E^{\otimes 4} \oplus \cdots$$

is the orthogonal sum of the $E^{\otimes k}$s, where $E^{\otimes k}$ is spanned by elements of the form $g_{i_1} \otimes g_{i_2} \otimes \cdots \otimes g_{i_k}$ such that $g_{i_1} g_{i_2} \cdots g_{i_k}$ is an allowable path in G. Each edge generates a left creation operator $L(g_i) : F(E) \to F(E)$ defined by

$$L(g_i)(g_{i_1} \otimes \cdots \otimes g_{i_k}) = \begin{cases} g_i \otimes g_{i_1} \otimes \cdots \otimes g_{i_k} & \text{if } g_i g_{i_1} g_{i_2} \cdots g_{i_k} \text{ is an allowable path in } G \\ 0 & \text{otherwise.} \end{cases}$$

The tensor algebra generated by G is the norm closure of the algebra generated by $L(g_1, \dots, L(g_N)$ and n orthogonal projections $Q_1, \dots, Q_n$ that add up to the identity and that satisfy the relation $L(g_i) = Q_{r(i)} L(g_i) Q_{s(i)}$. Function $r(i)$ is the range of and $s(i)$ is the source of the ith edge. Notice that we multiply from right to left.

A completely contractive covariant representation of $E = X(G)$ on $B(H)$ consists of N bounded linear operators $T_1, T_2, \dots, T_N$ in $B(H)$ that satisfy $\sum_{i=1}^N T_i T_i^* \leq I$ and n nonzero orthogonal projections $P_1, P_2, \dots, P_n$ in $B(H)$ that add up to the identity and that satisfy

$$T_i = P_{r(i)} T_i P_{s(i)}. \tag{1.1}$$

More abstractly, a completely contractive covariant representation of $E = X(G)$ on $B(H)$ consists of an ordered pair (T, σ) such that $T : E \to B(H)$ is completely contractive, $\sigma : l_\infty^n \to B(H)$ a faithful $*$-representation, and T and σ satisfy an algebraic condition that is equivalent to (1.1). Moreover, the faithful $*$-representation $\sigma : l_\infty^n \to B(H)$ induces the Hilbert space $F(E) \otimes_\sigma H$ and an isometric covariant representation of $E = X(G)$ on $F(E) \otimes_\sigma H$ with left creation operators $L_\sigma(g_1), \dots, L_\sigma(g_N)$ and projections $\mathrm{Ind}\,(\sigma)(e_1), \dots, \mathrm{Ind}\,(\sigma)(e_n)$.

The connection with full Fock spaces lies in the fact that we can identify $F(E) \otimes_\sigma H$ with an $*$-invariant subspace of $l_2(\mathbf{F}_N^+) \otimes H$ in such a way that the $L_\sigma(g_i)$s are compressions of left creation operators $L_i \otimes I_H$s on the full Fock space $l_2(\mathbf{F}_N^+) \otimes H$. We actually have that

$$L_\sigma (g_i)^* = (L_i \otimes I_H)^*_{|F(E) \otimes_\sigma H} \quad \text{for } i \leq N.$$

If graph G has no sinks, the projections $\mathrm{Ind}\,(\sigma)(e_j)$s belong to the double commutant of the $L_\sigma(g_i)$s. This implies for example that the module maps of the two Hilbert modules

$$(F(E) \otimes_\sigma H; L_\sigma(g_1), \dots, L_\sigma(g_N); \mathrm{Ind}\,(\sigma)(e_1), \dots, \mathrm{Ind}\,(\sigma)(e_n))$$

and

$$\left(F\left(E\right)\otimes_{\sigma}H;L_{\sigma}\left(g_{1}\right),\dots,L_{\sigma}\left(g_{N}\right)\right)$$

are equal. As a result, in many problems we ignore the projections and study only the left creation operators.

In this paper we work with $C_{\cdot 0}$ completely contractive representations (T,σ) of $E = X(G)$ on $B(H)$. These are completely contractive representations with an extra condition that for example allows us to define the Poisson kernel $K : H \to F(E)\otimes_{\sigma} H$. This map is an isometry and its adjoint satisfies $K^{*}L_{\sigma}(g_i) = T_iK^{*}$ for $i \le N$, and $K^{*}[\text{Ind}\,(\sigma)(e_j)] = \sigma(e_j)K^{*}$ for $j \le n$. This means that

$$K^{*}:F\left(E\right)\otimes_{\sigma}H\longrightarrow H$$

is a surjective coisometric module map.

In principle there are two possible Poisson kernels associated to the $C_{\cdot 0}$ completely contractive representation (T,σ) of $E = X(G)$ on $B(H)$. One is the Poisson kernel of the C^{*}-correspondence that we define in Section 2: $K_1 : H \to F(E)\otimes_{\sigma} H$, and the other one is the full Fock Poisson kernel $K_2 : H \to l_2(\mathbf{F}_N^{+})\otimes H$ associated to the maps $T_1,\dots,T_N$. It turns out that when we identify $F(E)\otimes_{\sigma} H$ with a subspace of $l_2(\mathbf{F}_N^{+})\otimes H$, the two maps are equal. This makes the connection with full Fock spaces even stronger. For example, we show that there exist partial isometric module maps Φ_1 and Φ_2 such that the following is a short exact sequence

(1.2)

$$0\longrightarrow F\left(E\right)\otimes_{\widehat{\sigma}}H_{2}\xrightarrow{\ \Phi_{2}\ }F\left(E\right)\otimes_{\sigma}H_{1}\xrightarrow{\ \Phi_{1}\ }H\longrightarrow 0.$$

The existence of (1.2) can be deduced from the dilation theorem of C^{*}-correspondences [**16**]. However, in this case, we deduce the existence of (1.2) easily from Poisson kernels on Fock spaces.

If (T_1,σ) is a $C_{\cdot 0}$ completely contractive representation of $E = X(G)$ on $B(H)$, (T_2,π) is a $C_{\cdot 0}$ completely contractive representation of $E = X(G)$ on $B(\mathcal{H})$, and $f : H \to \mathcal{H}$ is a module map, then there exist module maps $f_1 : F(E)\otimes_{\sigma} H_1 \to F(E)\otimes_{\pi}\mathcal{H}_1$ and $f_2 : F(E)\otimes_{\widehat{\sigma}} H_2 \to F(E)\otimes_{\widehat{\pi}}\mathcal{H}_2$ such that $\|f_2\| \le \|f_1\| \le \|f\|$ and such that the following diagram commutes:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & F(G_2)\otimes_{\widehat{\sigma}} H_2 & \xrightarrow{\Phi_2} & F(G_1)\otimes_{\sigma} H_1 & \xrightarrow{\Phi_1} & H & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle f_2} & & \downarrow{\scriptstyle f_1} & & \downarrow{\scriptstyle f} & & \\
0 & \longrightarrow & F(G_2)\otimes_{\widehat{\pi}} \mathcal{H}_2 & \xrightarrow{\Psi_2} & F(G_1)\otimes_{\pi} \mathcal{H}_1 & \xrightarrow{\Psi_1} & \mathcal{H} & \longrightarrow & 0.
\end{array}$$

This can be deduced from the general commutant lifting theorem of C^*-correspondences of [**16**], but we deduce it from the commutant lifting theorem of Fock spaces. We obtain precise information about the module maps f_1 and f_2. As in the classical case, they are associated to certain analytic "symbols." We use this to obtain the following rigidity result: Suppose that $\mathcal{M}_1$ and $\mathcal{M}_2$ are submodules of $F(E)\otimes_\sigma H_1$ and $F(E)\otimes_\pi H_2$ and that $\mathcal{M}_1^\perp$ is isomorphic to $\mathcal{M}_2^\perp$:

$$0 \longrightarrow \mathcal{M}_1 \longrightarrow F(E)\otimes_{\sigma_1} H_1 \xrightarrow{P^\perp_{\mathcal{M}_1}} \mathcal{M}_1^\perp \longrightarrow 0$$

$$0 \longrightarrow \mathcal{M}_2 \longrightarrow F(E)\otimes_{\sigma_2} H_2 \xrightarrow{P^\perp_{\mathcal{M}_2}} \mathcal{M}_2^\perp \longrightarrow 0.$$

We prove that if the inclusions $\mathcal{M}_i^\perp \subset F(E)\otimes_{\sigma_i} H_i$, $i=1,2$, are "minimal," then $\mathcal{M}_1$ is isomorphic to $\mathcal{M}_2$. We use this to show that the map $\Phi_2 : F(E)\otimes_{\widehat{\sigma}} H_2 \to F(E)\otimes_\sigma H_1$ of (1.2) determines $(H;T_1,\dots,T_N;P_1,\dots,P_n)$ up to unitary equivalence. Following the classical case, we say that Φ_2 is the characteristic function of H.

The directed graph G_2 is a deformation of G_1, or $G_1\le G_2$ in symbols, if G_2 can be obtained by identifying some vertices in G_1. Katsoulis and Kribs [**11**] proved that the left creation operators of G_2 "dominate" the left creation operators of G_1. We strengthen this result by proving that if $G_1\le G_2\le G_3\le\cdots$, then there exist partial isometric module maps $\Phi_i : F(G_{i+1})\otimes_{\sigma_{i+1}} H_{i+1} \to F(G_i)\otimes_{\sigma_i} H_i$ such that

$$\cdots \overset{\Phi_3}{\longrightarrow} F(G_3)\otimes_{\sigma_3} H_3 \overset{\Phi_2}{\longrightarrow} F(G_2)\otimes_{\sigma_2} H_2 \overset{\Phi_1}{\longrightarrow} F(G_1)\otimes_{\sigma_1} H_1 \longrightarrow 0$$

is an exact sequence. We also prove a stronger version of the commutant lifting theorem that allows us to conclude that if $f_1 : F(G_1)\otimes_{\sigma_1} H_1 \to F(G_1)\otimes_{\sigma_1} H_1$ is a module map, then there exist module maps f_i :

$F(G_i) \otimes_{\sigma_i} H_i \to F(G_i) \otimes_{\sigma_i} H_i$ such that $\|f_i\| \le \|f\|$ and such that the following diagram commutes:

$$\begin{array}{ccccccccc}
\cdots & \xrightarrow{\Phi_3} & F(G_3) \otimes_{\sigma_3} H_3 & \xrightarrow{\Phi_2} & F(G_2) \otimes_{\sigma_2} H_2 & \xrightarrow{\Phi_1} & F(G_1) \otimes_{\sigma_1} H_1 & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle f_3} & & \downarrow{\scriptstyle f_2} & & \downarrow{\scriptstyle f_1} & & \\
\cdots & \xrightarrow{\Phi_3} & F(G_3) \otimes_{\sigma_3} H_3 & \xrightarrow{\Phi_2} & F(G_2) \otimes_{\sigma_2} H_2 & \xrightarrow{\Phi_1} & F(G_1) \otimes_{\sigma_1} H_1 & \longrightarrow & 0.
\end{array}$$

In Section 4 we look at the weighted Fock spaces $\mathcal{F}^2(\omega_\alpha) \otimes H$ studied in [**1, 3**]. Given a faithful representation $\sigma : l_\infty^n \to B(H)$ we define $F_{\omega_\alpha}(G) \otimes_\sigma H$ to be a natural $*$-invariant subspace of $\mathcal{F}^2(\omega_\alpha) \otimes H$. The left creation operators $L_\sigma(g_i)$ are the compression of the left creation operators of $\mathcal{F}^2(\omega_\alpha) \otimes H$ to $F_{\omega_\alpha}(G) \otimes_\sigma H$, and we look at the Hilbert module

$$(F_{\omega_\alpha}(G) \otimes_\sigma H; L_\sigma(g_1), \ldots, L_\sigma(g_N); [\mathrm{Ind}\,(\sigma)\,(e_1)], \ldots, [\mathrm{Ind}\,(\sigma)\,(e_n)]),$$

where $[\mathrm{Ind}\,(\sigma)\,(e_j)]$'s are orthogonal projections that add up to the identity. The left creation operators are no longer partial isometries, but the main results of Section 3 are true. The proofs of these results are almost identical, and we only indicate the differences. We decided to give the complete proof for the full Fock spaces case. The addition of weights causes no difficulties. The main difference is that the projective resolutions of (1.2) are no longer finite.

2. Preliminaries and Poisson kernels. A Hilbert C^*-module E over the C^*-algebra A is a right A-module with an A-valued inner product $\langle \cdot, \cdot \rangle : E \times E \to A$ with the following properties: (i) $\langle x, \cdot \rangle$ is linear for every $x \in E$, $\langle x, y \rangle^* = \langle y, x \rangle$ for every $x, y \in E$, (ii) $\langle x, y \cdot a \rangle = \langle x, y \rangle a$ for every $x, y \in E$ and $a \in A$, and (iii) $\langle x, x \rangle \ge 0$ for every $x \in E$ (moreover, $\langle x, x \rangle = 0$ if and only if $x = 0$). We assume that E is a complete normed space with the norm $\|x\| = \sqrt{\|\langle x, x \rangle\|}$.

If E and F are Hilbert modules over the C^*-algebra A, a map $T : E \to F$ is adjointable if there exists a map $S : F \to E$ such that for every $x \in E$ and $y \in F$,

$$\langle Tx, y \rangle = \langle x, Sy \rangle.$$

The map S is denoted by T^*, and one easily checks that T is a bounded, linear and A-linear map, i.e., if $x \in E$ and $a \in A$, then $T(x \cdot a) = T(x) \cdot a$. The set of adjointable maps from E to F is denoted by $\mathcal{L}(E, F)$, and the set of adjointable maps from E to E is denoted by $\mathcal{L}(E)$. One can check that $\mathcal{L}(E)$ is a C^*-algebra.

Example 1. If $A = \mathbf{C}$, then any Hilbert space H is a Hilbert $\mathbf{C}$-module with the usual inner product. It is easy to check that $\mathcal{L}(H)$ is the usual $B(H)$.

Example 2 (Sums). Suppose that E_i is a Hilbert A-module for $i \in I$. Define $\oplus_{i \in I} E_i$ to be the set of $(x_i)_{i \in I}$ such that $\sum_{i \in I} \langle x_i, x_i \rangle$ converges in A. Then $\oplus_{i \in I} E_i$ is a Hilbert A-module with inner product given by $\langle (x_i), (y_i) \rangle = \sum_{i \in I} \langle x_i, y_i \rangle$.

A Hilbert module E over a C^*-algebra A has a natural operator space structure induced by linking algebra $\mathcal{L}$ (we refer to [**5**] for details). For example, if $A = \mathbf{C}$ any Hilbert $\mathbf{C}$-module E is a Hilbert space at the Banach space level and a column Hilbert space at the operator space level. That is, $E = C_N$ if E is N-dimensional or $E = C$ if E is separable and infinite-dimensional. We also mention that an adjointable map T between A-Hilbert modules is completely bounded and $\|T\|_{cb} = \|T\|$. We refer to [**14**] for additional information about Hilbert C^*-modules.

2.1. Tensor products and C^*-correspondences over A. Suppose that E is a Hilbert A-module and that F is a Hilbert B-module. For every $*$-representation $\sigma : A \to \mathcal{L}(F)$ one can construct the "balanced" tensor product $E \otimes_\sigma F$ (if there is no confusion about σ, the tensor product is also denoted $E \otimes_A F$) which is a Hilbert B-module that satisfies the following properties:

(1) If $x \in E$, $y \in F$, and $a \in A$, then $(x \cdot a) \otimes y = x \otimes (\sigma(a)y)$ (the tensor product is balanced over A).

(2) If $x_1, x_2 \in E$ and $y_1, y_2 \in F$, then $\langle x_1 \otimes y_1, x_2 \otimes y_2 \rangle = \langle y_1, \sigma(\langle x_1, x_2 \rangle) y_2 \rangle$.

Notice that if E is a Hilbert A-module and $\sigma : A \to B(H)$ is an $*$-representation, then $E \otimes_\sigma H$ is a Hilbert space. Indeed, H is a Hilbert

$\mathbf{C}$-module and $B(H) = \mathcal{L}(H)$. Then $E \otimes_\sigma H$ is a Hilbert $\mathbf{C}$-module, which makes it a Hilbert space.

A C^*-correspondence over A, or a Hilbert bimodule over A, is a Hilbert A-module E with an $*$-representation $\varphi : A \to \mathcal{L}(E)$. The map φ induces a "left" action of A on E given by $a \cdot x = \varphi(a)x$. If E is a C^*-correspondence over A we can construct the Hilbert A-module $E \otimes_A E$, and more generally, for every $n \geq 1$, we can construct the Hilbert A-module

$$E^{\otimes n} = \underbrace{E \otimes_A E \otimes_A \cdots \otimes_A E}_{n \text{ times}}.$$

Each of these Hilbert A-modules becomes a C^*-correspondence if $\varphi^{(n)} : A \to \mathcal{L}(E^{\otimes n})$ is defined by

$$\varphi^{(n)}(a)(x_1 \otimes x_2 \otimes \cdots x_n) = [\varphi(a)x_1] \otimes x_2 \otimes \cdots x_n.$$

The following example is due to Pimsner [**15**].

Example 3 (Fock module). Let E be a C^*-correspondence over A. The Fock module of E is the Hilbert A-module defined by

$$F(E) = A \oplus E \oplus E^{\otimes 2} \oplus E^{\otimes 3} \oplus \cdots \oplus E^{\otimes n} \oplus \cdots.$$

$F(E)$ is a C^*-correspondence over A if $\varphi_\infty : A \to \mathcal{L}(F(E))$ is defined by $\varphi_\infty(a) = \oplus_{n=0}^{\infty} \varphi^{(n)}(a)$.

2.2. Representations and tensor algebras: $\mathcal{T}_+(E)$**.** Let E be a C^*-correspondence over the C^*-algebra A. Muhly and Solel [**16**] defined a covariant representation of E to be an ordered pair (T, σ) such that

(1) $T : E \to B(H)$ is a bounded linear map,

(2) $\sigma : A \to B(H)$ is an injective $*$-representation, and

(3) For every $x \in E$ and $a \in A$,

$$T(x \cdot a) = T(x)\sigma(a) \quad \text{and} \quad T(\varphi(a)x) = \sigma(a)T(a).$$

The covariant representation is

(1) Completely bounded if $\|T\|_{cb} < \infty$.

(2) Completely contractive if $\|T\|_{cb} \leq 1$, and

(3) Isometric if $T(x)^*T(y) = \sigma(\langle x, y\rangle)$ for every $x, y \in E$.

Fowler and Raeburn [**8**] considered isometric covariant representation but they did not require σ to be faithful. They called such representations Toeplitz representations.

Example 4 (Fock representation). Suppose that E is a C^*-correspondence over A, and let $F(E)$ be the Fock module of Example 5. Define

$$L : E \longrightarrow \mathcal{L}\left(F\left(E\right)\right) \quad \text{by} \quad L(x)\left(\eta\right) = x \otimes \eta.$$

It is easy to check that (L, φ_∞) is an isometric covariant representation into $\mathcal{L}(F(E))$.

The tensor algebra of the C^*-correspondence E is denoted by $\mathcal{T}_+(E)$, and it is the norm-closed subalgebra of $\mathcal{L}(F(E))$ generated by $L(x)$ for $x \in E$ and $\varphi_\infty(a)$ for $a \in A$. This is the non self-adjoint part of an algebra introduced by Pimsner, see [**15**]: The Toeplitz algebra $\mathcal{T}_E$ is the C^*-algebra generated by $L(x)$ for $x \in E$ and $\varphi_\infty(a)$ for $a \in A$. Pimsner also defined $\mathcal{O}_E$ (the Cuntz-Pimsner algebra) as a quotient of $\mathcal{T}_E$. These algebras have been the object of much study recently.

Muhly and Solel proved that every completely contractive covariant representation of E induces a completely contractive representation on $\mathcal{T}_+(E)$, and it is easy to see that every completely contractive representation on $\mathcal{T}_+(E)$ induces a completely contractive covariant representation on E. Moreover, they proved that every completely contractive covariant representation has a unique minimal isometric covariant dilation [**16**].

The Fock representation is an isometric covariant representation of E into an abstract C^*-algebra. Every $*$-representation $\sigma : A \to B(H)$ induces an isometric covariant representation into a concrete Hilbert space:

Example 5 (Induced representation). Suppose that E is a C^*-correspondence over A and that $\sigma : A \to B(H)$ is an injective $*$-representation. Define

$$L_\sigma : E \longrightarrow B\left(F\left(E\right) \otimes_\sigma H\right) \qquad \text{by} \quad L_\sigma(x)\left(\eta\right) = L(x) \otimes \eta,$$

and

$$\text{Ind}\ (\sigma) : A \longrightarrow B\left(F\left(E\right) \otimes_\sigma H\right) \quad \text{by} \quad \text{Ind}\ (\sigma)\left(a\right) = \varphi_\infty\left(a\right) \otimes I.$$

It is easy to check that $(L_\sigma, \text{Ind}\,(\sigma))$ is an isometric covariant representation into the bounded linear operators of $F(E) \otimes_\sigma H$. Notice that since $F(E)$ is a Hilbert A-module, $F(E) \otimes_\sigma H$ is a Hilbert space.

Suppose that (T, σ) is a covariant representation of E into $B(H)$. Let $E \odot_\sigma H$ be the subspace of $E \otimes_\sigma H$ spanned by vectors of the form $x \otimes h$, where $x \in E$ and $h \in H$, and define

$$\widetilde{T} : E \odot_\sigma H \to H \quad \text{by} \quad \widetilde{T}\left(x \otimes h\right) = T\left(x\right) h.$$

The following lemma of Muhly and Solel is fundamental:

Lemma 1 ([**16**]). *T is completely bounded if and only if $\widetilde{T}$ is bounded. And, more precisely, $\|T\|_{cb} = \|\widetilde{T} : E \otimes_\sigma H \to H\|$. Moreover, (T, σ) is isometric if and only if $\widetilde{T} : E \otimes_\sigma H \to H$ is an isometry.*

2.3. Poisson kernels. In this section we define the Poisson kernels for Hilbert bimodules by adapting the definition of Poisson kernels for Fock spaces. These were introduced by Popescu in [**24**], where he used them to prove several versions of von Neumann inequalities. In [**2**] we used them to obtain noncommutative multi-variable interpolation results, and in this paper, we use them to study C^*-correspondences of directed graphs.

Let (T, σ) be a completely contractive covariant representation. For every $k \geq 1$, define

$$\widetilde{T^k} : E^{\otimes k} \otimes_\sigma H \longrightarrow H \quad \text{by} \quad \widetilde{T^k}\left(x_1 \otimes x_2 \otimes \cdots \otimes x_k \otimes h\right) = T\left(x_1\right) T\left(x_2\right) \cdots T\left(x_k\right) h.$$

Since $\widetilde{T^{k+1}}(\widetilde{T^{k+1}})^* = \widetilde{T^k}(I_{E^{\otimes k}} \otimes \widetilde{T})(I_{E^{\otimes k}} \otimes (\widetilde{T})^*)(\widetilde{T^k})^* \leq \widetilde{T^k}(\widetilde{T^k})^*$, it follows that

$$\widetilde{T}\big(\widetilde{T}\big)^* = \widetilde{T^1}\big(\widetilde{T^1}\big)^* \geq \widetilde{T^2}\big(\widetilde{T^2}\big)^* \geq \cdots \geq \widetilde{T^k}\big(\widetilde{T^k}\big)^* \geq 0.$$

The representation is called $C_{\cdot 0}$ if $\text{SOT} - \lim_{k\to\infty} \widetilde{T^k}\big(\widetilde{T^k}\big)^* = 0$, see [**19**].

Let $\Delta : H \to H$ be $\Delta = (I - \widetilde{T}(\widetilde{T})^*)^{1/2}$, and define

$$(2.1) \quad K : H \longrightarrow F\big(E\big) \otimes_\sigma H \quad \text{by} \quad K\big(h\big) = \sum_{k=0}^{\infty} \big(I_{E^{\otimes k}} \otimes \Delta\big)\big(\widetilde{T^k}\big)^* h.$$

Since
(2.2)

$$\begin{aligned} \Big\| \big(I_{E^{\otimes k}} \otimes \Delta\big)\big(\widetilde{T^k}\big)^* h \Big\|^2 &= \Big\langle \big(I_{E^{\otimes k}} \otimes \Delta\big)\big(\widetilde{T^k}\big)^* h, \big(I_{E^{\otimes k}} \otimes \Delta\big)\big(\widetilde{T^k}\big)^* h \Big\rangle \\ &= \Big\langle \big(I_{E^{\otimes k}} \otimes \Delta^2\big)\big(\widetilde{T^k}\big)^* h, \big(\widetilde{T^k}\big)^* h \Big\rangle \\ &= \Big\langle \big(I_{E^{\otimes k}} \otimes \Big[I - \widetilde{T}\big(\widetilde{T}\big)^*\Big]\big)\big(\widetilde{T^k}\big)^* h, \big(\widetilde{T^k}\big)^* h \Big\rangle \\ &= \Big\langle \big(\widetilde{T^k}\big)^* h, \big(\widetilde{T^k}\big)^* h \rangle - \langle \big(\widetilde{T^{k+1}}\big)^* h, \big(\widetilde{T^{k+1}}\big)^* h \Big\rangle \\ &= \Big\langle \Big[\widetilde{T^k}\big(\widetilde{T^k}\big)^* - \widetilde{T^{k+1}}\big(\widetilde{T^{k+1}}\big)^*\Big] h, h \Big\rangle, \end{aligned}$$

we obtain the following proposition:

Proposition 1. *Suppose that* (T, σ) *is a completely contractive covariant representation. Then*

(1) $K : H \to F(E) \otimes_\sigma H$ is a bounded map.

(2) $K : H \to F(E) \otimes_\sigma H$ *is an isometry if and only if* SOT-$\lim_{k\to\infty} \widetilde{T^k}(\widetilde{T^k})^* = 0$ *if and only if* (T, σ) *is a* $C_{\cdot 0}$ *representation.*

(3) $K^*(x_1 \otimes x_2 \otimes \cdots \otimes x_k \otimes h) = T(x_1)T(x_2)\cdots T(x_k)\Delta h$ *for every* $x_1 \otimes x_2 \otimes \cdots \otimes x_k \otimes h \in E^{\otimes k} \otimes_\sigma H$. *Consequently,*

(a) $K^* L_\sigma(x) = T(x)K^*$ *for every* $x \in E$, *and*

(b) $K^*[\text{Ind}\,(\sigma)(a)] = \sigma(a)K^*$ *for every* $a \in A$.

Proof. Parts (1) and (2) follow from (2.2), and (3) (a) and (b) follow easily from the formula $K^*(x_1 \otimes x_2 \otimes \cdots \otimes x_k \otimes h) = T(x_1)T(x_2)\cdots T(x_k)\Delta h$. So we only need to verify the formula. Let $h' \in H$. Then

$$\begin{aligned}
\langle K^*\left(x_1 \otimes x_2 \otimes \cdots \otimes x_k \otimes h\right), h'\rangle \\
&= \langle x_1 \otimes x_2 \otimes \cdots \otimes x_k \otimes h, K(h')\rangle \\
&= \left\langle x_1 \otimes x_2 \otimes \cdots \otimes x_k \otimes h, (I \otimes \Delta)\left(\widetilde{T^k}\right)^*(h')\right\rangle \\
&= \left\langle x_1 \otimes x_2 \otimes \cdots \otimes x_k \otimes \Delta h, \left(\widetilde{T^k}\right)^*(h')\right\rangle \\
&= \left\langle \widetilde{T^k}(x_1 \otimes x_2 \otimes \cdots \otimes x_k \otimes \Delta h), h'\right\rangle \\
&= \langle T(x_1)T(x_2)\cdots T(x_k)\Delta h, h'\rangle .
\end{aligned}$$

This proves the proposition. □

2.4. W^*-correspondences and $H^\infty(E)$. A Hilbert C^*-module E over the von Neumann algebra M is a W^*-module if E is self dual. This means that for every M-linear bounded map $f : E \to M$, there exists an $a \in E$ such that $f(x) = \langle a, x\rangle$. Paschke [**20**] proved that if E is a W^*-module, then E is a dual space and that $\mathcal{L}(E)$ is an abstract W^*-algebra. A W^*-correspondence over the von Neumann algebra M is a W^*-module E over M, and a w^*-continuous injective homomorphism $\varphi : M \to \mathcal{L}(E)$.

If (E, φ) is a W^*-correspondence, then so is $(F(E), \varphi_\infty)$. Muhly and Solel defined $H^\infty(E)$ to be the WOT closed subalgebra of $\mathcal{L}(F(E))$ generated by $L(x)$ for $x \in E$ and $\varphi_\infty(a)$ for $a \in A$. If $\sigma : A \to B(H)$ is a faithful normal $*$-representation, then $H^\infty(E)$ is isomorphic to the WOT closed subalgebra of $B(F(E) \otimes_\sigma H)$ generated by the induced representation, see [**18**]. We call this algebra $H^\infty_\sigma(E)$.

A covariant representation of E in $B(H)$ is an ordered pair (T, σ) such that $T : E \to B(E)$ is a w^*-continuous bounded linear map, $\sigma : M \to B(H)$ is a w^*-continuous injective homomorphism, and $T(x \cdot a) = T(x)\sigma(a)$ and $T(\varphi(a)x) = \sigma(a)T(a)$ for $x \in E$ and $a \in M$. As before, the covariant representation is completely bounded if T is completely bounded, completely contractive if T is completely contractive, and isometric if $T(x)^*T(y) = \sigma(\langle x, y\rangle)$ for every $x, y \in E$.

It is useful to note that a completely contractive $C_{\cdot 0}$-representation (T, σ) induces a representation in $H^\infty(E)$. Indeed, the map $\Phi : B(F(E) \otimes_\sigma H) \to B(H)$ defined by $\Phi(B) = KBK^*$ is WOT-continuous. Identify $H^\infty(E)$ with the WOT closed subalgebra of $B(F(E) \otimes_\sigma H)$ mentioned above, and notice that the restriction of Φ to this algebra is an isometric representation that is continuous with respect to w^*-topologies.

2.5. Noncommutative algebras on Fock spaces. One of the most important examples of C^*-correspondences is given by Popescu's noncommutative analytic algebras [**21**]. Let $A = \mathbf{C}$ and $E = l_2^N$, or $E = C_N$ if we look at the operator space structure of E. The left action is trivial and then $l_2^N \otimes_{\mathbf{C}} l_2^N \otimes_{\mathbf{C}} \cdots \otimes_{\mathbf{C}} l_2^N$ (k-times) is the usual tensor product of Hilbert spaces. Then $F(E)$ is identified with the full Fock space $l_2(\mathbf{F}_N^+)$, the Hilbert space with orthonormal basis $\{\delta_\alpha : \alpha \in \mathbf{F}_N^+\}$ indexed by the free semi-group on N-generators $g_1, g_2, \dots, g_N$. The left creation operators $L_i : l_2(\mathbf{F}_N^+) \to l_2(\mathbf{F}_N^+)$ are defined by $L_i(\delta_\alpha) = \delta_{g_i\alpha}$, for $i \leq N$. The norm closure of the algebra generated by the L_i's and the identity is a noncommutative analogue of the disc algebra $A(\mathbf{D})$, and it was denoted by Popescu by $\mathcal{A}_N$. The WOT closure of $\mathcal{A}_N$ is a noncommutative analogue of the Hardy space H^∞, and it was denoted by Popescu by F^∞. It can be easily checked that $\mathcal{T}_+(E) = \mathcal{A}_N$ and that $H^\infty(E) = F^\infty$.

In [**23**], Popescu proved that the commutant of F^∞ is generated by the right creation operators R_i (which are defined by $R_i(\delta_\alpha) = \delta_{\alpha g_i}$) and the identity. (In [**6**], the WOT closed subalgebra generated by the R_is and the identity were denoted by $\mathcal{R}_N$, and the WOT closed subalgebra generated by the L_i's and the identity were denoted by $\mathcal{L}_N$.)

Since $A = \mathbf{C}$, a completely contractive representation (T, σ) has trivial σ, and it is entirely determined by the completely contractive map $T : C_N \to B(H)$. Let δ_i, $i \leq N$, be the canonical basis of C_N, and let $T_i = T(\delta_i)$ for $i \leq N$. It follows that (T, σ) is completely contractive if and only if $\sum T_i T_i^* \leq I$, which is equivalent to $T = (T_1, T_2, \dots, T_N)$ being row contractive. And (T, σ) is $C_{\cdot 0}$ contractive if and only if $WOT\text{-}\lim_{k\to\infty} \sum_{\alpha \in \mathbf{F}_N^+, |\alpha| = k} T_\alpha T_\alpha^* = 0$ (we use the free semi-group on N generators $\mathbf{F}_N^+$ to represent arbitrary products: if $\alpha = g_{i_1} g_{i_2} \cdots g_{i_k} \in \mathbf{F}_N^+$, then $T_\alpha = T_{i_1} T_{i_2} \cdots T_{i_k}$). Popescu called this condition C_0, and he showed that whenever $T=(T_1,T_2,\dots,T_N)$ is a C_0 row contraction, there exists an isometry $K:H\to l_2(\mathbf{F}_N^+)\otimes H$ that satisfies $K^*(L_i\otimes I_H)=T_iK^*$ for $i\le N$. This is the construction that motivated the definition of Poisson kernels for C^*-correspondences.

The $*$-invariant subspaces play an important role in Fock spaces. Consider $l_2(\mathbf{F}_N^+)\otimes H$ with ampliation of the left creation operators: $L_1\otimes I_H,\dots,L_N\otimes I_H$. A subspace $\mathcal{M}\subset l_2(\mathbf{F}_N^+)\otimes H$ is called $*$-invariant if for every $i\le N$, $(L_i\otimes I_{l_2})^*\mathcal{M}\subset\mathcal{M}$. Define $V_i=P_{\mathcal{M}}(L_i\otimes I_{l_2})_{|\mathcal{M}}$ for $i\le N$, and notice that

$$\begin{aligned}V_{i_1}V_{i_2}\cdots V_{i_k}&=P_{\mathcal{M}}\left(L_{i_1}\otimes I_H\right)\left(L_{i_2}\otimes I_H\right)\cdots\left(L_{i_2}\otimes I_H\right)_{|\mathcal{M}}\\&=P_{\mathcal{M}}\left(L_{i_1}L_{i_2}\cdots L_{i_k}\otimes I_H\right)_{|\mathcal{M}}.\end{aligned}$$

This implies that, for every $i\le N$, $P_{\mathcal{M}}(L_i\otimes I_H)=V_iP_{\mathcal{M}}$. In the notation of the next section, it is stated that $(\mathcal{M};V_1,\dots,V_n)$ an $*$-submodule of $(l_2(\mathbf{F}_N^+)\otimes H;L_1\otimes I_H,\dots,L_N\otimes I_H)$ and that $P_{\mathcal{M}}$ is a module map; see [**1**] for related results.

The following summarizes some of the most important tools of F^∞.

Theorem 1 [**22, 23**]. *Suppose that $\mathcal{M}\subset l_2(\mathbf{F}_N^+)\otimes H$ is a subspace invariant under $(L_i\otimes I_H)^*$ for $i\le N$, and that $\mathcal{N}\subset l_2(\mathbf{F}_N^+)\otimes\mathcal{H}$ is invariant under $(L_i\otimes I_{\mathcal{H}})^*$ for $i\le N$. If $f:\mathcal{M}\to\mathcal{N}$ is a bounded linear module map, then there exists $g:l_2(\mathbf{F}_N^+)\otimes H\to l_2(\mathbf{F}_N^+)\otimes\mathcal{H}$ such that $\|g\|=\|f\|$, $f=P_{\mathcal{N}}g_{|\mathcal{M}}$, and $g(L_i\otimes I_H)=(L_i\otimes I_{\mathcal{H}})g$ for every $i\le N$. Such a map g is of the form $g=\sum_{\alpha\in F_N^+}R_\alpha\otimes A_\alpha$ for some $A_\alpha:H\to\mathcal{H}$. Furthermore, if $\mathcal{M}$ is invariant under $(R_i\otimes I_H)^*$ for $i\le N$, and $\mathcal{N}$ is invariant under $(R_i\otimes I_{\mathcal{H}})^*$ for $i\le N$, any map of the form $P_{\mathcal{N}}(R_\alpha\otimes A_\alpha)_{|\mathcal{M}}:\mathcal{M}\to\mathcal{N}$ is a module map.*

2.6. Hilbert modules. A Hilbert module over an index I is a Hilbert space H and a family of bounded linear maps $\{T_\alpha:H\to H\}_{\alpha\in I}$. If $(H,\{T_\alpha\}_{\alpha\in I})$ and $(K,\{S_\alpha\}_{\alpha\in I})$ are two Hilbert modules, a *module map* $f:H\to K$ is a bounded linear map that intertwines the T_αs with the S_αs. That is, $fT_\alpha=S_\alpha f$ for every $\alpha\in I$. The Hilbert modules are *isomorphic* if an isometric surjective module map exists between them.

For example, if E is a C^*-correspondence and (T,σ) is a completely contractive covariant representation on $B(H)$, H is the Hilbert module with maps $\{T(x)\}_{x\in E}$ and $\{\sigma(a)\}_{a\in A}$. The adjoint of the Poisson kernel is an example of a module map.

Proposition 2. *Suppose that* (T,σ) *is a* $C_{\cdot 0}$*-completely contractive covariant representation of the* C^**-correspondence* E *on* $B(H)$*, and let* $K : H \to F(E)\otimes_\sigma H$ *be the Poisson kernel. Then* $(\widehat{T},\widehat{\sigma})$ *is a* $C_{\cdot 0}$*-completely contractive covariant representation of* E *in* $B(K(H))$*, where* $\widehat{T} : E \to B(K(H))$ *is* $\widehat{T}(x) = P_{K(H)}L_\sigma(x)_{|K(H)}$ *and* $\widehat{\sigma} : A \to B(K(H))$ *is* $\widehat{\sigma}(a) = P_{K(H)}[\mathrm{Ind}\,(\sigma)(a)]_{|K(H)}$*. Moreover, Hilbert modules* $(H; T(x), x\in E; \sigma(a), a\in A)$ *and* $(K(H); \widehat{T}(x), x\in E; \widehat{\sigma}(a), a\in A)$ *are isomorphic.*

Proof. From Proposition 1 it follows that $\mathrm{Ind}\,(\sigma)(A)$ leaves $K(H)$ invariant and therefore $\widehat{\sigma} : A \to B(K(H))$ is a representation. Let $x\in E$ and $a\in A$. Then

$$\begin{aligned}\widehat{T}\,(x\cdot a) &= P_{K(H)}L_\sigma\,(x\cdot a)_{|K(H)} = P_{K(H)}L_\sigma\,(x)\,[\mathrm{Ind}\,(\sigma)\,(a)]_{|K(H)}\\ &= P_{K(H)}L_\sigma\,(a)\,P_{K(H)}\,[\mathrm{Ind}\,(\sigma)\,(a)]_{|K(H)} = \widehat{T}\,(x)\,\widehat{\sigma}\,(a)\,.\end{aligned}$$

To check the other direction, start from

$$\widehat{T}(\varphi(a)x) = P_{K(H)}[\mathrm{Ind}\,(\sigma)(a)]L_\sigma(x)_{|K(H)}.$$

Then

$$\begin{aligned}\left[\widehat{T}\,(\varphi\,(a)\,x)\right]^* &= P_{K(H)}\,[L_\sigma\,(x)]^*\,[\mathrm{Ind}\,(\sigma)\,(a^*)]_{|K(H)}\\ &= P_{K(H)}\,[L_\sigma\,(x)]^*\,P_{K(H)}\,[\mathrm{Ind}\,(\sigma)\,(a^*)]_{|K(H)}\\ &= \left[\widehat{T}\,(x)\right]^*[\widehat{\sigma}\,(a)]^* = \left[\widehat{\sigma}\,(a)\,\widehat{T}\,(x)\right]^*.\end{aligned}$$

Hence we obtain that $(\widehat{T},\widehat{\sigma})$ is a completely contractive covariant representation of E in $B(K(H))$. The isomorphism is implemented by $(K^*)_{|K(H)}$, and the module map properties follow from (3)(a) and (3)(b) of Proposition 1. □

Since H and $K(H)\subset F(E)\otimes_\sigma H$ are unitarily equivalent, the Poisson kernel can be used to show that a completely contractive $C_{\cdot 0}$ covariant representation can be dilated to an isometric induced representation. This is a result of [**16**].

A completely contractive covariant representation of the C^*-correspondence E on $B(H)$ induces two natural Hilbert modules:

$$\left(H; \{T(x)\}_{x\in E}; \{\sigma(a)\}_{a\in A}\right) \quad \text{and} \quad \left(H; \{T(x)\}_{x\in E}\right).$$

We will study Hilbert modules H that satisfy

$$\{h \in H : T(x)\, h = 0 \text{ for every } x \in E\} = \{0\}. \tag{2.3}$$

Proposition 3. *Suppose that, for $i = 1, 2$, (T_i, σ_i) is a completely contractive covariant representation of E in $B(H_i)$. Suppose also that H_2 satisfies (2.3). If $f : H_1 \to H_2$ satisfies $fT_1(x) = T_2(x)f$ for every $x \in E$, then for every $a \in A$, we have $f\sigma_1(a) = \sigma_2(a)f$.*

Proof. Fix $a \in A$ and $x \in E$. Then

$$\begin{aligned} T_2(x)\left[f\sigma_1(a) - \sigma_2(a)f\right] &= T_2(x)\, f\sigma_1(a) - T_2(x)\,\sigma_2(a)\, f \\ &= fT_1(x)\,\sigma_1(a) - T_2(x)\,\sigma_2(a)\, f \\ &= fT_1(x\cdot a) - T_2(x\cdot a)f \\ &= T_1 f(x\cdot a) - T_2(x\cdot a)f = 0. \end{aligned}$$

Since $x \in E$ is arbitrary, it follows from (2.3) that $f\sigma_1(a) = \sigma_2(a)f$. □

If the completely contractive covariant representation (T, σ) in $B(H)$ does not satisfy (2.3), we can find a largest $\mathcal{H} \subset H$ such that the compression of (T, σ) to $\mathcal{H}$ satisfies (2.3). Define $H_0 = \{h \in H : T(x)h = 0 \text{ for every } x \in E\}$. Notice that H_0 is invariant under $T(x)$ for $x \in E$ and under $\sigma(a)$ for $a \in A$. The first statement is clear. The second is easy: If $h \in H_0$, $a \in A$, and $x \in E$, then $T(x)[\sigma(a)h] = T(x\cdot a)h = 0$ because $x \cdot a \in E$. Since $x \in E$ is arbitrary, it follows that $\sigma(a)h \in H_0$. Define $\widehat{T} : E \to B(H_0^{\perp})$ by $\widehat{T}(x) = P_{H_0^{\perp}} T(x)_{|H_0^{\perp}}$, and define $\widehat{\sigma} : A \to B(H_0^{\perp})$ by $\widehat{\sigma}(a) = P_{H_0^{\perp}}\sigma(a)_{|H_0^{\perp}}$. We can easily check that $(\widehat{T}, \widehat{\sigma})$ is a completely contractive covariant representation that satisfies property (2.3).

Property (2.3) is satisfied if (T, σ) is an isometric covariant representation, E is a full Hilbert A-module and A is unital. This simplifies a little bit the proof of the commutant lifting theorem of [**16**], because one does not have to pay attention to the algebra A. We illustrate this in subsection 3.3.

3. C^*-correspondence of directed graphs. Let $G = (G^0, G^1, r, s)$ be a directed graph with vertex set G^0, edge set G^1, and range and source maps $r, s : G^1 \to G^0$. For simplicity, assume that G^0 and G^1 are finite, say $G^0 = \{v_1, \dots, v_n\}$ and $G^1 = \{\varepsilon_1, \dots, \varepsilon_N\}$. We view each edge ε_i as an arrow from $s(\varepsilon_i)$ to $r(\varepsilon_i)$. We multiply from right to left and we say that $\varepsilon_{i_k}\varepsilon_{i_{k-1}} \cdots \varepsilon_{i_2}\varepsilon_{i_1}$ is an allowable path in graph G if and only if $r(\varepsilon_{i_1}) = s(\varepsilon_{i_2})$, $r(\varepsilon_{i_2}) = s(\varepsilon_{i_3}) \dots$, and $r(\varepsilon_{i_{k-1}}) = s(\varepsilon_{i_k})$. The set of allowable paths of G is denoted by Γ_G, and it can be thought of as a subset of $\mathbf{F}_N^+$. We extend the definitions of $r, s : \Gamma_G \to G^0$ in the obvious way: $s(\varepsilon_{i_k}\varepsilon_{i_{k-1}} \cdots \varepsilon_{i_2}\varepsilon_{i_1}) = s(\varepsilon_{i_1})$ and $r(\varepsilon_{i_k}\varepsilon_{i_{k-1}} \cdots \varepsilon_{i_2}\varepsilon_{i_1}) = r(\varepsilon_{i_k})$.

Let $A = l_\infty(G^0) = l_\infty^n$, and define the Hilbert l_∞^n-module $E = X(G)$ to be the set of functions $x : G^1 \to \mathbf{C}$ with module action over l_∞^n given by

$$x \cdot a\,(\varepsilon) = x\,(\varepsilon)\, a\,(s\,(\varepsilon)) \quad \text{for } x \in E, a \in A, \text{ and } \varepsilon \in G^1,$$

and

$$\langle x, y\rangle\,(v) = \sum_{\{\varepsilon \in G^1 : s(\varepsilon) = v\}} \overline{x\,(\varepsilon)}\, y\,(\varepsilon) \ \text{ for } x, y \in E \text{ and } v \in G^1.$$

Let $e_i \in l_\infty^n$ be the canonical projection ($e_i(v_j) = 0$ if $i \neq j$ and $e_i(v_j) = 1$ if $i = j$), and define $g_i : G^1 \to \mathbf{C}$ by $g_i(\varepsilon_j) = \delta_{ij}$. Notice that $E = X(G)$ is the span of the g_is and that

$$\langle g_i, g_j\rangle = \begin{cases} 0 & \text{if } i \neq j \\ e_{s(\varepsilon_i)} & \text{if } i = j. \end{cases}$$

In the future, we will write $s(g_i)$ or $s(i)$ instead of $s(\varepsilon_i)$, and we will assume that the range of r and s is $\{1, 2, \dots, n\}$ instead of G^0.

The range function $r : G^1 \to \{1, \dots, n\}$ defines the $*$-representation $\varphi : l_\infty^n \to \mathcal{L}(E)$ by

$$\varphi(e_j)(g_i) = \begin{cases} 0 & \text{if } r(g_i) \neq j \\ g_i & \text{if } r(g_i) = j. \end{cases}$$

With the left action, the Hilbert l_∞^n-module $E = X(G)$ is a C^*-correspondence, and it is called a Cuntz-Krieger bimodule.

For $k \geq 1$, we construct $E^{\otimes k} = E \otimes_\varphi E \otimes_\varphi \cdots \otimes_\varphi E$ (k times), and then we define the Fock module of Example 3: $F(E) = l_\infty^n \oplus E \oplus E^{\otimes 2} \oplus \cdots$. To understand the structure of $F(E)$, it is instructive to look at a simple example. Let $g_i \otimes g_j \in E^{\otimes 2}$, and notice that

$$\begin{aligned} g_i \otimes g_j &= \left[g_i \cdot e_{s(g_i)}\right] \otimes g_j = g_i \otimes \varphi\left(e_{s(g_i)}\right)(g_j) \\ &= \begin{cases} 0 & \text{if } r(g_j) \neq s(g_i) \\ g_i \otimes g_j & \text{if } r(g_j) = s(g_i). \end{cases} \end{aligned}$$

Similarly, $g_{i_1} \otimes g_{i_2} \otimes \cdots \otimes g_{i_k} \in E^{\otimes k}$ is nonzero if and only if $g_{i_1} g_{i_2} \cdots g_{i_k} = \gamma$ is an allowable path in Γ_G. We denote $g_\gamma := g_{i_1} \otimes g_{i_2} \otimes \cdots \otimes g_{i_k}$. Moreover, we can check that

$$\langle g_\gamma, g_\alpha \rangle = \begin{cases} 0 & \text{if } \gamma \neq \alpha \\ e_{s(\varepsilon_\gamma)} & \text{if } \gamma = \alpha. \end{cases}$$

This implies that $\{g_\gamma : \gamma \in \Gamma_G \text{ and } |\gamma| = k\}$ is a basis of $E^{\otimes k}$. If $\sigma : l_\infty^n \to B(H)$ is an $*$-representation, then $g_\gamma \otimes_\sigma H$ is orthogonal to $g_\alpha \otimes_\sigma H$ if $\alpha \neq \gamma$.

3.1. Full Fock space representations. In this section, we study induced representations of the Cuntz-Krieger bimodule of Example 5. We will prove that they are associated to $*$-invariant subspaces of full Fock spaces. Consider a faithful $*$-representation $\sigma : l_\infty^n \to B(H)$. This decomposes $H = H_1 \oplus H_2 \oplus \cdots \oplus H_n$, where $H_j = \sigma(e_j)H$ for $j \leq n$. Notice that if $g_i \otimes x \in E \otimes_\sigma H$, then

$$g_i \otimes x = g_i \cdot e_{s(g_i)} \otimes x = g_i \otimes \sigma\left(e_{s(g_i)}\right)(x).$$

Hence, we can assume that $x \in H_{s(g_i)}$. Similarly, if $g_\gamma \otimes x \in E^{\otimes k} \otimes_\sigma H$ is nonzero, then $\gamma \in \Gamma_G$ and, since $g_\gamma \otimes x = g_\gamma \cdot e_{s(\gamma)} \otimes x = g_\gamma \otimes \sigma(e_\gamma)(x)$, we can assume that $x \in H_{s(\gamma)}$. It follows that $F(E) \otimes_\sigma H$ is spanned by vectors of the form $g_\gamma \otimes x$ for $\gamma \in \Gamma_G$ and $x \in H_{s(\gamma)}$ and by vectors of the form $1 \otimes x$ for $x \in H_j$, $j \leq n$. Consequently, we identify $F(E) \otimes_\sigma H$ with the closed subspace $\mathcal{M}_\sigma$ of $l_2(\mathbf{F}_N^+) \otimes H$ spanned by vectors of the form

$$\{\delta_\gamma \otimes x : \gamma \in \Gamma_G, x \in H_{s(\gamma)}\} \bigcup \{\delta_0 \otimes x : x \in H\}. \tag{3.1}$$

Proposition 4. *There exists an $*$-invariant subspace $\mathcal{M}_\sigma$ of $l_2(\mathbf{F}_N^+) \otimes H$ such that the Hilbert module $(F(E) \otimes_\sigma H; L_\sigma(g_1), \ldots, L_\sigma(g_N))$ is unitarily equivalent to the Hilbert module $(\mathcal{M}_\sigma; P_{\mathcal{M}_\sigma}(L_1 \otimes I)_{|\mathcal{M}_\sigma}, \ldots, P_{\mathcal{M}_\sigma}(L_N \otimes I)_{|\mathcal{M}_\sigma})$.*

Proof. Let $\mathcal{M}_\sigma$ be the subspace of $F(E) \otimes_\sigma H$ spanned by vectors of the form (3.1). If $\gamma = g_{i_1} g_{i_2} \cdots g_{i_k} \in \Gamma_G$ and $x \in H_{s(\gamma)}$, then

$$(L_i \otimes I_H)^* (\delta_\gamma \otimes x) = \begin{cases} \delta_{g_{i_2} \cdots g_{i_k}} \otimes x & \text{if } r(\gamma) = r(g_{i_1}) = i \\ 0 & \text{if } r(g_{i_1}) \neq i. \end{cases}$$

And, naturally, $(L_i \otimes I_H)^*(\delta_0 \otimes x) = 0$ for every $x \in H$. Thus, it follows that $\mathcal{M}_\sigma$ is $*$-invariant in $l_2(\mathbf{F}_N^+) \otimes H$. Define $u : F(E) \otimes_\sigma H \to l_2(\mathbf{F}_N^+) \otimes H$ by $u(g_\gamma \otimes x) = \delta_\gamma \otimes x$ if $\gamma \in \Gamma_G$ and $x \in H_{s(\gamma)}$ and by $u(1 \otimes x) = \delta_0 \otimes x$ if $x \in H$. The map u is unitary and we easily check that $u(L_\sigma(g_i)) = P_{\mathcal{M}_\sigma}(L_i \otimes I)_{|\mathcal{M}_\sigma} u$ for every $i \leq N$. This proves the result. □

For $j \leq n$, $\mathrm{Ind}(\sigma)(e_j)$ corresponds to projection $P_j : \mathcal{M}_\sigma \to \mathcal{M}_\sigma$, defined by

$$P_j(\delta_\gamma \otimes x) = \begin{cases} \delta_\gamma \otimes x & \text{if } r(\gamma) = j \\ 0 & \text{if } r(\gamma) \neq j \end{cases} \quad \text{for } \gamma \in \Gamma_G \text{ and } \delta x \in H_{s(\gamma)}$$
$$P_j(\delta_0 \otimes x) = \begin{cases} \delta_0 \otimes x & \text{if } j = k \\ 0 & \text{if } j \neq k. \end{cases} \quad \text{for } x \in H_k. \tag{3.2}$$

As a result, we simply identify $F(E) \otimes_\sigma H$ with $\mathcal{M}_\sigma$, and we assume that

(3.3)

$$\begin{cases} F\left(E\right)\otimes_{\sigma} H \subset l_2\left(\mathbf{F}_N^+\right)\otimes H & \text{is } *\text{-invariant,}\\ L_\sigma\left(g_i\right) = P_{F(E)\otimes_\sigma H}\left(L_i\otimes I\right)_{|F(E)\otimes_\sigma H} & \text{for } i\le N\text{, and that}\\ \text{Ind}\left(\sigma\right)\left(e_j\right) = P_j & \text{is given by formula (3.2).}\end{cases}$$

Notice that for every $i \le N$,

$$L_\sigma\left(g_i\right) = P_{r(g_i)} L_\sigma\left(g_i\right) P_{s(g_i)}.$$

Definition 1. A directed graph G has no sinks if every vertex is the source of a directed edge.

The following simple observation tells us that the two natural Hilbert modules associated to a directed graph are equivalent if the graph has no sinks.

Lemma 2. *Suppose that G has no sinks and that $\sigma : l_\infty^n \to B(H)$ is a faithful $*$-representation. Then $F(E) \otimes_\sigma H$ satisfies property* (2.3)*. That is, if $L_\sigma(g_i)z = 0$ for $i \le N$, then $z = 0$ in $F(E) \otimes_\sigma H$.*

Proof. It is enough to look at vectors of the form (3.1). First take $\gamma \in \Gamma_G$ and a nonzero $x \in H_{s(\gamma)}$. Since G has no sinks, there exists $j \le N$ such that $s(g_j) = r(\gamma)$. Then $L_\sigma(g_j)(\delta_\gamma \otimes x) = \delta_{g_j\gamma} \otimes x$ is nonzero. Now take a nonzero $x \in H_k$. Since G has no sinks, there exists a $j \le N$ such that $s(g_j) = k$. Then $L_\sigma(g_j)(\delta_0 \otimes x) = \delta_{g_j} \otimes x$ is nonzero. $\square$

From Proposition 3, we obtain

Corollary 1. *Suppose that G has no sinks, and let $\sigma : l_\infty^n \to B(H)$ and $\pi : l_\infty^n \to B(\mathcal{H})$ be injective $*$-representations. If $T : F(E) \otimes_\sigma H \to F(E) \otimes_\pi \mathcal{H}$ is a bounded linear map that satisfies $TL_\sigma(g_i) = L_\pi(g_i)T$ for $i \le N$, then $T[\text{Ind}\,(\sigma)(e_j)] = [\text{Ind}\,(\pi)(e_j)]T$ for $j \le n$.*

The condition that G has no sinks is necessary.

Example 6. Suppose that G is a directed graph with incidence matrix $\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$. Label the vertices $\{a, b\}$ and the edges $\{g_1, g_2\}$, where g_1 goes from a to a and g_2 goes from a to b. Define $T : F(E) \otimes_\sigma l_2^2 \to F(E) \otimes_\sigma l_2^2$ by the formula

$$T(\delta_0 \otimes \xi_a) = \delta_{g_2} \otimes \xi_a, T(\delta_0 \otimes \xi_b) = 0,$$

and

$$T(\delta_\gamma \otimes h) = 0 \quad \text{for } \gamma \in \Gamma_G.$$

We can easily check that $TL_\sigma(g_i) = L_\sigma(g_i)T = 0$ for $i = 1, 2$, and this implies that T intertwines the $L_\sigma(g_i)$s. However, T does not intertwine the projections. If P_a is the projection induced by vertex a, then $TP_a(\delta_0 \otimes \xi_a) = T(\delta_0 \otimes \xi_a) = \delta_{g_2} \otimes \xi_a$. But since $r(g_2) = b$, $P_a T(\delta_0 \otimes \xi_a) = P_a(\delta_{g_2} \otimes \xi_a) = 0$. Therefore, the condition that G has no sinks is necessary.

3.2. Poisson kernels and von Neumann inequalities. Suppose that (T, σ) is a $C_{\cdot 0}$-completely contractive representation of $E = X(G)$ in a Hilbert space H. We start by giving the explicit form of the Poisson kernel of (T, σ). Notice that the representation is characterized by the bounded linear maps $T(g_1), \ldots, T(g_N)$ and by the orthogonal projections $\sigma(e_1), \ldots, \sigma(e_n)$. For simplicity, we will denote $T(g_i) = T_i$ for $i \leq N$. Recall that $\widetilde{T^k} : E^{\otimes k} \otimes_\sigma H \to H$ is given by $\widetilde{T^k}(g_{i_1} \otimes g_{i_2} \otimes \cdots \otimes g_{i_k} \otimes h) = T_{i_1} T_{i_2} \cdots T_{i_k}(h)$. If $\gamma = g_{i_1} g_{i_2} \cdots g_{i_k} \in \mathbf{F}_N^+$, we can write this as $\widetilde{T^k}(\delta_\gamma \otimes h) = T_\gamma(h)$, where $T_\gamma = T_{i_1} T_{i_2} \cdots T_{i_k}$.

Lemma 3. *If (T, σ) is a completely bounded covariant representation of $E = X(G)$ in a Hilbert space H, then for every $k \geq 1$, $(\widetilde{T^k})^*(h) = \sum_{\gamma \in \Gamma_G, |\gamma| = k} \delta_\gamma \otimes (T_\gamma)^*(h)$. If $k = 0$, then $\widetilde{T^k}$ is the identity from H to H.*

Proof. Let $h \in H$. Then $(\widetilde{T^k})^*(h) = \sum_{\gamma \in \Gamma_G, |\gamma| = k} \delta_\gamma \otimes x_\gamma$ for some $x_\gamma \in H_{s(\gamma)}$. To see that $x_\alpha = (T_\alpha)^*(h)$ for $\alpha \in \Gamma_G$ with $|\alpha| = k$, take $h' \in H_{s(\alpha)}$ arbitrary and compute

$$\left\langle \left(\widetilde{T^k}\right)^* (h), \delta_\alpha \otimes h' \right\rangle = \left\langle h, \widetilde{T^k}(\delta_\alpha \otimes h') \right\rangle = \langle h, T_\alpha h' \rangle = \left\langle (T_\alpha)^* h, h' \right\rangle.$$

On the other hand,

$$\left\langle \left(\widetilde{T^k}\right)^* (h), \delta_\alpha \otimes h' \right\rangle = \left\langle \sum_{\substack{\gamma \in \Gamma_G \\ |\gamma| = k}} \delta_\gamma \otimes x_\gamma, \delta_\alpha \otimes h' \right\rangle = \langle x_\alpha, h' \rangle .$$

Since h' is arbitrary, we conclude that $x_\alpha = (T_\alpha)^* h$. □

It is well known that if (T, σ) is completely contractive, then $T = (T_1, \dots, T_N)$ is a row contraction. You can see this from Lemma 3: $\Delta^2 h = (I - \widetilde{T}\widetilde{T}^*)h = h - \sum_{i=1}^N T_i T_i^* h$, and hence $\Delta^2 = I - \sum_{i=1}^N T_i T_i^* \geq 0$.

Proposition 5. *The Poisson kernel of the $C_{\cdot 0}$-completely contractive representation of $E = X(G)$ in a Hilbert space H is the isometry $K : H \to F(E) \otimes_\sigma H$ given by the formula*

$$K(h) = \delta_0 \otimes \Delta h + \sum_{k=1}^{\infty} \sum_{\substack{\gamma \in \Gamma_G \\ |\gamma| = k}} \delta_\gamma \otimes \Delta (T_\gamma)^* h.$$

Proof. The formula follows from (2.1). We only need to comment on the $k = 0$ case. Since A is unital, we identify $A \otimes_\sigma H$ with H in the natural way: $1 \otimes h \mapsto h$. Recall that we identify $1 \otimes h$ with $\delta_0 \otimes h$ when we see $1 \otimes h$ as an element of the $*$-invariant subspace of $l_2(\mathbf{F}_N^+) \otimes H$. □

Suppose that (T, σ) is a $C_{\cdot 0}$-completely contractive covariant representation of $E = X(G)$ in a Hilbert space H. This representation induces two natural Poisson kernels on H. The first one is the Poisson kernel of the C^*-correspondence of subsection 2.3. This is the isometry

$$K_C : H \to F(E) \otimes_\sigma H \text{ defined by } K_C(h) = \delta_0 \otimes \Delta h + \sum_{k=1}^{\infty} \sum_{\substack{\gamma \in \Gamma_G \\ |\gamma| = k}} \delta_\gamma \otimes \Delta (T_\gamma)^* h.$$

We use the subscript K_C to indicate this is the Poisson kernel of the C^*-correspondence.

On the other hand, we can look only at the row contraction $T = (T_1, T_2, \dots, T_N)$. Since this is a C_0-row contraction, we have the Fock space Poisson kernel associated to T. This is the isometry

$$K_F : H \to l_2\left(\mathbf{F}_N^+\right) \otimes H \quad \text{defined by} \quad K_F(h) = \sum_{\alpha \in \mathbf{F}_N^+} \delta_\alpha \otimes \Delta (T_\alpha)^* h,$$

where Δ and T_α have the same meaning as in the Poisson kernel of the C^*-correspondence. We used the subscript K_F to indicate this is the Poisson kernel of T and that it takes values in the Fock space.

Proposition 6. *If we use* (3.3) *to identify $F(E) \otimes_\sigma H$ with an $*$-invariant subspace of $l_2(\mathbf{F}_N^+) \otimes H$, the Poisson kernel of the $C_{.0}$-completely contractive covariant representation of $E = X(G)$ in a Hilbert space H is exactly the Poisson kernel of the C_0-row contraction $T = (T_1, \dots, T_N)$. In particular, $K_F(H) \subset F(E) \otimes_\sigma H$.*

Proof. The proof of this result is simple. We only need to show that if $\alpha \notin \Gamma_G$, then $T_\alpha = 0$. Recall that, for every $i \leq N$, $T_i = P_{r(i)} T_i P_{s(i)}$. If $\alpha = g_{i_1} g_{i_2} \cdots g_{i_k}$, we have

$$\begin{aligned} T_\alpha &= T_{i_1} T_{i_2} \cdots T_{i_k} \\ &= P_{r(i_1)} T_{i_1} P_{s(i_1)} P_{r(i_2)} T_{i_2} P_{s(i_2)} \cdots P_{r(i_{k-1})} T_{i_{k-1}} P_{s(i_{k-1})} P_{r(i_k)} T_{i_k} P_{s(i_k)}. \end{aligned}$$

Then it follows that $T_\alpha \neq 0$ only if $s(i_1) = r(i_2)$, $s(i_2) = r(i_3), \dots$, and $s(i_{k-1}) = r(i_k)$. But this means that $\alpha \in \Gamma_G$. Therefore, the two maps are equal and they have the same range. We just note that Δ commutes with the projections. □

In [**11**], Katsoulis and Kribs proved the following von Neumann inequality associated to directed graphs: Suppose that $T_1, T_2, \dots, T_N$ are bounded linear operators in $B(H)$ such that $T = (T_1, T_2, \cdots, T_N)$ is a row contraction and that $P_1, P_2, \dots, P_n$ is a family of orthogonal projections on H that add up to the identity and that stabilize T in the following way:

$$T_i P_j, P, T_i \in \{T_i, 0\} \text{ for } i \leq N \text{ and } j \leq n.$$

They noted that this relation determines a directed graph with n vertices and N edges, whose source and range maps are determined by the relation

$$T_i = P_{r(i)} T_i P_{s(i)}.$$

They proved that if $p(z_1, z_2, \ldots, z_N)$ is a noncommutative polynomial in N variables, then

$$\|p(T_1, T_2, \ldots, T_N)\| \leq \|p(L_\sigma(g_1), L_\sigma(g_2), \ldots, L_\sigma(g_N))\|, \tag{3.4}$$

where $\sigma : l_\infty^n \to B(H)$ is the $*$-representation generated by the P_js. They denoted $\|p\|_{G,\infty} = \|p(L_{\sigma_2}(g_1), L_{\sigma_2}(g_2), \ldots, L_{\sigma_2}(g_i))\|$ (notice that this has the same value for any faithful $*$-representation $\sigma : l_\infty^n \to B(H)$).

We will show that this inequality can be deduced from the usual von Neumann inequality on Fock spaces. Assume first that the norm of $[T_1 T_2 \cdots T_N]$ is smaller than one, and hence that $T_1, \ldots, T_N$ is a C_0 contraction. (If this is not the case, replace T_i with rT_i for $0 < r < 1$. Obtain inequality (3.4) for this r and then let $r \to 1$.) Then construct the Fock space Poisson kernel $K : H \to l_2(\mathbf{F}_N^+) \otimes H$ of the C_0 row contraction T. Map K has the property $K^*(L_i \otimes I_H) = T_i K^*$ for $i \leq N$, which implies that, for every $\alpha \in \mathbf{F}_N^+$, $K^*(L_\alpha \otimes I_H)K = T_\alpha$. Popescu's noncommutative von Neumann inequality follows from this. Indeed, since

$$p(T_1, T_2, \ldots, T_N) = K^* p(L_1 \otimes I_H, L_2 \otimes I_H, \ldots, L_N \otimes I_H) K,$$

$$\begin{aligned} \|p(T_1, T_2, \ldots, T_N)\| &\leq \|p(L_1 \otimes I_H, L_2 \otimes I_H, \ldots, L_N \otimes I_H)\| \\ &= \|p(L_1, L_2, \ldots, L_N)\|. \end{aligned}$$

We can strengthen this result because K takes values in $F(E) \otimes_\sigma H$. Indeed,

$$\begin{aligned} T_\alpha = K^*(L_\alpha \otimes I_H) K &= K^* P_{F(E)\otimes_\sigma H} (L_\alpha \otimes I_H) P_{F(E)\otimes_\sigma H} K \\ &= K^* L_\sigma(g_\alpha) K. \end{aligned}$$

Hence, we have that

$$p(T_1, T_2, \ldots, T_N) = K^* p(L_\sigma(g_1), L_\sigma(g_2), \ldots, L_\sigma(g_i)) K,$$

and this implies that

$$\|p(T_1, T_2, \dots, T_N)\| \leq \|p(L_\sigma(g_1), L_\sigma(g_2), \dots, L_\sigma(g_N))\|.$$

Kribs and Katsoulis defined an order of directed graphs. Suppose that G_1 is a graph with N edges and n_1 vertices and G_2 is a graph with N and n_2 vertices. G_2 is a deformation of G_1, or $G_1 \leq G_2$ in symbols, if we can obtain G_2 from G_1 by identifying some vertices in G_1. For example, the graph with one vertex and N edges—which corresponds to the usual full Fock space—is the largest graph of N edges.

They proved that if $G_1 \leq G_2$, then the left creation operators of G_2 dominate the left creation operators of G_1. That is, if $p(z_1, z_2, \dots, z_N)$ is a noncommutative polynomial in N variables, then

$$\|p\|_{G_1,\infty} \leq \|p\|_{G_2,\infty}.$$

The following proposition gives an alternative proof of this result. In subsection 3.4 we will strengthen this result using homological language.

Proposition 7. *Suppose that $G_1 \leq G_2$, and let $\sigma_1 : l_\infty^{n_1} \to B(H)$ be a faithful representation. Then there exists a faithful representation $\sigma_2 : l_\infty^{n_2} \to B(H)$ such that $F(G_1) \otimes_{\sigma_1} H \subset F(G_2) \otimes_{\sigma_2} H \subset l_2(\mathbf{F}_N^+) \otimes H$. This implies that $F(G_1) \otimes_{\sigma_1} H$ is an $*$-invariant subspace of $F(G_2) \otimes_{\sigma_2} H$. Consequently, $L_{\sigma_1}(g_i)^* = L_{\sigma_2}(g_i)^*_{|F(E)\otimes_\sigma H}$ for $i \leq N$, and $p(L_{\sigma_1}(g_1), L_{\sigma_1}(g_2), \dots, L_{\sigma_1}(g_i)) = P_{\mathcal{M}_{\sigma_1}} p(L_{\sigma_2}(g_1), L_{\sigma_2}(g_2), \dots, L_{\sigma_2}(g_i))_{|\mathcal{M}_{\sigma_1}}$.*

Proof. Assume that the equivalent edges of G_1 and G_2 are indexed by the same $i \in \{1, \dots, n\}$. Let $\sigma_1 : l_\infty^{n_1} \to B(H)$ be a faithful representation. This decomposes $H = H_1 \oplus \cdots \oplus H_n$. The equivalence relation that identifies vertices in G_1 to obtain G_2 induces a partition on $\{1, 2, \dots, n\}$. Let $A_1, A_2, \dots, A_{n_2}$ be the set of equivalent vertices, and define $\sigma_2 : l_\infty^{n_2} \to B(H)$ by $\sigma_2(e_j)$, where $\sigma_2(e_j)$ is the orthogonal projection onto $\mathcal{H}_j = \oplus_{i \in A_j} H_j$. Clearly, $\Gamma_{G_1} \subset \Gamma_{G_2}$ and if $\gamma \in \Gamma_G$, we have $H_{s_1(\gamma)} \subset \mathcal{H}_{s_2(\gamma)}$. Therefore, every element in $F(G_1) \otimes_{\sigma_1} H$ is an element in $F(G_2) \otimes_{\sigma_2} H$. $\square$

Popescu pointed out to us that the Wold decomposition of Jury and Kribs [**9**] is a consequence of the Wold decomposition for isometries with orthogonal ranges. The argument goes like this: Let $T_1, T_2, \dots, T_N$ be partial isometries on $B(H)$ satisfying (1.1). Let $V_1, V_2, \dots, V_N$ be the minimal isometric dilation on the Hilbert space $K \supset H$. Take the Wold decomposition of the V_is on K and decompose $K = K_s \oplus K_c$. Denote $H_s = H \cap K_s$ and $H_c = H \cap K_c$. It is easy to check that H_s and H_c are invariant under the T_is. Moreover, the restriction of the T_is to H_s is a C_0-contraction of partial isometries that satisfies (1.1), and the restriction of the T_is to H_c is a Cuntz contraction. The Poisson kernels can be used to prove that the restriction of the T_is to H_s is a direct sum of the shifts $L_\sigma(g_1), L_\sigma(g_2), \dots, L_\sigma(g_N)$.

3.3. Intertwining spaces and commutant lifting theorems. Muhly and Solel [**18**] developed a general duality theory for W^*-correspondences that says $H^\infty_\sigma(E)$ is its double commutant. This was also proved by Kribs and Power [**13**] for free semi-groupoid algebras. We will use Fock space techniques to characterize the maps that intertwine the left creation operators of two induced representations. We will obtain the specific form of these maps. This is useful in subsection 3.4. Furthermore, the methods of this section generalize to the weighted graphs that we study in Section 4.

Theorem 2. *Suppose that G has no sinks, and let $\sigma : l^n_\infty \to B(H)$ and $\pi : l^n_\infty \to B(\mathcal{H})$ be faithful $*$-representations. If $T : F(E) \otimes_\sigma H \to F(E) \otimes_\pi \mathcal{H}$ satisfies $TL_\sigma(g_i) = L_\pi(g_i)T$ for $i \leq N$, then there exist operators $A_\alpha \in B(H, \mathcal{H})$ satisfying $A_\alpha = P_{\mathcal{H}_{s(\alpha)}} A_\alpha P_{H_{r(\alpha)}}$ for $\alpha \in \Gamma_G$ and operators $A_j \in B(H, \mathcal{H})$ satisfying $A_j = P_{\mathcal{H}_j} A_j P_{H_j}$ for $j \leq n$ such that*

$$T = \sum_{j=1}^{n} (I \otimes A_j)_{|\mathcal{M}_\sigma} + \sum_{\alpha \in \Gamma_G} (R_\alpha \otimes A_\alpha)_{|\mathcal{M}_\sigma}.$$

Let $T : F(E) \otimes_\sigma H \to F(E) \otimes_\pi \mathcal{H}$ be a bounded linear map that satisfies $TL_\sigma(g_i) = L_\pi(g_i)T$ for $i \leq N$. By Theorem 1 there exists $\widehat{T} : l_2(\mathbf{F}^+_N) \otimes H \to l_2(\mathbf{F}^+_N) \otimes \mathcal{H}$ such that $\|\widehat{T}\| = \|T\|$, $\widehat{T}(L_i \otimes I) = (L_i \otimes I)\widehat{T}$ for every $i \leq N$, and $P_{\mathcal{M}_\pi} \widehat{T}_{|\mathcal{M}_\sigma} = T$. Moreover, $\widehat{T} = \sum_{\alpha \in \mathbf{F}^+_N} R_\alpha \otimes A_\alpha$

for some $A_\alpha \in B(H, \mathcal{H})$ (recall that $R_\alpha(\delta_\beta) = \delta_{\beta\alpha}$). Hence,

$$\widehat{T} = \sum_{\alpha\in\mathbf{F}_N^+} R_\alpha \otimes A_\alpha = \sum_{\alpha\in\mathbf{F}_N^+} \sum_{j=1}^{n} \sum_{k=1}^{n} R_\alpha \otimes P_{\mathcal{H}_j} A_\alpha P_{H_k}. \tag{3.5}$$

Lemma 4. *Let $\alpha \in \mathbf{F}_N^+$. Then $P_{\mathcal{M}_\pi}[R_\alpha \otimes P_{\mathcal{H}_j} A_\alpha P_{H_k}]_{|\mathcal{M}_\sigma}$ is nonzero only if $\alpha \in \Gamma_G$ and $j = s(\alpha)$. Moreover, the maps of the form $[R_\alpha \otimes P_{\mathcal{H}_{s(\alpha)}} A_\alpha P_{H_{r(\alpha)}}]_{|\mathcal{M}_\sigma}$ for $\alpha \in \Gamma_G$, and $[I \otimes P_{\mathcal{H}_j} A_\alpha P_{H_j}]_{|\mathcal{M}_\sigma}$ for $j \leq n$ map $\mathcal{M}_\sigma$ into $\mathcal{M}_\pi$, intertwine $L_\sigma(g_i)$ with $L_\pi(g_i)$ and intertwine* $\mathrm{Ind}\,(\sigma)(a)$ *with* $\mathrm{Ind}\,(\pi)(a)$*, even if G has sinks.*

Proof. Suppose that $P_{\mathcal{M}_\pi}[R_\alpha \otimes P_{\mathcal{H}_j} A_\alpha P_{H_k}]_{|\mathcal{M}_\sigma} \neq 0$ for $|\alpha| \geq 1$. Then either there exists $\gamma \in \Gamma_G$ and $x \in H_{s(\gamma)}$ such that $P_{\mathcal{M}_\pi}[R_\alpha \otimes P_{\mathcal{H}_j} A_\alpha P_{H_k}]_{|\mathcal{M}_\sigma}(\delta_\gamma \otimes x) = (\delta_{\gamma\alpha} \otimes P_{\mathcal{H}_j} A_\alpha P_{H_k} x) \neq 0$, or there exists $x \in H_k$ such that $P_{\mathcal{M}_\pi}[R_\alpha \otimes P_{\mathcal{H}_j} A_\alpha P_{H_k}]_{|\mathcal{M}_\sigma}(\delta_0 \otimes x) = \delta_\alpha \otimes P_{\mathcal{H}_j} A_\alpha P_{H_k} x \neq 0$. In either case it is clear that $\alpha \in \Gamma_G$ and that $P_{\mathcal{H}_j} A_\alpha P_{H_k} x \neq 0$, which implies that $s(\alpha) = j$.

Suppose now that $\alpha \in \Gamma_G$ and define $W_\alpha = [R_\alpha \otimes P_{\mathcal{H}_{s(\alpha)}} A_\alpha P_{H_{r(\alpha)}}]_{|\mathcal{M}_\sigma}$. We show first that W_α maps $\mathcal{M}_\sigma = F(E) \otimes_\sigma H$ into $\mathcal{M}_\pi = F(E) \otimes_\sigma \mathcal{H}$. We only need to check the vectors of the form (3.1). If $\gamma \in \Gamma_G$ and $x \in H_{s(\gamma)}$, then

$$W_\alpha\,(\delta_\gamma \otimes x) = \begin{cases} \delta_{\gamma\alpha} \otimes P_{\mathcal{H}_{s(\alpha)}} A_\alpha P_{H_{r(\alpha)}} x \in \mathcal{M}_\pi & \text{if } s\,(\gamma) = r\,(\alpha) \\ 0 & \text{if } s\,(\gamma) \neq r\,(\alpha). \end{cases}$$

And if $x \in H_j$, then

$$W_\alpha\,(\delta_0 \otimes x) = \begin{cases} \delta_\alpha \otimes P_{\mathcal{H}_{s(\alpha)}} A_\alpha P_{H_{r(\alpha)}} x \in \mathcal{M}_\pi & \text{if } j = r\,(\alpha) \\ 0 & \text{if } j \neq r\,(\alpha). \end{cases}$$

These two formulas prove that $W_\alpha \mathcal{M}_\sigma \subset \mathcal{M}_\pi$. We will check now that W_α intertwines $L_\sigma(g_i)$ and $L_\pi(g_i)$ for $i \leq N$. As before, we only check vectors of the form (3.1). Let $\gamma \in \Gamma_G$ and $x \in H_{s(\gamma)}$. Then $W_\alpha L_\sigma(g_i)(\delta_\gamma \otimes x) = \delta_{g_i\gamma\alpha} \otimes P_{\mathcal{H}_{s(\alpha)}} A_\alpha P_{H_{r(\alpha)}} x$ if $s(g_i) = r(\gamma)$ and $r(\alpha) = s(\gamma)$, and $W_\alpha L_\sigma(g_i)(\delta_\gamma \otimes x) = 0$ otherwise. On the other hand, $L_\pi(g_i) W_\alpha(\delta_\gamma \otimes x) = \delta_{g_i\gamma\alpha} \otimes P_{\mathcal{H}_{s(\alpha)}} A_\alpha P_{H_{r(\alpha)}} x$ if $r(\alpha) = s(\gamma)$

and $s(g_i) = r(\alpha)$ and $L_\pi(g_i)W_\alpha(\delta_\gamma \otimes x) = 0$ otherwise. Now let $x \in H_j$. Then $W_\alpha L_\sigma(g_i)(\delta_0 \otimes x) = \delta_{g_i\alpha} \otimes P_{\mathcal{H}_{s(\alpha)}} A_\alpha P_{H_{r(\alpha)}} x$ if $s(g_i) = j$ and $r(\alpha) = j$ and $W_\alpha L_\sigma(g_i)(\delta_0 \otimes x) = 0$ otherwise. On the other hand, $L_\pi(g_i)W_\alpha(\delta_0 \otimes x) = \delta_{g_i\alpha} \otimes P_{\mathcal{H}_{s(\alpha)}} A_\alpha P_{H_{r(\alpha)}} x$ if $r(\alpha) = j$ and $r(\alpha) = s(g_i)$ and $L_\pi(g_i)W_\alpha(\delta_0 \otimes x) = 0$ otherwise. This proves that $W_\alpha L_\sigma(g_i) = L_\pi(g_i)W_\alpha$ for every $i \leq N$.

We will now check that W_α intertwines $\text{Ind}\,(\sigma)(e_j)$ and $\text{Ind}\,(\sigma)(e_j)$. As before we look at vectors of the form (3.1). Let $\gamma \in \Gamma_G$ and $x \in H_{s(\gamma)}$. Then $W_\alpha[\text{Ind}\,(\sigma)(e_j)](\delta_\gamma \otimes x) = \delta_{\gamma\alpha} \otimes \xi_{s(\alpha)}$ if $r(\gamma) = j$ and $r(\alpha) = s(\gamma)$ and $W_\alpha[\text{Ind}\,(\sigma)(e_j)](\delta_\gamma \otimes x) = 0$ otherwise. Now let $x \in H_j$. Then $[\text{Ind}\,(\pi)(e_j)]W_\alpha(\delta_\gamma \otimes x) = \delta_{\gamma\alpha} \otimes \xi_{s(\alpha)}$ if $r(\alpha) = s(\gamma)$ and $r(\gamma) = j$, and $[\text{Ind}\,(\pi)(e_j)]W_\alpha(\delta_\gamma \otimes x) = 0$ otherwise. This proves that $[\text{Ind}\,(\pi)(e_j)]W_\alpha = W_\alpha[\text{Ind}\,(\sigma)(e_j)]$.

Now define $Q_j = [I \otimes P_{\mathcal{H}_j} A_\alpha P_{H_j}]_{|\mathcal{M}_\sigma}$. A similar proof will give that $Q_j\mathcal{M}_\sigma \subset \mathcal{M}_\pi$, $Q_j L_\sigma(g_i) = L_\pi(g_i)Q_j$ for $i \leq N$, and $Q_j[\text{Ind}\,(\sigma)(e_j)] = [\text{Ind}\,(\sigma)(e_j)]Q_j$ for $j \leq n$. □

It follows from Lemma 4 that

$$T = T_1 + T_2,$$

where

$$T_1 = \sum_{\alpha \in \Gamma_G} P_{\mathcal{M}_\pi}(R_\alpha \otimes P_{\mathcal{H}_{s(\alpha)}} A_\alpha P_{H_{r(\alpha)}})_{|\mathcal{M}_\sigma} + \sum_{j=1}^{n} P_{\mathcal{M}_\pi}(I \otimes P_{\mathcal{H}_j} A_0 P_{H_j})_{|\mathcal{M}_\sigma}$$

$$T_2 = \sum_{\alpha \in \Gamma_G} \sum_{k \neq r(\alpha)} P_{\mathcal{M}_\pi}(R_\alpha \otimes P_{\mathcal{H}_{s(\alpha)}} A_\alpha P_{H_k})_{|\mathcal{M}_\sigma} + \sum_{j=1}^{n} \sum_{k \neq j} P_{\mathcal{M}_\pi}(I \otimes P_{\mathcal{H}_j} A_0 P_{H_k})_{|\mathcal{M}_\sigma}.$$

An averaging argument shows that T_1 and T_2 are bounded.

Lemma 5. $\|T_1\| \leq \|T\|$.

Proof. Let

$$z=\sum_{\alpha\in\Gamma_G}(\delta_\beta\otimes b_\beta)+\sum_{j=1}^{n}(\delta_0\otimes b_j)\in F(E)\otimes_\sigma H,$$

where $b_\beta\in H_{s(\beta)}$ and $b_j\in H_j$. Then

$$\begin{aligned}&P_{\mathcal{M}_\pi}\widehat{T}_{|\mathcal{M}_\sigma}(z)\\&=\sum_{\alpha\in\Gamma_G}\delta_\alpha\otimes\left[\sum_{\substack{k=1\\k\neq r(\alpha)}}^{n}P_{\mathcal{H}_{r(\alpha)}}A_\alpha b_k+P_{\mathcal{H}_{r(\alpha)}}A_\alpha b_{r(\alpha)}+\sum_{\substack{\gamma,\beta\in\Gamma_G\\ \beta\gamma=\alpha}}P_{\mathcal{H}_{r(\gamma)}}A_\gamma b_\beta\right]\\&\quad+\sum_{j=1}^{n}\delta_0\otimes\left[\sum_{\substack{k=1\\k\neq j}}^{n}P_{\mathcal{H}_j}A_0b_k+P_{\mathcal{H}_j}A_0b_j\right].\end{aligned}$$

Let $\varepsilon=(\varepsilon_1,\varepsilon_2,\cdots,\varepsilon_n)\in\{-1,1\}^n$ be a fixed sequence of signs. Define the unitary map Φ_ε on $F(E)\otimes_\sigma H$ by

$$\Phi_\varepsilon(\delta_\gamma\otimes x)=\begin{cases}\varepsilon_j(\delta_\gamma\otimes x) & \text{if } |\gamma|=0 & \text{and } x\in H_j\\ \varepsilon_{r(\gamma)}(\delta_\gamma\otimes x) & \text{if } |\gamma|\geq 1,\gamma\in\Gamma_G, & \text{and } \in H_{s(\gamma)},\end{cases}$$

and define a similar unitary map Φ_ε on $F(E)\otimes_\sigma\mathcal{H}$. Then we have

$$\begin{aligned}&\Phi_\varepsilon\circ P_{\mathcal{M}_\pi}\widehat{T}_{|\mathcal{M}_\sigma}\circ\Phi_\varepsilon(z)\\&=\sum_{\alpha\in\Gamma_G}\delta_\alpha\otimes\left[\begin{array}{r}\sum_{\substack{k=1\\k\neq r(\alpha)}}^{n}\left(\varepsilon_k\varepsilon_{r(\alpha)}\right)P_{\mathcal{H}_{r(\alpha)}}A_\alpha b_k+\left(\varepsilon_{r(\alpha)}\varepsilon_{r(\alpha)}\right)P_{\mathcal{H}_{r(\alpha)}}A_\alpha b_{r(\alpha)}\\+\sum_{\substack{\gamma,\beta\in\Gamma_G\\ \beta\gamma=\alpha}}\left(\varepsilon_{r(\beta)}\varepsilon_{r(\alpha)}\right)P_{\mathcal{H}_{r(\gamma)}}A_\gamma b_\beta\end{array}\right]\\&\quad+\sum_{j=1}^{n}\delta_0\otimes\left[\sum_{\substack{k=1\\k\neq j}}^{n}(\varepsilon_k\varepsilon_j)P_{\mathcal{H}_j}A_0b_k+(\varepsilon_j\varepsilon_j)P_{\mathcal{H}_j}A_0b_j\right].\end{aligned}$$

Notice that if $\beta\gamma=\alpha$ then $\varepsilon_{r(\beta)}=\varepsilon_{r(\alpha)}$. This means that the signs $(\varepsilon_k\varepsilon_{r(\alpha)})$, $(\varepsilon_{r(\alpha)}\varepsilon_{r(\alpha)})$, and $(\varepsilon_j\varepsilon_j)$ in (3.6) are equal to $+1$, and $(\varepsilon_k\varepsilon_{r(\alpha)})$ and $(\varepsilon_k\varepsilon_j)$ are sometimes equal to $+1$ and sometimes equal to -1. If we look at all possible signs $\varepsilon=(\varepsilon_1,\varepsilon_2,\cdots,\varepsilon_n)\in\{-1,1\}^n$, then half the time $(\varepsilon_k \varepsilon_{r(\alpha)})$ and $(\varepsilon_k \varepsilon_j)$ will be equal to $+1$ and half the time they will be equal to -1. When we take the average over all signs, the terms that have $(\varepsilon_k \varepsilon_{r(\alpha)})$ and $(\varepsilon_k \varepsilon_j)$ in front of them disappear and (3.6) becomes

$$\begin{aligned}
&\text{Average}_{\varepsilon_i=\pm 1}\left[\Phi_\varepsilon \circ P_{\mathcal{M}_\pi} \widehat{T}_{|\mathcal{M}_\sigma} \circ \Phi_\varepsilon(z)\right] \\
&\quad = \sum_{\alpha\in\Gamma_G} \delta_\alpha \otimes \left[P_{\mathcal{H}_{r(\alpha)}} A_\alpha b_{r(\alpha)} + \sum_{\substack{\gamma,\beta\in\Gamma_G \\ \beta\gamma=\alpha}} P_{\mathcal{H}_{r(\gamma)}} A_\gamma b_\beta\right] \\
&\qquad + \sum_{j=1}^{n} \delta_0 \otimes \left[P_{\mathcal{H}_j} A_0 b_j\right].
\end{aligned}$$

Since this is equal to $T_1(z)$, we obtain that $\|T_1\| \le \|P_{\mathcal{M}_\pi} \widehat{T}_{|\mathcal{M}_\sigma}\| \le \|\widehat{T}\| = \|T\|$. □

Notice that it follows from Lemma 4 that T_1 commutes with $\{L_\sigma(g_1), \dots, L_\sigma(g_N)\}$.

Lemma 6. *For every $\gamma \in \Gamma_G$ and $x \in H_{s(\gamma)}$, $T_2(\delta_\gamma \otimes \xi_{s(\gamma)}) = 0$. Consequently, if $F_0(E) \otimes_\sigma H$ is the closed span of $\delta_\gamma \otimes x$ for $\gamma \in \Gamma_G$ and $x \in H_{s(\gamma)}$, we have that $T_{2|F_0(E)\otimes_\sigma l_2^n} = 0$. Moreover, if G has no sinks, $T_2 = 0$.*

Proof. Let $\gamma \in \Gamma_G$ and $x \in H_{s(\gamma)}$. We will prove that $T_2(\delta_\gamma \otimes x) = 0$ by checking that each term in T_2 annihilates $\delta_\gamma \otimes x$. Take first $\alpha \in \Gamma_G$ and $k \ne r(\alpha)$. Then

$$\begin{aligned}
&P_{\mathcal{M}_\pi}(R_\alpha \otimes P_{\mathcal{H}_{s(\alpha)}} A_\alpha P_{H_k})_{|\mathcal{M}_\sigma}(\delta_\gamma \otimes x) \\
&\qquad = \begin{cases} 0 & \text{if } s(\gamma) \ne k \\ P_{\mathcal{M}_\pi}\left(\delta_{\gamma\alpha} \otimes P_{\mathcal{H}_{s(\alpha)}} A_\alpha x\right) & \text{if } s(\gamma) = k. \end{cases}
\end{aligned}$$

Now, $P_{\mathcal{M}_\pi}(\delta_{\gamma\alpha} \otimes P_{\mathcal{H}_{s(\alpha)}} A_\alpha x) \ne 0$ only if $r(\alpha) = s(\gamma)$. But this implies that $r(\alpha) = k$, which is not possible. Hence $P_{\mathcal{M}_\pi}(R_\alpha \otimes$

$P_{\mathcal{H}_{s(\alpha)}} A_\alpha P_{H_k})_{|\mathcal{M}_\sigma}(\delta_\gamma \otimes x) = 0$. Fix now $j, k \leq n$ with $k \neq j$. We easily see that $P_{\mathcal{M}_\pi}(I \otimes P_{\mathcal{H}_j} A_0 P_{H_k})_{|\mathcal{M}_\sigma}(\delta_\gamma \otimes x) \neq 0$ only if $k = s(\gamma)$ and $\delta_\gamma \otimes P_{\mathcal{H}_j} A_0 P_{H_k} x \in \mathcal{M}_\pi$. But this implies that $j = s(\gamma)$, which is not possible since $k \neq j$. This proves the first part of Lemma 6.

Notice that T and T_1 intertwine $L_\sigma(g_i)$ and $L_\sigma(g_i)$, and $T = T_1 + T_2$. Hence, we have that T_2 also intertwines $L_\sigma(g_i)$ and $L_\sigma(g_i)$. That is, for each $i \leq N$, $L_\pi(g_i) T_2 = T_2 L_\sigma(g_i) = 0$. And since G has no sinks, it follows from Lemma 2 that $F(E) \otimes_\pi \mathcal{H}$ has property (2.3) and this implies that $T_2 = 0$. □

This completes the proof of Theorem 2. □

We now obtain a particular case of the duality of [**13, 18**]. We will use the rank one maps $\theta_{x,y} : H \to H$ that map y to x and $\Theta_{x,y} = I \otimes \theta_{x,y}$.

Corollary 2. *If G has no sinks, the double commutant of $\{L_\sigma(g_1), \dots, L_\sigma(g_N)\}$ is generated by $\{L_\sigma(g_i) : i \leq N\} \cup \{\mathrm{Ind}\,(\sigma)(e_j) : j \leq n\}$. Consequently, $\{L_\sigma(g_1), \dots, L_\sigma(g_N)\}'' \cong H^\infty_\sigma(E)$.*

Proof. Let $T : \mathcal{M}_\sigma \to \mathcal{M}_\sigma$ be an element of $\{L_\sigma(g_1), \dots, L_\sigma(g_N)\}''$. By Theorem 2 applied to $\pi = \sigma$, we have that T commutes with maps of the form

$$\left\{[R_\alpha \otimes A]_{|\mathcal{M}_\sigma} : \alpha \in \Gamma_G \text{ and } A = P_{H_{s(\alpha)}} A P_{H_{r(\alpha)}}\right\},$$

and

$$\left\{[I \otimes A]_{|\mathcal{M}_\sigma} : A = P_{H_j} A P_{H_j} \text{ and } j \leq n\right\}.$$

Fix $x \in H_j$. Then $T(\delta_0 \otimes x) = T\Theta_{x,x}(\delta_0 \otimes x) = \Theta_{x,x} T(\delta_0 \otimes x) = \psi_x \otimes x$ for some $\psi_x \in l_2(\mathbf{F}_N^+)$. If $x, y \in H_j$, then $T(\delta_0 \otimes y) = T\Theta_{y,x}(\delta_0 \otimes x) = \Theta_{y,x} T(\delta_0 \otimes x) = \psi_x \otimes y$, and $T(\delta_0 \otimes y) = \psi_y \otimes y$. This gives that $\psi_x = \psi_y$, and since x and y are arbitrary, we conclude that there exists $\psi_j \in l_2(\mathbf{F}_N^+)$ such that

$$T\,(\delta_0 \otimes x) = \psi_j \otimes x \quad \text{for every } x \in H_j. \tag{3.7}$$

Moreover, for $x \in H_j$, $\psi_j \otimes x = \psi_j \otimes \sigma(e_j) x = \psi_j \cdot e_j \otimes x$, and we can assume that $\psi_j = \psi_j \cdot e_j$. The map T is determined by (3.7). Indeed, let $\gamma \in \Gamma_G, x \in H_{s(\gamma)}$ and $y \in H_{r(\gamma)}$. Since $\theta_{x,y} = P_{H_{s(\alpha)}}\theta_{x,y}P_{H_{s(\alpha)}}$, we have that

$$\begin{aligned} T\left(\delta_\gamma \otimes x\right) &= T\left[R_\gamma \otimes \theta_{x,y}\right]\left(\delta_0 \otimes y\right) = \left[R_\gamma \otimes \theta_{x,y}\right] T\left(\delta_0 \otimes y\right) \\ &= \left[R_\gamma \otimes \theta_{x,y}\right]\left(\psi_{r(\gamma)} \otimes y\right) = R_\gamma\left(\psi_{r(\gamma)}\right) \otimes x. \end{aligned}$$

Fix $j \leq n$, and consider $T_j = T[\mathrm{Ind}\,(\sigma)(e_j)] : \mathcal{M}_\sigma \to \mathcal{M}_\sigma$. Since both T and $[\mathrm{Ind}\,(\sigma)(e_j)]$ belong to the double commutant of the $L_\sigma(g_i)$s, so does T_j. Moreover, T_j is determined by

$$T_j\left(\delta_0 \otimes x\right) = \begin{cases} \psi_j \otimes x & \text{if } k = j \\ 0 & \text{if } k \neq j \end{cases} \quad \text{for } x \in H_k.$$

Then it follows that T_j is the left multiplication by ψ_j. Using an argument similar to [**13**] we obtain that T_j is in the WOT closure of the span of $\{L_\sigma(g_i) : i \leq N\}$. And since $T = T_1 + \cdots + T_n$, we have that T is the WOT closure of the span of $\{L_\sigma(g_i) : i \leq N\} \cup \{\mathrm{Ind}\,(\sigma)(e_j) : j \leq n\}$ □

The commutant lifting theorem for Fock spaces can be used to obtain several versions of the commutant lifting theorem for directed graphs. We use the following version in the next section.

Theorem 3. *Suppose that G is a graph with no sinks, and let $\sigma, \pi : A \to B(H)$ be faithful $*$-representations. Suppose that $(\mathcal{H}; T_1, \dots, T_N)$ is a Hilbert module satisfying*

$$\{h \in \mathcal{H} : T_i h = 0 \text{ for every } i \leq N\} = (0) \tag{3.8}$$

and that $\Phi : F(E) \otimes_\pi H \to \mathcal{H}$ is a coisometric module map. Then, for every module map $f : F(E) \otimes_\sigma H \to \mathcal{H}$ there exists $f_1 : F(E) \otimes_\sigma H \to F(E) \otimes_\pi H$ satisfying $\|f\| = \|f_1\|$, $f = \Phi \circ f_1$, and $f_1 L_\sigma(x) = L_\pi(x) f_1$ for $x \in E$.

Proof. Our goal is to define $f_1 : F(E) \otimes_\sigma H \to F(E) \otimes_\pi H$. To do so, we first extend the diagram to $l_2(\mathbf{F}_N^+) \otimes H$.

$$
\begin{array}{ccccc}
 & & & & l_2(\mathbf{F}_N^+) \otimes H \\
 & & & & \big\downarrow P_{\mathcal{M}_\pi} \\
 & & & & F(E) \otimes_\pi H \\
 & & & & \big\downarrow \Phi \\
l_2(\mathbf{F}_N^+) \otimes H & \xrightarrow{P_{\mathcal{M}_\sigma}} & F(E) \otimes_\sigma H & \xrightarrow{\quad f \quad} & \mathcal{H}.
\end{array}
$$

Since $f \circ P_{\mathcal{M}_\sigma}$ is a module map and $\Phi \circ P_{\mathcal{M}_\pi}$ is a coisometric module map, by Popescu's commutant lifting theorem, there exists $g : l_2(\mathbf{F}_N^+) \otimes H \to l_2(\mathbf{F}_N^+) \otimes \mathcal{H}$ such that $\|g\| = \|f\|$, $g(L_i \otimes I_H) = (L_i \otimes H)g$ for every $i \leq N$ and $\Phi \circ P_{\mathcal{M}_\pi}(g)_{|\mathcal{M}_\sigma} = f$. Such a function g is of the form

$$g = \sum_{\alpha \in \mathbf{F}_N^+} R_\alpha \otimes A_\alpha = \sum_{\alpha \in \mathbf{F}_N^+} \sum_{j=1}^{n} \sum_{k=1}^{n} R_\alpha \otimes P_{H'_j} A_\alpha P_{H_k}$$

for some $A_\alpha : H \to H$.

The compression $P_{\mathcal{M}_\pi}(g)_{|\mathcal{M}_\sigma} : F(E) \otimes_\sigma H \to F(E) \otimes_\pi H$ satisfies $\|P_{\mathcal{M}_\pi}(g)_{|\mathcal{M}_\sigma}\| = \|f\|$ and $\Phi \circ [P_{\mathcal{M}_\pi}(g)_{|\mathcal{M}_\sigma}] = f$, but it doesn't necessarily intertwine the $L_\sigma(g_i)$s and the $L_\pi(g_i)$s. (According to Theorem 1, this would be formal if $F(E) \otimes_\sigma H$ and $F(E) \otimes_\pi H$ were invariant under the $(R_i \otimes I_H)^*$ and $(R_i \otimes I_{\mathcal{H}})^*$ but they are not).

Similarly as in Lemma 4, we have that for $\alpha \in \mathbf{F}_N^+$, $P_{\mathcal{M}_\pi}[R_\alpha \otimes P_{H'_j} A_\alpha P_{H_k}]_{|\mathcal{M}_\sigma}$ is nonzero only if $\alpha \in \Gamma_G$ and $j = s(\alpha)$. Moreover, maps of the form

$$\Big[R_\alpha \otimes P_{H'_{s(\alpha)}} A_\alpha P_{H_{r(\alpha)}}\Big]_{|\mathcal{M}_\sigma} \quad \text{for } \alpha \in \Gamma_G,$$

and

$$\Big[I \otimes P_{H'_j} A_\alpha P_{H_j}\Big]_{|\mathcal{M}_\sigma} \quad \text{for } j \leq n$$

intertwine $L_\sigma(g_i)$ with $L_\pi(g_i)$ and $\mathrm{Ind}\,(\sigma)(a)$ with $\mathrm{Ind}\,(\pi)(a)$.

Then we have that

$$P_{\mathcal{M}_\pi} g_{|\mathcal{M}_\sigma} = f_1 + f_2,$$

where

$$f_1 = \sum_{\alpha\in\Gamma_G} P_{\mathcal{M}_\pi}(R_\alpha \otimes P_{H'_{s(\alpha)}} A_\alpha P_{H_{r(\alpha)}})_{|\mathcal{M}_\sigma} + \sum_{j=1}^{n} P_{\mathcal{M}_\pi}(I \otimes P_{H'_j} A_0 P_{H_j})_{|\mathcal{M}_\sigma}$$

$$f_2 = \sum_{\alpha\in\Gamma_G} \sum_{k\neq r(\alpha)} P_{\mathcal{M}_\pi}(R_\alpha \otimes P_{H'_{s(\alpha)}} A_\alpha P_{H_k})_{|\mathcal{M}_\sigma} + \sum_{j=1}^{n} \sum_{k\neq j} P_{\mathcal{M}_\pi}(I \otimes P_{H'_j} A_0 P_{H_k})_{|\mathcal{M}_\sigma}.$$

The maps f_1 and f_2 are bounded and $\|f_1\| \leq \|f\|$ by Lemma 5. In general, we do not know if $f_2 = 0$, but we will show that $\Phi \circ f_2 = 0$. This implies that $\Phi \circ f_1 = f$. The map $f_1 : F(E) \otimes_\sigma H \to F(E) \otimes_\pi H$ satisfies all conditions of Theorem 3.

Similarly, as in Lemma 6, we check that for every $\gamma \in \Gamma_G$ and $x \in H_{s(\gamma)}$, $f_2(\delta_\gamma \otimes x) = 0$. This means that $f_{2|F_0(E)\otimes_\sigma H} = 0$. Since f and $\Phi \circ f_1$ intertwine $L_\sigma(g_i)$ with T_i, we have that $\Phi \circ f_2$ also intertwines $L_\sigma(g_i)$ with T_i. Let $z \in F(E) \otimes_\sigma H$. For every $i \leq N$, $T_i \circ \Phi \circ f_2(z) = \Phi \circ f_2 \circ L_\sigma(g_i)(z) = 0$, because $L_\sigma(g_i)(z) \in F_0(E) \otimes_\sigma H$. By (3.8) we have that $\Phi \circ f_2(z) = 0$, which is what we wanted to prove. □

The next example shows that condition (3.8) is necessary.

Example 7. Let G be a graph with no sinks, and suppose that (T, π) is a $C_{\cdot 0}$ completely contractive covariant representation of $E = X(G)$ in $B(\mathcal{H})$ that does not satisfy property (3.8). Then there exists a nonzero $h \in H$ such that $T(g_i)h = 0$ for every $i \leq N$. Find $j \leq n$ such that $h \neq \pi(e_j)h$, and define $f : F(E) \otimes_\sigma H \to H$ by

$$f(\delta_0 \otimes \xi_k) = \begin{cases} h & \text{if } k = j \\ 0 & \text{if } k \neq j, \end{cases}$$

and $f(\delta_\gamma \otimes \xi_{s(\gamma)}) = 0$ for $\gamma \in \Gamma_G$. We easily check that f is bounded and that $fL_\sigma(x) = T(x)f = 0$ for every $x \in E$. Moreover, $h = f[\text{Ind}\,(\sigma)(e_j)](\delta_0 \otimes \xi_j)$, but $\pi(e_j)f(\delta_0\xi_j) = \pi(e_j)h \neq h$.

Property (3.8) is not necessary if we include the projections in Hilbert modules. If (T, σ) is a $C_{\cdot 0}$ completely contractive covariant representation of $E = X(G)$ in $B(H)$, we look at the Hilbert module: $(H; L_\sigma(g_1), \ldots, L_\sigma(g_1);\ \text{Ind}\,(\sigma)(e_1), \ldots,\ \text{Ind}\,(\sigma)(e_1))$. The following is a proof of a particular case of the general commutant lifting theorem of Muhly and Solel [**16**].

Theorem 4. *Suppose that (T_1, π_1) and (T_2, π_2) are $C_{\cdot 0}$ completely contractive covariant representations of $E = X(G)$ in $B(H_1)$ and $B(H_2)$, respectively, and that $\Phi : H_2 \to H_1$ is a coisometric module map. Then, if $\sigma : A \to B(H)$ is a faithful $*$-representation and $f : F(E) \otimes_\sigma H \to H_1$ is a module map, then there exists a module map $\widehat{f} : F(E) \otimes_\sigma H \to H_2$ such that $\Phi \circ \widehat{f} = f$ and $\|\widehat{f}\| = \|f\|$. That is, the following diagram commutes*

$$\begin{array}{ccc} & & H_2 \\ & \overset{\widehat{f}}{\nearrow} & \big\downarrow \Phi \\ F(E) \otimes_\sigma H & \xrightarrow{\ f\ } & H_1. \end{array}$$

Proof. Let $K_2 : H_2 \to F(E) \otimes_{\pi_2} H_2$ be the Poisson kernel of (T_2, π_2). and define $\Psi : F(E) \otimes_{\pi_2} H_2 \to H_1$ by $\Psi = \Phi \circ K_2^*$. Following the proof of Theorem 2, we find $f_1, f_2 : F(E) \otimes_\sigma H \to F(E) \otimes_{\pi_2} H_2$ of the form

$$\begin{aligned} f_1 = &\sum_{\alpha \in \Gamma_G} P_{\mathcal{M}_\pi}(R_\alpha \otimes P_{H_{s(\alpha)}} A_\alpha P_{H_{r(\alpha)}})_{|\mathcal{M}_\sigma} \\ &+ \sum_{j=1}^{n} P_{\mathcal{M}_\pi}(I \otimes P_{H_j} A_0 P_{H_j})_{|\mathcal{M}_\sigma} \end{aligned}$$

$$f_2 = \sum_{\alpha\in\Gamma_G}\sum_{k\neq r(\alpha)} P_{\mathcal{M}_\pi}(R_\alpha \otimes P_{H_{s(\alpha)}} A_\alpha P_{H_k})_{|\mathcal{M}_\sigma}$$
$$+\sum_{j=1}^{n}\sum_{k\neq j} P_{\mathcal{M}_\pi}(I \otimes P_{H_j} A_0 P_{H_k})_{|\mathcal{M}_\sigma},$$

such that $\Psi\circ(f_1+f_2)=f$.

Notice that f and $\Psi\circ f_1$ are module maps, and hence $\Psi\circ f_2$ is also a module map. We need to prove that $\Psi\circ f_2=0$. Since $f_{2|F(E)\otimes_\sigma H}=0$, it is enough to prove that $\Psi\circ f_2(\delta_0\otimes h)=0$ for $h\in H_k$ and $k\le n$. So fix $k\le n$ and take $h\in H_k$. Then

$$\begin{aligned} f_2\left(\delta_0\otimes h\right) &= \sum_{\substack{\alpha\in\Gamma_G\\ r(\alpha)\neq k}} \delta_\alpha\otimes P_{H_{s(\alpha)}}A_\alpha h + \sum_{\substack{j=1\\ j\neq k}}^{n}\delta_0\otimes P_{H_j}A_0 h \\ &= \sum_{\substack{\alpha\in\Gamma_G\\ r(\alpha)\neq k}} \left[\text{Ind }(\sigma)\left(e_{r(\alpha)}\right)\right]\left(\delta_\alpha\otimes P_{H_{s(\alpha)}}A_\alpha h\right) \\ &\quad + \sum_{\substack{j=1\\ j\neq k}}^{n}\left[\text{Ind }(\sigma)\left(e_j\right)\right]\left(\delta_0\otimes P_{H_j}A_0 h\right), \end{aligned}$$

and

$$\begin{aligned} \Psi\circ f_2\left(\delta_0\otimes h\right) &= \sum_{\substack{\alpha\in\Gamma_G\\ r(\alpha)\neq k}} \pi_1\left(e_{r(\alpha)}\right)\Psi\left(\delta_\alpha\otimes P_{H_{s(\alpha)}}A_\alpha h\right) \\ &\quad + \sum_{\substack{j=1\\ j\neq k}}^{n}\pi_1\left(e_j\right)\Psi\left(\delta_0\otimes P_{H_j}A_0 h\right). \end{aligned} \tag{3.9}$$

On the other hand,

$$\Psi\circ f_2\left(\delta_0\otimes h\right)=\Psi\circ f_2\left[\text{Ind }(\sigma)\left(e_k\right)\right]\left(\delta_0\otimes h\right)=\pi_1\left(e_k\right)\Psi\circ f_2\left(\delta_0\otimes h\right).$$

Since $\pi_1(e_k)\pi_1(e_{r(\alpha)})=0$ if $r(\alpha)\neq k$ and $\pi_1(e_k)\pi_1(e_j)=0$ if $k\neq j$, we see from (3.9) that $\pi_1(e_k)\Psi\circ f_2(\delta_0\otimes h)=0$, which is what we wanted to prove. To finish, define $\widehat{f}=K_2^*\circ f_1$. $\square$

Suppose that we work on a category of Hilbert modules. A Hilbert module H is called strongly orthogonally projective if, whenever there exist Hilbert modules H_1, H_2, a surjective coisometric module map $\Phi : H_2 \to H_1$ and a module map $f : H \to H_1$, there exists a module map $f_1 : H \to H_2$ satisfying $\|f_1\| = \|f\|$ and $f_1 \circ \Phi = f$

$$\begin{array}{ccc} & & H_2 \\ & \overset{f_1}{\nearrow} & \big\downarrow \Phi \\ H & \xrightarrow{\ f\ } & H_1. \end{array}$$

This property was introduced by Douglas and Paulsen [**7**] with the name "hypoprojective," and it was later renamed "strongly orthogonally projective" by Muhly and Solel.

Theorem 4 says that the Hilbert modules

$$\left(F\left(E\right)\otimes_\sigma H; L_\sigma\left(g_1\right), \dots , L_\sigma\left(g_N\right); \mathrm{Ind}\ (\sigma)\left(e_1\right), \dots , \mathrm{Ind}\ (\sigma)\left(e_n\right)\right)$$

are strongly orthogonally projective in the category of $C_{\cdot 0}$ completely contractive covariant representations of $E = X(G)$. The following proposition characterizes the projective elements in this category:

Proposition 8. *Let G be a graph with no sinks. If (T, σ) is a $C_{\cdot 0}$ completely contractive covariant representation of $E = X(G)$ on $B(H)$ such that H is strongly orthogonally projective, then H is isomorphic $F(E) \otimes_\sigma \mathcal{L}$ for some $\mathcal{L} \subset H$.*

Proof. Let $K : H \to F(E) \otimes_\sigma H$ be the Poisson kernel of (T, σ). Since H is isomorphic to $K(H)$, we have that $K(H)$ is strongly orthogonally projective too. Consider the diagram

$$\begin{array}{ccc} & & F\left(E\right)\otimes_\sigma H \\ & & \big\downarrow P_{K(H)} \\ K(H) & \xrightarrow{\ \mathrm{Id}\ } & K\left(H\right), \end{array}$$

where $P_{K(H)}$ is the orthogonal projection and Id is the identity. Recall that $K(H)$ is $*$-invariant and that $P_{K(H)}$ is a module map. Then there exists a contractive module map $\Phi : K(H) \to F(E) \otimes_\sigma H$ such that $P_{K(H)} \circ \Phi = \mathrm{Id}$. The condition $\|\Phi\| = 1$ forces Φ to be the identity, and it follows that $K(H)$ is a submodule of $F(E) \otimes_\sigma H$. This implies that $K(H)$ is invariant under the $L_\sigma(g_i)$s and $L_\sigma(g_i)^*$s for $i \leq N$. In particular, $K(H)$ is invariant under $[I - L_\sigma(g_1)L_\sigma(g_1)^* - \cdots - L_\sigma(g_1)L_\sigma(g_1)^*]$. Since $[I - \sum_{i\leq n} L_\sigma(g_i)L_\sigma(g_i)^*]$ is the orthogonal projection in $F(E) \otimes_\sigma H$ onto the subspace $\delta_0 \otimes H$, it follows that

$$\left[I - L_\sigma(g_1)\, L_\sigma(g_1)^* - \cdots - L_\sigma(g_1)\, L_\sigma(g_1)^*\right] K(H) = \delta_0 \otimes \mathcal{L}$$

for some $\mathcal{L} \subset H$.

Arguing as in the proof of Theorem 7 of the next section, we see that $\mathcal{L}$ is the "right slice" of $K(H)$. This implies that $K(H) \subset F(E) \otimes_\sigma \mathcal{L}$. On the other hand, we have that $F(E) \otimes_\sigma \mathcal{L} \subset K(H)$ because $K(H)$ is invariant under the $L_\sigma(g_i)$s. Therefore, we have proved that H is isomorphic to $F(E) \otimes_\sigma \mathcal{L}$. □

3.4. Hilbert module formulation. In this section assume that the graph G has no sinks. Let $\sigma : l_\infty^n \to B(H)$ be a faithful $*$-representation. We view $F(E) \otimes_\sigma H$ as the Hilbert module $(F(E) \otimes_\sigma H; L_\sigma(g_1), \dots, L_\sigma(g_1);\ \mathrm{Ind}(\sigma)(e_1), \dots, \mathrm{Ind}(\sigma)(e_1))$, but sometimes we ignore the projections and look only at the Hilbert module $(F(E) \otimes_\sigma H; L_\sigma(g_1), \dots, L_\sigma(g_1))$. Suppose that $\mathcal{M} \subset F(E) \otimes_\sigma H$. We say that $\mathcal{M}$ is an *$*$-submodule* of $F(E) \otimes_\sigma H$ if

$$L_\sigma(g_i)^*\, \mathcal{M} \subset \mathcal{M} \quad \text{for every } i \leq N,$$

and

$$[\mathrm{Ind}\ (\sigma)\,(e_j)]^*\, \mathcal{M} \subset \mathcal{M} \quad \text{for every } j \leq n.$$

And we say that $\mathcal{M}$ is a *submodule* of $F(E) \otimes_\sigma H$ if

$$L_\sigma(g_i)\, \mathcal{M} \subset \mathcal{M} \quad \text{for every } i \leq N,$$

and

$$[\mathrm{Ind}\ (\sigma)\,(e_j)]\, \mathcal{M} \subset \mathcal{M} \quad \text{for every } j \leq n.$$

If $\mathcal{M}$ is either a submodule or $*$-submodule of $F(E) \otimes_\sigma H$, define $T : E \to B(\mathcal{M})$ by $T(x) = P_\mathcal{M} L_\sigma(x)_{|\mathcal{M}}$ and $\widehat{\sigma} : l_\infty^n \to B(\mathcal{M})$ by $\widehat{\sigma}(a) = P_\mathcal{M}[\mathrm{Ind}\,(\sigma)(a)]_{|\mathcal{M}}$, and notice that $(T, \widehat{\sigma})$ is a $C_{\cdot 0}$-completely contractive covariant representation.

The $*$-invariant submodules play a very important role. Proposition 2 can be restated to say

Proposition 9. *Suppose that (T, σ) is a $C_{\cdot 0}$ completely contractive covariant representation of $E = X(G)$ on $B(H)$. Then the Hilbert module $(H; T(x), x \in E; \sigma(a), a \in l_\infty^n)$ is isomorphic to an $*$-submodule of $F(E) \otimes_\sigma H$.*

If $\mathcal{M}$ is a submodule, the representation $(T, \widehat{\sigma})$ is isometric. Then Δ^2 is a projection and $\Delta^2 = \Delta$. The set $\Delta(\mathcal{M}) = \mathcal{L}$ is the wandering subspace. The Poisson kernel of T is an isometry $K : \mathcal{M} \to F(E) \otimes_{\hat{\sigma}} \mathcal{M}$. However, since Δ is the orthogonal projection onto $\mathcal{L}$, we see from (2.1) that $K(\mathcal{M}) \subset F(E) \otimes_{\hat{\sigma}} \mathcal{L}$.

Proposition 10. *Suppose that $\mathcal{M}$ is a submodule of $F(E) \otimes_\sigma H$ with wandering subspace $\mathcal{L}$ and Poisson kernel $K : \mathcal{M} \to F(E) \otimes_{\hat{\sigma}} \mathcal{M}$. Then $K(\mathcal{M}) = F(E) \otimes_{\hat{\sigma}} \mathcal{L}$. Consequently, the invariant subspaces of $F(E) \otimes_\sigma H$ are of the form $F(E) \otimes_{\hat{\sigma}} \mathcal{L}$.*

Proposition 10 was proved by Muhly and Solel in the general setting of $C_{\cdot 0}$-completely contractive covariant representations [**17**] and by Kribs and Power for free semi-groupoid algebras in [**13**]. We sketch a proof of this result using Poisson kernels. The same proof works in the general setting of Muhly and Solel. Notice that, for every $i \leq N$, $(T_i)^* \Delta^2 = \Delta^2 T_i = 0$. Hence, it follows that if $\gamma \in \Gamma_G$, $|\gamma| = j$ and $z \in \mathcal{L}$, $(T_\gamma)^* z = 0$ and $\Delta T_\gamma z = 0$. Recall that $K^*(\delta_\gamma \otimes z) = T_\gamma z$. Then

$$\begin{aligned} K\left(T_\gamma z\right) &= \delta_0 \otimes \Delta\left(T_\gamma z\right) + \sum_{k=1}^{\infty} \left(I_{E^{\otimes k}} \otimes \Delta\right) \left(\widetilde{T^k}\right)^* \left(T_\gamma z\right) \\ &= \left(\widetilde{T^j}\right)^* \left(T_\gamma z\right) = \left(\widetilde{T^j}\right)^* \widetilde{T^j}\left(\delta_\gamma \otimes z\right) = \delta_\gamma \otimes z. \end{aligned}$$

This shows that $F(E) \otimes_{\hat{\sigma}} \mathcal{L} \subset K(\mathcal{M})$, which is what we wanted to show.

We can reformulate the last two propositions in the following way:

Theorem 5. *Suppose that (T,σ) is a $C_{\cdot 0}$ completely contractive covariant representation of $E = X(G)$ on $B(H)$. Then there exists a Hilbert space $\mathcal{L}$, an isometric module map $\Phi_1 : F(E) \otimes_\sigma \mathcal{L} \to F(E) \otimes_\sigma H$, and a surjective coisometric module map $\Phi_2 : F(E) \otimes_\sigma H \to H$, such that the following is a short exact sequence*

$$0 \longrightarrow F(E) \otimes_{\hat{\sigma}} \mathcal{L} \xrightarrow{\Phi_2} F(E) \otimes_\sigma H \xrightarrow{\Phi_1} H \longrightarrow 0.$$

Proof. Suppose that $K_1 : H \to F(E) \otimes_\sigma H$ is the Poisson kernel of H, and let $\Phi_1 = K_1^*$. Since $\ker \Phi_1$ is a submodule of $F(E) \otimes_\sigma H$, it is of the form $F(E) \otimes_{\hat{\sigma}} \mathcal{L}$ for some $\mathcal{L} \subset F(E) \otimes_\sigma H$. The map that sends $F(E) \otimes_{\hat{\sigma}} \mathcal{L}$ onto $\ker \Phi_1$ is an isometric module map. □

The sequence $0 \longrightarrow F(E) \otimes_{\hat{\sigma}} \mathcal{L} \xrightarrow{\Phi_2} F(E) \otimes_\sigma H \xrightarrow{\Phi_1} H \longrightarrow 0$ is called a projective resolution of Hilbert module H. From Proposition 8, we obtain the following theorem:

Theorem 6. *If (T_1,σ) is a $C_{\cdot 0}$ completely contractive representation of $E = X(G)$ on $B(H)$, (T_2,π) is a $C_{\cdot 0}$ completely contractive representation of $E = X(G)$ on $B(\mathcal{H})$ and $f : H \to \mathcal{H}$ is a module map, then there exist module maps $f_1 : F(E) \otimes_\sigma H_1 \to F(E) \otimes_\pi \mathcal{H}_1$ and $f_2 : F(E) \otimes_{\hat{\sigma}} H_2 \to F(E) \otimes_{\hat{\pi}} \mathcal{H}_2$ satisfying $\|f_2\| \le \|f_1\| \le \|f\|$ and such that the following diagram commutes*:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & F(G_2) \otimes_{\hat{\sigma}} H_2 & \xrightarrow{\Phi_2} & F(G_1) \otimes_\sigma H_1 & \xrightarrow{\Phi_1} & H & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle f_2} & & \downarrow{\scriptstyle f_1} & & \downarrow{\scriptstyle f} & & \\
0 & \longrightarrow & F(G_2) \otimes_{\hat{\pi}} \mathcal{H}_2 & \xrightarrow{\Psi_2} & F(G_1) \otimes_\pi \mathcal{H}_1 & \xrightarrow{\Psi_1} & \mathcal{H} & \longrightarrow & 0.
\end{array}$$

Proof. Since $f \circ \Phi_1 : F(G_1) \otimes_\sigma H_1 \to \mathcal{H}$ is a module map and $\Psi_1 : F(G_1) \otimes_\pi \mathcal{H}_1 \to \mathcal{H}$ is a coisometric module map, it follows from Theorem 4 that there exists $f_1 : F(G_1) \otimes_\sigma H_1 \to F(G_1) \otimes_\pi \mathcal{H}_1$ that satisfies $\|f_1\| = \|f \circ \Phi_1\| = \|f\|$ and $f \circ \Phi_1 = \Psi_1 \circ f_1$.

We claim that f_1 maps $\ker \Phi_1$ into $\ker \Psi_1$. To see this take $x \in \ker \Phi_1$ and compute $\Psi_1(f_1(x)) = f(\Phi_1(x)) = 0$. Then we have the diagram

$$\begin{array}{ccc} F(G_2) \otimes_{\hat{\sigma}} H_2 & \xrightarrow{\Phi_2} & \ker \Phi_1 \\ & & \Big\downarrow f_{1|\ker \Phi_1} \\ F(G_2) \otimes_{\hat{\pi}} \mathcal{H}_2 & \xrightarrow{\Psi_2} & \ker \Phi_2. \end{array}$$

Applying Theorem 4 again, we see that since $f_{1|\ker \Phi_1} \circ \Phi_2 : F(G_2) \otimes_{\hat{\sigma}} H_2 \to \ker \Phi_2$ is a module map and $\Psi_2 : F(G_2) \otimes_{\hat{\pi}} \mathcal{H}_2 \to \ker \Phi_2$ is a surjective coisometric module map, there exists a module map $f_2 : F(G_2) \otimes_{\hat{\sigma}} H_2 \to F(G_2) \otimes_{\hat{\pi}} \mathcal{H}_2$ satisfying the required properties. □

We need the following definition:

Definition 2. If $\mathcal{E} \subset F(E) \otimes_\sigma l_2$, the "right slice" of $\mathcal{E}$ is the smallest Hilbert space $\mathcal{H} \subset l_2$ such that $\mathcal{E} \subset F(E) \otimes_\sigma \mathcal{H}$.

It is easy to see that the right slice of $\mathcal{E}$ is the closed span of vectors of the form $\{x_j : j \leq n\} \cup \{x_\gamma : \gamma \in \Gamma_G\}$ where $x = \sum_{j=1}^n \delta_0 \otimes x_j + \sum_{\gamma \in \Gamma_G} \delta_\gamma \otimes x_\gamma \in \mathcal{E}$.

If $\Phi : F(E) \otimes_\sigma \mathcal{H} \to H$ is a surjective coisometric module map, then $\Phi^*(H)$ is an $*$-invariant submodule of $F(E) \otimes_\sigma \mathcal{H}$ isomorphic to H. We say that Φ is a *minimal resolution* of H if $\mathcal{H}$ is the right slice map of $\Phi^*(H)$.

We now prove a rigidity result:

Theorem 7. *Suppose that $\mathcal{M}_1 \subset F(E) \otimes_\sigma H$ and $\mathcal{M}_2 \subset F(E) \otimes_\pi \mathcal{H}$ are submodules such that $\mathcal{M}_1^\perp$ and $\mathcal{M}_2^\perp$ are isomorphic, H is the right slice of $\mathcal{M}_1^\perp$, and $\mathcal{H}$ is the right slice of $\mathcal{M}_2^\perp$. Then $\mathcal{M}_1$ and $\mathcal{M}_2$ are isomorphic via a map of the form $I \otimes U$, where $U : H \to \mathcal{H}$ is unitary, and $UH_j = \mathcal{H}_j$ for $j \leq n$. The subspaces H_j and $\mathcal{H}_j$ correspond to the decomposition $H = H_1 \oplus \cdots \oplus H_n$ induced by $\sigma : l_\infty^n \to B(H)$ and the decomposition $\mathcal{H} = \mathcal{H}_1 \oplus \cdots \oplus \mathcal{H}_n$ induced by $\pi : l_\infty^n \to B(\mathcal{H})$.*

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & \mathcal{M}_1 & \longrightarrow & F(E)\otimes_\sigma H & \xrightarrow{P_{\mathcal{M}_1^\perp}} & \mathcal{M}_1^\perp & \longrightarrow & 0 \\
 & & & & \big\downarrow{\scriptstyle f_1} & & \big\downarrow{\scriptstyle u} & & \\
0 & \longrightarrow & \mathcal{M}_2 & \longrightarrow & F(E)\otimes_\pi \mathcal{H} & \xrightarrow{P_{\mathcal{M}_2^\perp}} & \mathcal{M}_2^\perp & \longrightarrow & 0 \\
 & & & & \big\downarrow{\scriptstyle f_2} & & \big\downarrow{\scriptstyle u^{-1}} & & \\
0 & \longrightarrow & \mathcal{M}_1 & \longrightarrow & F(E)\otimes_\sigma H & \xrightarrow{P_{\mathcal{M}_1^\perp}} & \mathcal{M}_1^\perp & \longrightarrow & 0.
\end{array}
\tag{3.10}
$$

Proof. Suppose that there exists a $u : \mathcal{M}_1^\perp \to \mathcal{M}_2^\perp$ such that u and u^{-1} are isometric module maps. Since $uP_{\mathcal{M}_1^\perp} : F(E)\otimes_\sigma H \to \mathcal{M}_2^\perp$ is a module map and $P_{\mathcal{M}_2^\perp} : F(E)\otimes_\pi \mathcal{H} \to M_2^\perp$ is a surjective coisometric module map, it follows from Theorem 3 that there exists a module map $f_1 : F(E)\otimes_\sigma H \to F(E)\otimes_\pi \mathcal{H}$ such that $\|f_1\| = 1$ and $P_{\mathcal{M}_2^\perp} f_1 = uP_{\mathcal{M}_1^\perp}$. Similarly, there exists a module map $f_2 : F(E)\otimes_\pi \mathcal{H} \to F(E)\otimes_\sigma H$ such that $\|f_2\| = 1$ and $P_{\mathcal{M}_1^\perp} f_2 = u^{-1}P_{\mathcal{M}_2^\perp}$.

The maps f_1 and f_2 are of the form

$$
\begin{aligned}
f_1 &= \sum_{j=1}^n I\otimes A_j + \sum_{\gamma\in\Gamma_G} R_\gamma \otimes A_\gamma \\
f_2 &= \sum_{j=1}^n I\otimes B_j + \sum_{\gamma\in\Gamma_G} R_\gamma \otimes B_\gamma,
\end{aligned}
$$

where $A_j = P_{\mathcal{H}_j} A_j P_{H_j}$ for $j \le n$, $A_\gamma = P_{\mathcal{H}_{s(\gamma)}} A_\gamma P_{H_{r(\gamma)}}$ for $\gamma \in \Gamma_G$, $B_j = P_{H_j} B_j P_{\mathcal{H}_j}$ for $j \le n$, and $B_\gamma = P_{H_{s(\gamma)}} B_\gamma P_{\mathcal{H}_{r(\gamma)}}$ for $\gamma \in \Gamma_G$. Since $\|f_1\| = \|f_2\| = 1$, it follows that $\|A_j\| \le 1, \|B_j\| \le 1$ for $j \le n$, and $\|A_\gamma\| \le 1$, $\|B_\gamma\| \le 1$ for $\gamma \in \Gamma_G$. We verify that

$$
\begin{aligned}
f_2 f_1 = {} & \left(\sum_{j=1}^n I\otimes B_j A_j\right) \\
& + \sum_{\gamma\in\Gamma_G} R_\gamma \otimes \left(B_\gamma A_{r(\gamma)} + B_{s(\gamma)} A_\gamma + \sum_{\substack{\beta\alpha=\gamma \\ \alpha,\beta\in\Gamma_G}} B_\alpha A_\beta \right).
\end{aligned}
\tag{3.11}
$$

We will check that for every $z \in \mathcal{M}_1^\perp$, $f_2 f_1(z) = z$. Notice that $\|z\| = \|u^{-1}uz\| = \|P_{\mathcal{M}_1^\perp} f_2 P_{\mathcal{M}_2^\perp} f_1(z)\| \leq \|f_2 P_{\mathcal{M}_2^\perp} f_1(z)\| \leq \|P_{\mathcal{M}_2^\perp} f_1(z)\| \leq \|f_1(z)\| \leq \|z\|$. Then it follows that $\|P_{\mathcal{M}_2^\perp} f_1(z)\| = \|f_1(z)\|$, which implies that $P_{\mathcal{M}_2^\perp} f_1(z) = f_1(z)$. Moreover, we have that $\|f_2 f_1 z\| = \|P_{\mathcal{M}_1^\perp} f_2 f_1 z\|$, and this implies that $P_{\mathcal{M}_1^\perp} f_2 f_1(z) = f_2 f_1(z)$. Combining these two equalities we conclude that $f_2 f_1(z) = z$.

Let

$$z = \sum_{j=1}^{n} \delta_0 \otimes x_j + \sum_{\gamma \in \Gamma_G} \delta_\gamma \otimes x_\gamma \in \mathcal{M}_1^\perp. \tag{3.12}$$

Since H is the right slice of $\mathcal{M}_1^\perp$, H is the closed span of the vectors of the form x_j, $j \leq n$, and x_γ, $\gamma \in \Gamma_G$, where $x \in \mathcal{M}_1^\perp$. However, a moment's thought indicates that H is the span of vectors of the form x_j, $j \leq n$, for $x \in \mathcal{M}_1^\perp$. Indeed, if $\gamma \in \Gamma_G$, then $L_\sigma(\gamma)^* z = \delta_0 \otimes x_\gamma + \sum_{i=1, i \neq s(\gamma)}^{n} \delta_0 \otimes y_i +$ [higher order terms]. And since $\mathcal{M}_1^\perp$ is $*$-invariant, $L_\sigma(\gamma)^* z \in \mathcal{M}_1^\perp$.

We now claim that

$$B_j A_j x = x \quad \text{for every } j \leq n \text{ and every } x \in H_j. \tag{3.13}$$

By the previous paragraph, it is enough to take $x_j \in H_j$ where x_j comes from (3.12). From (3.11) we have that $f_2 f_1(z) = \sum_{j=1}^{n} \delta_0 \otimes B_j A_j x_j +$ [higher order terms] and since this is equal to (3.12), we conclude that $B_j A_j x_j = x_j$ for $j \leq n$.

Since A_j and B_j are contractions, it follows from (3.13) that A_j is a partial isometry with initial space H_j. Indeed, if $x \in H_j$, $\|x\| = \|B_j A_j x\| \leq \|A_j x\| \leq \|x\|$. Similarly we obtain that B_j is a partial isometry with initial space $\mathcal{H}_j$ and final space H_j and that A_j has final space $\mathcal{H}_j$.

Now we will prove that, for every $\alpha \in \Gamma_G$, $A_\alpha = 0$. Recall that $A_\alpha = P_{\mathcal{H}_{s(\alpha)}} A_\alpha P_{H_{r(\alpha)}}$, and take $x \in H_{r(\alpha)}$. Then $f_1(\delta_0 \otimes x) = \delta_0 \otimes A_{r(\alpha)} x + \sum_{\gamma \in \Gamma_G} \delta_\gamma \otimes A_\gamma x$, and hence

$$\begin{aligned}
\|x\|^2 = \left\|A_{r(\alpha)} x\right\|^2 &= \left\|\delta_0 \otimes A_{r(\alpha)} x\right\|^2 \\
&\leq \left\|\delta_0 \otimes A_{r(\alpha)} x\right\|^2 + \sum_{\gamma \in \Gamma_G} \|\delta_\gamma \otimes A_\gamma x\|^2 \\
&= \|f_1(\delta_0 \otimes x)\|^2 \leq \|\delta_0 \otimes x\|^2 = \|x\|^2 .
\end{aligned}$$

This implies that $A_\gamma x = 0$ for every $\gamma \in \Gamma_G$. In particular, we have that $A_\alpha x = 0$.

Similarly, we prove that, for $\gamma \in \Gamma_G$, $B_\gamma = 0$, and we conclude that

$$f_1 = \sum_{j=1}^{n} I \otimes A_j = I \otimes \left(\sum_{j=1}^{n} A_j \right) = I \otimes U, \quad \text{and} \quad f_2 = \sum_{j=1}^{n} I \otimes B_j.$$

The map $U : H \to \mathcal{H}$ defined by $U = A_1 + A_2 + \cdots + A_n$ is unitary. To finish the proof, we check that f_1 maps $\mathcal{M}_1$ into $\mathcal{M}_2$ and that f_2 maps $\mathcal{M}_2$ into $\mathcal{M}_1$. But this follows from diagram (3.10). □

Remark 1. Notice that the proof of Theorem 7 gives that $F(E) \otimes_\sigma H$ is isomorphic to $F(E) \otimes_\pi \mathcal{H}$. The map $f_1 = I \otimes U : F(E) \otimes_\sigma H \to F(E) \otimes_\pi \mathcal{H}$ is a unitary module map.

Two module maps $\Phi : H_2 \to H_1$ and $\Psi : H_4 \to H_3$ are unitarily equivalent if unitary module maps $U_1 : H_1 \to H_3$ and $U_2 : H_2 \to H_4$ exist such that $U_1 \circ \Phi = \Psi \circ U_2$.

Definition 3. Suppose that (T, σ) is a $C_{\cdot 0}$ completely contractive representation of $E = X(G)$ on $B(H)$ with minimal resolution $0 \longrightarrow F(E) \otimes_{\hat{\sigma}} H_2 \xrightarrow{\Phi_2} F(E) \otimes_\sigma H_1 \xrightarrow{\Phi_1} H \longrightarrow 0$. The map $\Phi_2 : F(E) \otimes_{\hat{\sigma}} H_2 \longrightarrow F(E) \otimes_\sigma H_1$ is called the *characteristic function* of H.

The next result indicates that the characteristic function is essentially unique.

Theorem 8. *Suppose that (T, σ) and (T', π) are $C_{\cdot 0}$ completely contractive representations of $E = X(G)$ on $B(H)$ and $B(\mathcal{H})$, respectively, with minimal resolutions*:

$$0 \longrightarrow F(E) \otimes_{\hat{\sigma}} H_2 \xrightarrow{\Phi_2} F(E) \otimes_\sigma H_1 \xrightarrow{\Phi_1} H \longrightarrow 0$$

$$0 \longrightarrow F(E) \otimes_{\hat{\pi}} \mathcal{H}_2 \xrightarrow{\Psi_2} F(E) \otimes_\pi \mathcal{H}_1 \xrightarrow{\Psi_1} \mathcal{H} \longrightarrow 0.$$

Then H and $\mathcal{H}$ are isomorphic if and only if $\Phi_2 : F(E) \otimes_{\hat{\sigma}} H_2 \longrightarrow F(E) \otimes_\sigma H_1$ is unitarily equivalent to $\Psi_2 : F(E) \otimes_{\hat{\pi}} \mathcal{H}_2 \to F(E) \otimes_\pi \mathcal{H}_1$.

Proof. Suppose first that a unitary module map $u : H \to \mathcal{H}$ exists. Since H is isomorphic to the $*$-invariant submodule $(\Phi_1)^*(H) \subset F(E) \otimes_\sigma H_1$ and $\mathcal{H}$ is isomorphic to the $*$-invariant submodule $(\Psi_1)^*(\mathcal{H}) \subset F(E) \otimes_\pi \mathcal{H}_1$, it follows that $(\Phi_1)^*(H)$ and $(\Psi_1)^*(\mathcal{H})$ are isomorphic to each other. Applying Theorem 7 to these modules, we conclude that a unitary module map $U_1 : F(E) \otimes_\sigma H_1 \to F(E) \otimes_\pi \mathcal{H}_1$ exists that satisfies $u \circ \Phi_1 = \Psi_1 \circ U_1$ and that maps $\ker \Phi_1 = ((\Phi_1)^*(H))^\perp$ onto $\ker \Psi_1 = ((\Psi_1)^*(\mathcal{H}))^\perp$. This implies that $U_{1|\ker \Phi_1} : \ker \Phi_1 \to \ker \Psi_1$ is a unitary module map. Since $\Phi_2 : F(E) \otimes_{\hat{\sigma}} H_2 \to \ker \Phi_1$ and $\Psi_2 : F(E) \otimes_{\hat{\pi}} \mathcal{H}_2$ are coisometric module maps, we can apply Theorem 7 again to conclude that a unitary module map $U_2 : F(E) \otimes_{\hat{\sigma}} H_2 \to F(E) \otimes_\pi \mathcal{H}_2$ exists that satisfies $U_{1|\ker \Phi_1} \circ \Phi_2 = \Psi_2 \circ U_2$. But this implies that $U_1 \circ \Phi_2 = \Psi_2 \circ U_2$, which is what we wanted to prove.

Suppose now that unitary module maps $U_1 : F(E) \otimes_\sigma H_1 \to F(E) \otimes_\pi \mathcal{H}_1$ and $U_2 : F(E) \otimes_{\hat{\sigma}} H_2 \to F(E) \otimes_{\hat{\pi}} \mathcal{H}_2$ exist such that $U_1 \circ \Phi_2 = \Psi_2 \circ U_2$. Since $\ker \Phi_1 = \operatorname{Im} \Phi_2$ and $\ker \Psi_1 = \operatorname{Im} \Psi_2$, we easily check that $U_1(\ker \Phi_1) = \ker \Psi_1$ and $U_1(\ker \Phi_1)^\perp = (\ker \Psi_1)^\perp$. We need to check that $U_1 : (\ker \Phi_1)^\perp \to (\ker \Psi_1)^\perp$ is a module map, but this follows formally from the next lemma. □

Lemma 7. *Suppose that H_1, H_2 are Hilbert spaces, $T : H_1 \to H_1$, $S : H_2 \to H_2$ are bounded linear maps, and $E_1 \subset H_1$, $E_2 \subset H_2$ are subspaces satisfying $T^* E_1 \subset E_1$ and $S^* E_2 \subset E_2$. Then, if $U : H_1 \to H_2$ is an isometry satisfying $UT = SU$ and $UE_1 = E_2$, it follows that $(P_{E_2} U_{|E_1})(P_{E_1} T_{|E_1}) = (P_{E_2} S_{|E_2})(P_{E_2} U_{|E_1})$.*

Proof. We compute $T^*(U^*)_{|E_2}$ in two ways:

$$\begin{aligned} T^* \left(U^*\right)_{|E_2} &= T^* P_{E_1} \left(U^*\right)_{|E_2} = P_{E_1} T^* P_{E_1} \left(U^*\right)_{|E_2} \\ &= \left(P_{E_1} T_{|E_1}\right)^* \left(P_{E_2} U_{|E_1}\right)^*, \end{aligned}$$

and

$$\begin{aligned} T^* \left(U^*\right)_{|E_2} &= U^* \left(S^*\right)_{|E_2} = U^* P_{E_2} \left(S^*\right)_{|E_2} = P_{E_1} U^* P_{E_2} \left(S^*\right)_{|E_2} \\ &= \left(P_{E_2} U_{|E_1}\right)^* \left(P_{E_2} S_{|E_2}\right)^*. \end{aligned}$$

The result follows by dualizing the two equalities. □

We will now strengthen Proposition 7.

Theorem 9. *Suppose that $G_1 \leq G_2 \leq \cdots \leq G_m$ are directed graphs with N edges such that G_{i+1} is a deformation of G_i and that $\sigma_1 : l_\infty^{n_1} \to B(H_1)$ is a faithful representation. For $2 \leq i \leq m$, there exist Hilbert spaces H_i, faithful representations $\sigma_i : l_\infty^{n_i} \to B(H_i)$, a surjective coisometric module map $\Phi_1 : F(G_2) \otimes_{\sigma_2} H_2 \to F(G_1) \otimes_{\sigma_1} H_1$ and partial isometric module maps $\Phi_i : F(G_{i+1}) \otimes_{\sigma_{i+1}} H_{i+1} \to F(G_i) \otimes_{\sigma_i} H_i$ such that $\operatorname{Im} \Phi_{i.} = \ker \Phi_{i-1}$. Moreover, $G_{m+1} = G_m$ and Φ_m is an isometry. That is, if $P_i = F(G_i) \otimes_{\sigma_i} H_i$, we have the following short exact sequence*:

$$0 \longrightarrow P_{m+1} \xrightarrow{\Phi_m} P_m \xrightarrow{\Phi_{m-1}} \cdots \xrightarrow{\Phi_3} P_3 \xrightarrow{\Phi_2} P_2 \xrightarrow{\Phi_1} P_1 \longrightarrow 0.$$

Proof. By Proposition 7 a faithful representation $\sigma_2 : l_\infty^{n_2} \to B(H_2)$ exists such that $H_2 = H_1$ and $F(G_1) \otimes_{\sigma_1} H_1 \subset F(G_2) \otimes_{\sigma_2} H_2$. Since $F(G_1) \otimes_{\sigma_1} H_1$ is $*$-invariant under the $L_{\sigma_2}(g_i)$s, define $\Phi_1 : F(G_2) \otimes_{\sigma_2} H_2 \to F(G_1) \otimes_{\sigma_1} H_1$ to be an orthogonal projection. Φ_1 is a coisometric module map and $\ker \Phi_1$ is a submodule of $F(G_2) \otimes_{\sigma_2} H_2$. By Proposition 10, $\ker \Phi_1 = F(G_2) \otimes_{\hat{\sigma}_2} \mathcal{L}$, where $\mathcal{L}$ is the wandering subspace of $\ker \Phi_1$. Let $H_3 = \mathcal{L}$ and by Proposition 7 again a faithful representation $\sigma_3 : l_\infty^{n_3} \to B(H_3)$ exists such that $F(G_2) \otimes_{\hat{\sigma}_2} H_2 \subset F(G_3) \otimes_{\sigma_3} H_3$. Define $\Psi_2 : F(G_3) \otimes_{\sigma_3} H_3 \to F(G_2) \otimes_{\hat{\sigma}_1} H_2$ to be the orthogonal projection, and let $\Phi_2 = \iota_2 \circ \Psi_2$, where $\iota_2 : F(G_2) \otimes_{\hat{\sigma}_1} H_2 \to F(G_1) \otimes_{\sigma_1} H_1$ is the inclusion. Since Ψ_2 and ι_2 are module maps, Φ_2 satisfies the required properties. Continue in this way until a partial isometric module map $\Phi_{m-1} : F(G_m) \otimes_{\sigma_m} H_m \to F(G_{m-1}) \otimes_{\sigma_{m-1}} H_{m-1}$ satisfying $\operatorname{Im} \Phi_{m-1.} = \ker \Phi_{m-2}$ is found. As before, we have that $\ker \Phi_{m-1}$ is a submodule of $F(G_m) \otimes_{\sigma_m} H_m$, and hence it can be written in the form $F(G_m) \otimes_{\hat{\sigma}_m} \mathcal{L}$ where $\mathcal{L}$ is the wandering subspace of $\ker \Phi_{m-1}$. Let $G_{m+1} = G_m$, $H_{m+1} = \mathcal{L}$, and define $\Phi_m : F(G_m) \otimes_{\hat{\sigma}_m} \mathcal{L} \to F(G_m) \otimes_{\sigma_m} H_m$ to be the inclusion. This proves the result. □

Theorem 3 can be used to complete the following diagram.

Theorem 10. *Use the notation of Theorem 9, and suppose that $f_1 : F(G_1) \otimes_{\sigma_1} H_1 \to F(G_1) \otimes_{\sigma_1} H_1$ is a module map. Then, for*

$i \le m+1$, there exist module maps $f_i : F(G_i) \otimes_{\sigma_i} H_i \to F(G_i) \otimes_{\sigma_i} H_i$ such that $\|f_i\| \le \|f_1\|$ and $f_i \circ \Phi_i = \Phi_i \circ f_{i+1}$. That is, the following diagram commutes:

$$\begin{array}{ccccccccccccc}
0 & \longrightarrow & P_{m+1} & \xrightarrow{\Phi_m} & \cdots & \xrightarrow{\Phi_3} & P_3 & \xrightarrow{\Phi_2} & P_2 & \xrightarrow{\Phi_1} & P_1 & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle f_{m+1}} & & & & \downarrow{\scriptstyle f_3} & & \downarrow{\scriptstyle f_2} & & \downarrow{\scriptstyle f_1} & & \\
0 & \longrightarrow & P_{m+1} & \xrightarrow{\Phi_m} & \cdots & \xrightarrow{\Phi_3} & P_3 & \xrightarrow{\Phi_2} & P_2 & \xrightarrow{\Phi_1} & P_1 & \longrightarrow & 0.
\end{array}$$

Proof. We look at $F(G_1) \otimes_{\sigma_1} H_1$ and $F(G_2) \otimes_{\sigma_2} H_2$ as the Hilbert modules:

$$(F(G_2) \otimes_{\sigma_2} H_2; L_{\sigma_2}(g_1), \dots, L_{\sigma_2}(g_n))$$

and

$$(F(G_1) \otimes_{\sigma_1} H_1; L_{\sigma_1}(g_1), \dots, L_{\sigma_1}(g_n)).$$

Notice that $f_1 \circ \Phi_1 : F(G_2) \otimes_{\sigma_2} H_2 \to F(G_1) \otimes_{\sigma_1} H_1$ is a module map, $\Phi_1 : F(G_2) \otimes_{\sigma_2} H_2 \to F(G_1) \otimes_{\sigma_1} H_1$ is a coisometric module map, and that $F(G_1) \otimes_{\sigma_1} H_1$ satisfies (3.8). Then it follows from Theorem 3 that a module map $f_2 : F(G_2) \otimes_{\sigma_2} H_2 \to F(G_2) \otimes_{\sigma_2} H_2$ exists such that $\Phi_1 \circ f_2 = f_1 \circ \Phi_2$ and $\|f_2\| = \|f_1 \circ \Phi_2\| \le \|f_1\|$. Proceeding in this way we find $f_3, f_4, \dots, f_{m+1}$, and we prove the result. $\square$

4. Weighted graphs. In [**3**], we studied a large family of weighted Fock spaces and their quotients that includes as special cases: the full Fock space, the symmetric Fock space, the antisymmetric Fock space, the Dirichlet algebra, and the reproducing kernel Hilbert spaces with a complete Nevanlinna-Pick kernel. We constructed Poisson kernels for these spaces, and we proved that they satisfy commutative and non-commutative interpolation theorems. In [**1**], we used Hilbert module language to study this family. We proved a commutant lifting theorem, and then we characterized the strongly orthogonally projective objects of each family.

In this section we define weighted graphs, and then we use the techniques from [**1**] to show that results from the previous section apply to weighted graphs. The proofs are almost identical, and we only sketch their proof. We decided to give the complete proof for the full Fock spaces case. We felt it was more instructive. The addition of weights causes no difficulties.

Given a family of weights $(\omega_\alpha)_{\alpha\in\mathbf{F}_N^+}$ satisfying the three conditions listed below, we define the weighted Fock space $\mathcal{F}^2(\omega_\alpha)$ to be the Hilbert space with complete orthogonal basis $(\delta_\alpha)_{\alpha\in\mathbf{F}_N^+}$ satisfying $(\delta_a,\delta_\alpha)=\omega_\alpha$. The left creation operators $L_i:\mathcal{F}^2(\omega_\alpha)\to\mathcal{F}^2(\omega_\alpha)$ are given by $L_i\delta_\alpha=\delta_{g_i\alpha}$ for $i\le N$.

We choose weights (ω_α) satisfying

$$(\omega_1)\qquad \omega_\alpha>0\quad\text{for every }\alpha\in\mathbf{F}_N^+\text{ and }\omega_0=1,$$

$$(\omega_2)\qquad \frac{\omega_{g_i\alpha g_j}}{\omega_{\alpha g_j}}\le\frac{\omega_{g_i\alpha}}{\omega_\alpha}\quad\text{for all }i,j\le N\text{ and }\alpha\in\mathbf{F}_N^+,$$

and there exist scalars $(a_\alpha)_{\alpha\in\mathbf{F}_N^+}$ such that for every $(\lambda_1,\dots,\lambda_N)$ satisfying $\sum_{i\le N}|\lambda_i|^2<1$,

$$(\omega_3)\qquad \left(\sum_{\alpha\in\mathbf{F}_N^+}\frac{\lambda_\alpha}{\omega_\alpha}L_\alpha\right)^{-1}=\sum_{\alpha\in\mathbf{F}_N^+}a_\alpha\lambda_\alpha L_a.$$

The motivation for these weights comes from a paper by Quiggin [**25**]. The first condition is clear. The second one implies that the maps L_i and R_i are bounded. And the most useful fact of the third condition is that $a_0>0$ and $a_\alpha\le 0$ if $|\alpha|\ge 1$. For example, if $\omega_\alpha=|\alpha|+1$, the three conditions are satisfied. In the one-dimensional case, this corresponds to the Dirichlet algebra.

Suppose that G is a directed graph with N edges and n vertices and that $\sigma:l_\infty^n\to B(H)$ is a faithful representation. As before, σ decomposes $H=H_1\oplus\cdots\oplus H_n$. We define the weighted graph induced by σ to be

$$\begin{aligned}F_{\omega_\alpha}(G)\otimes_\sigma H=\overline{\operatorname{span}}\{&\{\delta_\gamma\otimes x:\gamma\in\Gamma_G,x\in H_{s(\gamma)}\}\\&\cup\{\delta_0\otimes x:x\in H\}\}\\&\subset\mathcal{F}^2(\omega_\alpha)\otimes H.\end{aligned}$$

For $i \leq N$, define

$$L_\sigma(g_i) = P_{F_{\omega_\alpha}(G) \otimes_\sigma H} (L_i \otimes I_H)_{|F_{\omega_\alpha}(G) \otimes_\sigma H}.$$

The proof of Proposition 4 can be applied to obtain

Proposition 11. *$F_{\omega_\alpha}(G) \otimes_\sigma H$ is invariant under $(L_1 \otimes I_H)^*, (L_2 \otimes I_H)^*, \ldots, (L_N \otimes I_H)^*$. Consequently, $(F_{\omega_\alpha}(G) \otimes_\sigma H; L_\sigma(g_1), \ldots, L_\sigma(g_N))$ is an $*$-submodule of $\mathcal{F}^2(\omega_\alpha) \otimes H$.*

The orthogonal projections $\operatorname{Ind}(\sigma)(e_j) : F_{\omega_\alpha}(G) \otimes_\sigma H \to F_{\omega_\alpha}(G) \otimes_\sigma H$ are defined by formula (3.2). Following the proof of Theorem 2, we characterize the maps that intertwine the left creation operators:

Theorem 11. *Suppose that G has no sinks, and let $\sigma : l_\infty^n \to B(H)$ and $\pi : l_\infty^n \to B(\mathcal{H})$ be faithful $*$-representations. If $T : F_{\omega_\alpha}(E) \otimes_\sigma H \to F_{\omega_\alpha}(E) \otimes_\pi \mathcal{H}$ satisfies $TL_\sigma(g_i) = L_\pi(g_i)T$ for $i \leq N$, then there exist operators $A_\alpha \in B(H, \mathcal{H})$ satisfying $A_\alpha = P_{\mathcal{H}_{s(\alpha)}} A_\alpha P_{H_{r(\alpha)}}$ for $\alpha \in \Gamma_G$, and operators $A_j \in B(H, \mathcal{H})$ satisfying $A_j = P_{\mathcal{H}_j} A_j P_{H_j}$ for $j \leq n$ such that*

$$T = \sum_{j=1}^{n} (I \otimes A_j)_{|\mathcal{M}_\sigma} + \sum_{\alpha \in \Gamma_G} (R_\alpha \otimes A_\alpha)_{|\mathcal{M}_\sigma}.$$

Furthermore, $T[\operatorname{Ind}(\sigma)(e_j)] = [\operatorname{Ind}(\pi)(e_j)]T$ for $j \leq n$.

Proof. We start by taking $T : F_{\omega_\alpha}(G) \otimes_\sigma H \to F_{\omega_\alpha}(G) \otimes_\pi \mathcal{H}$ satisfying $TL_\sigma(g_i) = L_\pi(g_i)T$ for $i \leq N$. To simplify notation, let $\mathcal{M}_\sigma = F_{\omega_\alpha}(G) \otimes_\sigma H$ and $\mathcal{M}_\pi = F_{\omega_\alpha}(G) \otimes_\pi \mathcal{H}$. By the commutant lifting theorem of [**1**], $\widehat{T} : \mathcal{F}^2(\omega_\alpha) \otimes H \to \mathcal{F}^2(\omega_\alpha) \otimes \mathcal{H}$ exists such that $\|\widehat{T}\| = \|T\|$, $\widehat{T}(L_i \otimes I) = (L_i \otimes I)\widehat{T}$ for every $i \leq N$, and $P_{\mathcal{M}_\pi} \widehat{T}_{|\mathcal{M}_\sigma} = T$. Moreover, by [**1**, Corollary 3.7], $\widehat{T} = \sum_{\alpha \in \mathbf{F}_N^+} R_\alpha \otimes A_\alpha$ for some $A_\alpha \in B(H, \mathcal{H})$. Hence,

$$\widehat{T} = \sum_{\alpha \in \mathbf{F}_N^+} R_\alpha \otimes A_\alpha = \sum_{\alpha \in \mathbf{F}_N^+} \sum_{j=1}^{n} \sum_{k=1}^{n} R_\alpha \otimes P_{\mathcal{H}_j} A_\alpha P_{H_k}.$$

The same proof as in Lemma 4 gives that $P_{\mathcal{M}_\pi}[R_\alpha \otimes P_{\mathcal{H}_j} A_\alpha P_{H_k}]_{|\mathcal{M}_\sigma}$ is nonzero only if $\alpha \in \Gamma_G$ and $j = s(\alpha)$. Moreover, the maps of form $[R_\alpha \otimes P_{\mathcal{H}_{s(\alpha)}} A_\alpha P_{H_{r(\alpha)}}]_{|\mathcal{M}_\sigma}$ for $\alpha \in \Gamma_G$, and $[I \otimes P_{\mathcal{H}_j} A_\alpha P_{H_j}]_{|\mathcal{M}_\sigma}$ for $j \leq n$ map $\mathcal{M}_\sigma$ into $\mathcal{M}_\pi$, intertwine $L_\sigma(g_i)$ with $L_\pi(g_i)$, and intertwine $\mathrm{Ind}\,(\sigma)(e_j)$ with $\mathrm{Ind}\,(\pi)(e_j)$. Then we write $T = T_1 + T_2$ where

$$\begin{aligned} T_1 = &\sum_{\alpha \in \Gamma_G} P_{\mathcal{M}_\pi}(R_\alpha \otimes P_{\mathcal{H}_{s(\alpha)}} A_\alpha P_{H_{r(\alpha)}})_{|\mathcal{M}_\sigma} \\ &+ \sum_{j=1}^{n} P_{\mathcal{M}_\pi}(I \otimes P_{\mathcal{H}_j} A_0 P_{H_j})_{|\mathcal{M}_\sigma} \end{aligned}$$

$$\begin{aligned} T_2 = &\sum_{\alpha \in \Gamma_G} \sum_{k \neq r(\alpha)} P_{\mathcal{M}_\pi}(R_\alpha \otimes P_{\mathcal{H}_{s(\alpha)}} A_\alpha P_{H_k})_{|\mathcal{M}_\sigma} \\ &+ \sum_{j=1}^{n} \sum_{k \neq j} P_{\mathcal{M}_\pi}(I \otimes P_{\mathcal{H}_j} A_0 P_{H_k})_{|\mathcal{M}_\sigma}. \end{aligned}$$

The averaging argument of Lemma 5 works in weighted Fock spaces, and we get that $\|T_1\| \leq \|T\|$. The proof of Lemma 6 applied to weighted Fock spaces gives that $T_2 = 0$, and this proves Theorem 11. □

The proof of Theorem 11 with the changes we just outlined gives a commutant lifting theorem for weighted graphs.

Theorem 12. *Suppose that G is a graph with no sinks, and let $\sigma, \pi : A \to B(H)$ be faithful $*$-representations. Suppose that $(\mathcal{H}; T_1, \dots, T_N)$ is a Hilbert module satisfying*

$$\{h \in \mathcal{H} : T_i h = 0 \textit{ for every } i \leq N\} = (0)$$

and that $\Phi : F_{\omega_\alpha}(G) \otimes_\pi H \to \mathcal{H}$ is a coisometric module map. Then, for every module map $f : F_{\omega_\alpha}(G) \otimes_\sigma H \to \mathcal{H}$ there exists $f_1 : F_{\omega_\alpha}(G) \otimes_\sigma H \to F_{\omega_\alpha}(G) \otimes_\sigma H$ satisfying $\|f\| = \|f_1\|$, $f = \Phi \circ f_1$, and $f_1 L_\sigma(g_i x) = L_\pi(g_i) f_1$ for $i \leq N$.

Proof. Suppose that $\mathcal{E} \subset F_{\omega_\alpha}(G) \otimes_\sigma H$ and that $\Phi_{\mathcal{E}} : B(F_{\omega_\alpha}(G) \otimes_\sigma H) \to B(\mathcal{E})$ is the compression operator $\Phi_{\mathcal{E}}(T) = P_{\mathcal{E}} T_{|\mathcal{E}}$. We look at the subspaces $\mathcal{E}$ that satisfy the following conditions

$$\begin{aligned}(4.1)\quad &\Phi_{\mathcal{E}}\left(L_{\sigma}\left(g_{i_1}\right)L_{\sigma}\left(g_{i_2}\right)\cdots L_{\sigma}\left(g_{i_k}\right)\right)\\ &\quad=\Phi_{\mathcal{E}}\left(L_{\sigma}\left(g_{i_1}\right)\right)\Phi_{\mathcal{E}}\left(L_{\sigma}\left(g_{i_2}\right)\right)\cdots\Phi_{\mathcal{E}}\left(L_{\sigma}\left(g_{i_k}\right)\right)\quad\text{for } k\in\mathbf{N},\end{aligned}$$

and

$$\begin{aligned}(4.2)\quad &\Phi_{\mathcal{E}}\left(\left[\operatorname{Ind}\left(\sigma\right)\left(a\right)\right]\left[\operatorname{Ind}\left(\sigma\right)\left(b\right)\right]\right)\\ &\quad=\Phi_{\mathcal{E}}\left(\left[\operatorname{Ind}\left(\sigma\right)\left(a\right)\right]\right)\Phi_{\mathcal{E}}\left(\left[\operatorname{Ind}\left(\sigma\right)\left(b\right)\right]\right)\quad\text{for } a,b\in l_{\infty}^{n}.\end{aligned}$$

Condition (4.2) implies that $\mathcal{E}$ is invariant under maps of the form $\operatorname{Ind}(\sigma)(e_j)$, and then we have that $\widehat{\sigma}:l_{\infty}^{n}\to B(\mathcal{E})$ defined by $\widehat{\sigma}(a)=P_{\mathcal{E}}[\operatorname{Ind}(\sigma)(a)]_{|\mathcal{E}}$ is an $*$-representation. For each $i\le N$, define $T_i=P_{\mathcal{E}}L_{\sigma}(g_i)_{|\mathcal{E}}$ and obtain the Hilbert module

$$\left(\mathcal{E};T_1,\dots,T_N;\widehat{\sigma}\left(e_1\right),\dots,\widehat{\sigma}\left(e_n\right)\right).$$

Suppose that $\mathcal{E}\subset F_{\omega_\alpha}(G)\otimes_{\sigma}H$ satisfies (4.2). Then

(1) $\mathcal{E}$ is a submodule if $L_{\sigma}(g_i)\mathcal{E}\subset\mathcal{E}$ for every $i\le N$,

(2) $\mathcal{E}$ is an $*$-submodule if $L_{\sigma}(g_i)^{*}\mathcal{E}\subset\mathcal{E}$ for every $i\le N$, and

(3) $\mathcal{E}$ is a subquotient if it satisfies property (4.1).

It is clear that submodules and $*$-submodules are subquotients. Sarason [**26**] proved that $\mathcal{E}$ is a subquotient if and only if two submodules $\mathcal{E}_1$ and $\mathcal{E}_2$ exist such that $\mathcal{E}\oplus\mathcal{E}_1=\mathcal{E}_2$.

In [**3**], we proved that if $\mathcal{E}$ is a subquotient of $F_{\omega_\alpha}(G)\otimes_{\sigma}H$, an isometry $K:\mathcal{E}\to\mathcal{F}^2(\omega_\alpha)\otimes\mathcal{E}$ exists satisfying $K^{*}(L_i\otimes I_H)=T_iK^{*}$ for $i\le N$. The map K is defined by

$$K\left(x\right)=\sum_{\alpha\in\mathbf{F}_N^{+}}\frac{\delta_\alpha}{\omega_\alpha}\otimes\Delta T_{\alpha}^{*}x,\tag{4.3}$$

where

$$\Delta^2=\sum_{\alpha\in\mathbf{F}_N^{+}}a_\alpha T_\alpha T_a^{*}.\tag{4.4}$$

We proved in [**3**] that $\Delta^2 \geq 0$.

Since $L_\sigma(g_i) = [\mathrm{Ind}\,(\sigma)(e_{r(i)})]T[\mathrm{Ind}\,(\sigma)(e_{s(i)})]$, we have that

$$\begin{aligned} T_i &= P_{\mathcal{E}} L_\sigma\left(g_i\right)_{|\mathcal{E}} \\ &= P_{\mathcal{E}}\left[\mathrm{Ind}\;(\sigma)\left(e_{r(i)}\right)\right] L_\sigma\left(g_i\right)\left[\mathrm{Ind}\;(\sigma)\left(e_{s(i)}\right)\right]_{|\mathcal{E}} \\ &= P_{\mathcal{E}}\left[\mathrm{Ind}\;(\sigma)\left(e_{r(i)}\right)\right] L_\sigma\left(g_i\right) P_{\mathcal{E}}\left[\mathrm{Ind}\;(\sigma)\left(e_{s(i)}\right)\right]_{|\mathcal{E}}. \end{aligned}$$

Dualizing this equality, we obtain

$$\begin{aligned} T_i^* &= P_{\mathcal{E}}\left[\mathrm{Ind}\;(\sigma)\left(e_{s(i)}\right)\right]^* P_{\mathcal{E}} L_\sigma\left(g_i\right)^*\left[\mathrm{Ind}\;(\sigma)\left(e_{r(i)}\right)\right]_{|\mathcal{E}}^* \\ &= P_{\mathcal{E}}\left[\mathrm{Ind}\;(\sigma)\left(e_{s(i)}\right)\right]^* P_{\mathcal{E}} L_\sigma\left(g_i\right)^* P_{\mathcal{E}}\left[\mathrm{Ind}\;(\sigma)\left(e_{r(i)}\right)\right]_{|\mathcal{E}}^* \\ &= \widehat{\sigma}\left(e_{s(i)}\right)^* T_i^* \widehat{\sigma}\left(e_{r(i)}\right)^*, \end{aligned}$$

and this implies that

$$T_i = \widehat{\sigma}\left(e_{r(i)}\right) T_i \widehat{\sigma}\left(e_{s(i)}\right) \quad \text{for } i \leq N. \tag{4.5}$$

Arguing as in Proposition 5, we conclude that $T_\alpha \neq 0$ only if $\alpha \in \Gamma_G$. Therefore, we conclude from (4.3) that K takes values in $F_{\omega_\alpha}(G) \otimes_{\hat{\sigma}} H$. □

Proposition 12. *Suppose that $\mathcal{E}$ is a subquotient of $F_{\omega_\alpha}(G) \otimes_\sigma H$. Then there exists an isometry $K : \mathcal{E} \to F_{\omega_\alpha}(G) \otimes_{\hat{\sigma}} \mathcal{E}$ defined by*

$$K\left(x\right) = \delta_0 \otimes \Delta x + \sum_{\alpha \in \mathbf{\Gamma}_G} \frac{\delta_\alpha}{\omega_\alpha} \otimes \Delta T_\alpha^* x$$

such that $K^* L_\sigma(g_i) = T_i K^*$ for $i \leq N$ and $K^*[\mathrm{Ind}\,(\sigma)(e_j)] = \widehat{\sigma}(e_j)K^*$ for $j \leq n$.

Proof. We only need to check the last statements. For $i \leq N$,

$$\begin{aligned} K^* L_\sigma\left(g_i\right) &= K^* P_{F_{\omega_\alpha}(G)\otimes_\sigma H}\left(L_i \otimes I_H\right)_{|F_{\omega_\alpha}(G)\otimes_\sigma H} \\ &= K^*\left(L_i \otimes I_H\right)_{|F_{\omega_\alpha}(G)\otimes_\sigma H} \\ &= T_i K^*_{|F_{\omega_\alpha}(G)\otimes_\sigma H} = T_i K^*. \end{aligned}$$

The representation $\widehat{\sigma} : l_\infty^n \to B(\mathcal{E})$ decomposes $\mathcal{E}$ into $\mathcal{E} = \mathcal{E}_1 \oplus \mathcal{E}_2 \oplus \cdots \oplus \mathcal{E}_n$. Suppose now that $\alpha \in \Gamma_G$ and $x \in \mathcal{E}_{s(\alpha)}$. It is not hard to see from (4.3) that $K^*(\delta_\alpha \otimes x) = T_\alpha \Delta(x)$. Moreover, it follows from (4.4) that $\Delta x \in \mathcal{E}_{s(\alpha)}$, and from (4.5) that $T\Delta(x) \in \mathcal{E}_{r(\alpha)}$. This implies that $K^*[\mathrm{Ind}\,(\sigma)(e_j)](\delta_\alpha \otimes x) = \widehat{\sigma}(e_j)K^*(\delta_\alpha \otimes x)$ for $j \le n$. Moreover, we also have that $K^*[\mathrm{Ind}\,(\sigma)(e_j)](\delta_0 \otimes x) = \widehat{\sigma}(e_j)K^*(\delta_0 \otimes x)$ for $x \in \mathcal{E}_k$ where $k, j \le n$. Hence, $K^*[\mathrm{Ind}\,(\sigma)(e_j)] = \widehat{\sigma}(e_j)K^*$ for $j \le n$. □

With the Poisson kernels of Proposition 12 and Theorem 12, we follow the proofs of Theorem 5 and Theorem 6 to obtain projective resolutions for subquotients of $F_{\omega_\alpha}(G) \otimes_\sigma H$:

Theorem 13. *Suppose that $\mathcal{E}$ is a subquotient of $F_{\omega_\alpha}(G) \otimes_\sigma H$. Then a family $P_i = F_{\omega_\alpha}(G_i) \otimes_{\sigma_i} H_i$ and partial isometric module maps Φ_i exist such that the following sequence is exact*:

$$\cdots \longrightarrow P_4 \xrightarrow{\Phi_4} P_3 \xrightarrow{\Phi_3} P_2 \xrightarrow{\Phi_2} P_1 \xrightarrow{\Phi_1} \mathcal{E} \longrightarrow 0.$$

Moreover, if $\mathcal{E}_1$ and $\mathcal{E}_2$ are subquotients of $F_{\omega_\alpha}(G) \otimes_\sigma H$ with projective resolutions

$$\cdots \longrightarrow P_2 \xrightarrow{\Phi_2} P_1 \xrightarrow{\Phi_1} \mathcal{E}_1 \longrightarrow 0$$

and

$$\cdots \longrightarrow Q_2 \xrightarrow{\Psi_2} Q_1 \xrightarrow{\Psi_1} \mathcal{E}_1 \longrightarrow 0,$$

and $f : \mathcal{E}_1 \to \mathcal{E}_2$ is a module map, then there exist module maps $f_i : P_i \to Q_i$ satisfying $\|f_i\| \le \|f\|$ for every $i \in \mathbf{N}$ such that the following diagram commutes:

$$\begin{array}{ccccccccccc}
\cdots \longrightarrow & P_4 & \xrightarrow{\Phi_4} & P_3 & \xrightarrow{\Phi_3} & P_2 & \xrightarrow{\Phi_2} & P_1 & \xrightarrow{\Phi_1} & \mathcal{E}_1 & \longrightarrow 0 \\
& \downarrow{\scriptstyle f_4} & & \downarrow{\scriptstyle f_3} & & \downarrow{\scriptstyle f_2} & & \downarrow{\scriptstyle f_1} & & \downarrow{\scriptstyle f} & \\
\cdots \longrightarrow & Q_4 & \xrightarrow{\Psi_4} & Q_3 & \xrightarrow{\Psi_3} & Q_2 & \xrightarrow{\Psi_2} & Q_1 & \xrightarrow{\Psi_1} & \mathcal{E}_2 & \longrightarrow 0.
\end{array}$$

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