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CUNTZ-PIMSNER C^*-ALGEBRAS AND CROSSED PRODUCTS BY HILBERT C^*-BIMODULES

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ABSTRACT. Given a correspondence X over a C^*-algebra A, we construct a C^*-algebra A_∞^X and a Hilbert C^*-bimodule X_∞ over A_∞^X such that the augmented Cuntz-Pimsner C^*-algebras $\widetilde{\mathcal{O}}_X$ and the crossed product $A_\infty^X \rtimes X_\infty$ are isomorphic. This construction enables us to establish a condition for two augmented Cuntz-Pimsner C^*-algebras to be Morita equivalent.

1. Introduction and preliminaries. The augmented Cuntz-Pimsner C^*-algebra $\widetilde{\mathcal{O}}_X$ defined in [**9**] is a C^*-algebra associated to an A-correspondence (X, ϕ_X) that is universal for certain covariance conditions, see [**9**, 3.12], when ϕ_X is injective and X is full as a right Hilbert C^*-module.

On the other hand, when X is also a Hilbert C^*-bimodule over the C^*-algebra A, the crossed product $A \rtimes X$ defined in [**1**] is universal for covariance conditions that agree with those for which $\widetilde{\mathcal{O}}_X$ is universal under the assumptions mentioned above.

Thus, both constructions can be carried out when X is a Hilbert C^*-bimodule, and they agree when X is full on the right and the action on the left is faithful. But this may fail if the condition of faithfulness of the left action is dropped, as the following example, shown to us by Søren Eilers, proves.

Let $A = \mathbf{C} \oplus \mathbf{C}$ and $X = \mathbf{C}$ be the Hilbert C^*-bimodule over A obtained by setting:

$$(\lambda, \mu) \cdot x = \lambda x, \quad x \cdot (\lambda, \mu) = x\mu, \quad \langle x, y\rangle_L = (x\overline{y}, 0)$$

and

$$\langle x, y\rangle_R = (0, \overline{x}y).$$

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Then $X \otimes X = 0$ because $x \otimes y = x \cdot (0,1) \otimes y = x \otimes (0,1) \cdot y = x \otimes 0$. This implies that $\widetilde{\mathcal{O}}_X = \{0\}$ whereas $A \rtimes X$ is isomorphic to $M_2(\mathbf{C})$. This last statement can be checked directly by verifying that the *-homomorphism induced by the covariant pair of maps $(i_A, i_X) : (A, X) \to M_2(\mathbf{C})$ given by $i_A(\lambda, \mu) = \begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix}$, $i_X(x) = \begin{pmatrix} 0 & x \\ 0 & 0 \end{pmatrix}$ is an isomorphism, or by noting that X is the bimodule associated (as described in [**1**, 3.2]) to the partial action on A given by $I = \mathbf{C} \oplus 0$, $J = 0 \oplus \mathbf{C}$, $\theta(x, 0) = (0, x)$.

This shows that $A \rtimes X$ and $\widetilde{\mathcal{O}}_X$ may not agree for a Hilbert C^*-bimodule X over A. On the other hand, as mentioned in [**1**], for any A-correspondence (X, ϕ_X) the algebra $\widetilde{\mathcal{O}}_X$ is a crossed product $A_\infty \rtimes X_\infty$. The example above shows that the algebra A_∞ and the bimodule X_∞ do not necessarily agree with the original A and X when X is a Hilbert C^*-bimodule over A. In this work we give an abstract construction of A_∞ and X_∞, out of an A-correspondence (X, ϕ_X). Both A_∞ and X_∞ are described as direct limits of nicely related directed sequences in their respective categories.

We apply this construction to the discussion of Morita equivalence of augmented Cuntz-Pimsner C^*-algebras. One of our tools is a result from [**1**, 4.2]: if X and Y are Hilbert C^*-bimodules over C^*-algebras A and B, respectively, and M is an $A-B$ Morita equivalence bimodule such that the $A-B$ Hilbert C^*-bimodules $X \otimes_A M$ and $M \otimes_B Y$ are isomorphic, then the crossed products $A \rtimes X$ and $B \rtimes Y$ are Morita equivalent. In Theorem 4.7 we establish a condition of this kind for the Morita equivalence of two augmented Cuntz-Pimsner C^*-algebras. Muhly and Solel showed in [**8**, 3.3, 3.5] a similar result for Cuntz-Pimsner C^*-algebras and for correspondences (X, ϕ_X) and (Y, ϕ_Y) such that the maps ϕ_X and ϕ_Y are injective and the correspondences are nondegenerate, that is, $\phi_X(A)X = X$ and similarly for Y. Our result for augmented C^*-algebras does not require the action to be injective, but a condition related to nondegeneracy, see Remark 4.8, has to be met.

This work is organized as follows. Section 2 deals with the notion of direct limit of Hilbert C^*-modules and proves some basic results that will be further required. In Section 3 we construct, for an A-correspondence (X, ϕ_X), a C^*-algebra A_∞ and a Hilbert C^*-bimodule X_∞ over A_∞ such that $\widetilde{\mathcal{O}}_X$ and $A_\infty \rtimes X_\infty$ are isomorphic. In Section 4 we use that construction together with [**1**, 4.2] to give a sufficient condition for the Morita equivalence of two augmented Cuntz-Pimsner C^*-algebras.

We start by recalling some definitions and by setting some notation.

Notation 1.1. Let A and B be C^*-algebras. If $\phi : A \to B$ is a *-homomorphism, we denote by $\phi^{(k)}$ the *-homomorphism $\phi^{(k)} : M_k(A) \to M_k(B)$ defined by $(\phi^{(k)}(M))_{ij} = \phi(M_{ij})$.

Given a right Hilbert C^*-module X over a C^*-algebra A, we denote by $\mathcal{L}(X)$ and $\mathcal{K}(X)$, respectively, the C^*-algebras of adjointable and compact maps. For $x, y \in X$, we write $\theta_{x,y}$ to denote the map $\theta_{x,y} \in \mathcal{K}(X)$ defined by $\theta_{x,y}(z) = x\langle y, z\rangle$. For $x \in X$, $|x|$ denotes the element $|x| \in A$ defined by $|x| = \langle x, x\rangle^{1/2}$.

Given subsets S and T of X, we write $\langle S, T\rangle$ to denote the set $\langle S, T\rangle = \overline{\text{span}}\{\langle s, t\rangle : s \in S,\ t \in T\}$. If $S \subset \mathcal{L}(X)$, we denote by SX the set $SX = \overline{\text{span}}\{s(x) : s \in S,\ x \in X\}$. Given a C^*-subalgebra C of $\mathcal{L}(X)$, we denote by $L_{C,X}$ the right Hilbert C^*-module homomorphism $L_{C,X} : C \otimes_C X \to X$ defined by $L_{C,X}(c \otimes x) = c(x)$. Note that $L_{C,X}$ is an isomorphism when $CX = X$.

When X is a right Hilbert C^*-module over A, the map $x \otimes a \mapsto xa$, for $x \in X$ and $a \in A$, is an isomorphism of right A-Hilbert C^*-modules between $X \otimes A$ and X that associates the map $T \in \mathcal{L}(X)$ to the map $T \otimes \text{id}_A \in \mathcal{L}(X \otimes A)$. Often in this work we will identify X with $X \otimes A$ and $T \in \mathcal{L}(X)$ with $T \otimes \text{id}_A$ as above without further warning. For $T \in \mathcal{L}(X)$ we will understand that $T^{\otimes 0}$ is id_A.

We next recall some of the terminology in [**8**] that we will adopt. Given C^*-algebras A and B, an $A - B$ correspondence (X, ϕ_X) consists of a right Hilbert C^*-module X over B together with a $*$-homomorphism $\phi_X : A \to \mathcal{L}(X)$. We will denote the correspondence by X and drop the reference to the map ϕ_X when it does not lead to confusion. Besides, we will write $a \cdot x$ to denote $[\phi_X(a)](x)$.

Let X_i be an $A_i - B_i$ correspondence, for $i = 1, 2$. A homomorphism of correspondences (σ, ϕ, π) consists of C^*-algebra homomorphisms $\sigma : A_1 \to A_2$ and $\pi : B_1 \to B_2$ and a linear map $\phi : X_1 \to X_2$ such that

$$\phi(a \cdot xb) = \sigma(a) \cdot \phi(x)\pi(b) \quad \text{and} \quad \langle \phi(x), \phi(y) \rangle = \pi\big(\langle x, y \rangle\big),$$

for all $x, y \in X_1$, $a \in A_1$, $b \in B_1$.

Whenever $A_1 = A_2$, respectively $B_1 = B_2$, and there is no reference to the map σ, respectively π, we assume it is the identity map. Two $A - B$ correspondences X and Y are said to be isomorphic if there is a homomorphism $(\mathrm{id}_A, J, \mathrm{id}_B)$ where $J : X \to Y$ is invertible.

Note that the map $L_{C,X}$ defined above for a C^*-subalgebra C of $\mathcal{L}(X)$ is a homomorphism of $C - A$ correspondences, for C acting on $C \otimes X$ via $l \otimes \mathrm{id}_X$, l being left multiplication.

Homomorphisms of right Hilbert C^*-modules and homomorphisms of Hilbert C^*-bimodules are defined in the obvious analogous way, Hilbert C^*-bimodules being defined as in [**3**, 1.8].

Lemma 1.2. *Let $(\phi, \sigma) : (X, A) \to (Y, B)$ be a homomorphism of right Hilbert C^*-modules. Then ϕ is norm-decreasing, and it induces a C^*-algebra homomorphism $\phi_* : \mathcal{K}(X) \to \mathcal{K}(Y)$ such that $\phi_*(\theta_{x_1,x_2}) = \theta_{\phi(x_1),\phi(x_2)}$, for x_1, $x_2 \in X$.*

Proof. If $x \in X$, then

$$\|\phi(x)\|^2 = \|\langle \phi(x), \phi(x) \rangle\| = \|\sigma(\langle x, x \rangle)\| \leq \|\langle x, x \rangle\| = \|x\|^2.$$

As for the second statement, if $x_i, y_i \in X$ for $i = 1, 2, \ldots, n$, then by [**6**, 2.1] we have

$$\Big\| \sum \theta_{\phi(x_i),\phi(y_i)} \Big\| = \|S^{1/2} T^{1/2}\|_{M_n(B)},$$

where $S_{ij} = \langle \phi(x_i), \phi(x_j) \rangle$ and $T_{ij} = \langle \phi(y_i), \phi(y_j) \rangle$. Now, $S = \sigma^{(n)}(M)$ and $T = \sigma^{(n)}(N)$, where $M_{ij} = \langle x_i, x_j \rangle$ and $N_{ij} = \langle y_i, y_j \rangle$. Therefore,

$$\|S^{1/2} T^{1/2}\| = \|\sigma^{(n)}\big(M^{1/2} N^{1/2}\big)\| \leq \|M^{1/2} N^{1/2}\| = \Big\| \sum \theta_{x_i,y_i} \Big\|,$$

which shows that ϕ_* extends to a continuous map on $\mathcal{K}(X)$. Finally, it is straightforward to check that ϕ_* is a $*$-homomorphism from the fact that (σ, ϕ) is a homomorphism of right Hilbert C^*-modules. □

2. Directed sequences of right Hilbert C^*-modules. In this section we discuss a procedure to get, for a given A-correspondence (X, ϕ_X), a Hilbert C^*-bimodule X_∞ over a C^*-algebra A_∞. We will show in the next section that $A_\infty \rtimes X_\infty$ is isomorphic to $\widetilde{\mathcal{O}}_X$.

In order to get a left inner product on X one needs to add to $\operatorname{Im}\phi$ the compact operators $\mathcal{K}(X)$. If one lets $A_1 \subset \mathcal{L}(X)$ be defined by $A_1 = \operatorname{Im}\phi + \mathcal{K}(X)$, then X is an $A_1 - A$ Hilbert C^*-bimodule, but there is no clear right action of A_1 on X. This suggests replacing X by $X_1 := X \otimes_A A_1$. Thus, we end up with an A_1-correspondence X_1, and the procedure can be iterated. We show how this iteration yields directed sequences $\{A_n\}$ and $\{X_n\}$ whose limits A_∞ and X_∞ are such that X_∞ is a Hilbert C^*-bimodule over A_∞. We will develop this procedure in a somewhat more general context that will be of use in the discussion of Morita equivalence in the last section.

Definition 2.1. A directed sequence $\{(X_n, A_n, \phi_n^X, \phi_n^A)\}$ of right Hilbert C^*-modules consists of a directed sequence $\{(A_n, \phi_n^A)\}$ of C^*-algebras together with a directed sequence $\{(X_n, \phi_n^X)\}$ of vector spaces such that X_n is a right Hilbert C^*-module over A_n and (ϕ_n^X, ϕ_n^A) is a homomorphism of right Hilbert C^*-modules for each $n \in \mathbf{N}$.

Remark 2.2. Let $\{(X_n, A_n, \phi_n^X, \phi_n^A)\}$ be a directed sequence of right Hilbert C^*-modules. Since the maps ϕ_n^X are norm decreasing by Lemma 1.2, the sequence $\{X_n, \phi_n^X\}$ has a direct limit $(X_\infty, \{\lambda_n^X\})$ that can be described as follows. Let Y_0 be the vector space

$$Y_0 = \Big\{ x \in \prod X_n : \text{ there exists } n_0 \text{ such that } x_{n+1} = \phi_n(x_n) \quad \text{for all} \quad n \geq n_0 \Big\},$$

and let $Y = \{x \in Y_0 : \lim_n \|x_n\| = 0\}$. Then X_∞ is the completion of Y_0/Y for the norm $\|x\| = \lim_n \|x_n\|$. The canonical maps $\lambda_n^X : X_n \to X_\infty$ are given by $\lambda_n^X = \pi \circ \tilde{\lambda}_n^X$, where $\pi : Y_0 \to X_\infty$ is the canonical projection and $\tilde{\lambda}_n^X(x_n)(k) = \phi_{n,k}^X(x_n)$, for $\phi_{n,k}^X : X_n \to X_k$ given by

$$\phi_{n,k}^X = \begin{cases} 0 & \text{if } k < n; \\ \text{id} & \text{if } k = n; \\ \phi_{k-1}^X \circ \phi_{k-2}^X \cdots \circ \phi_n^X & \text{if } k > n. \end{cases}$$

Note that $\|\lambda_n^X(x)\|_{X_\infty} = \lim_m \|\phi_{n,m}^X(x_n)\|_{X_m} = \inf_m \|\phi_{n,m}^X(x_n)\|_{X_m}$.

If $\{x_n\} \in Y_0$, and n_0 is such that $x_{n+1} = \phi_n(x_n)$ for all $n \geq n_0$, then $\lambda^X_{n_0}(x_{n_0}) = \pi\big(\{x_n\}\big)$, which shows that $\cup \lambda^X_n(X_n)$ is dense in X_∞.

It is well known that a similar description holds for the direct limit $(A_\infty, \{\lambda^A_n\})$ of the directed sequence of C^*-algebras $\{A_n, \phi^A_n\}$ and that $\cup \lambda^A_n(A_n)$ is dense in A_∞. Note that $(\phi^A_{n,k}, \phi^X_{n,k})$ is a homomorphism of right Hilbert C^*-modules from (A_n, X_n) to (A_k, X_k).

We will say that $(X_\infty, A_\infty, \{\lambda^X_n\}, \{\lambda^A_n\})$ is the direct limit of the directed sequence $\{(X_n, A_n, \phi^X_n, \phi^A_n)\}$.

As the referee pointed out to us, a nicer approach to the discussion of the direct limit in the category of right Hilbert C^*-modules consists of showing that this category is equivalent to that of pairs (A, e) where A is a C^*-algebra and e is a projection in its multiplier algebra and morphisms $\phi : A \to B$ such that ϕ is a C^*-algebra morphism and $\phi(ex) = f\phi(x)$, $\phi(xe) = \phi(x)f$ for all $x \in X$.

The equivalence can be obtained by mapping a right A-Hilbert C^*-module X to the pair $(\mathcal{K}(X \oplus A), e)$, where $X \oplus A$ has the usual structure of right Hilbert C^*-module over A and e is the projection onto X. Morphisms are handled as in Lemma 1.2 for $\phi_X \oplus \phi_A$. At this point, the existence of the direct limit in the latter category can be easily obtained. However, since later in this work we will be making use of the specific construction in the next proposition, we show the existence of the direct limit in the category of right Hilbert C^*-modules.

Proposition 2.3. *Let $\{(X_n, A_n, \phi^X_n, \phi^A_n)\}$ be a directed sequence of right Hilbert C^*-modules with direct limit $(X_\infty, A_\infty, \{\lambda^X_n\}, \{\lambda^A_n\})$. Then X_∞ can be made into a right Hilbert C^*-module over A_∞ by setting:*

$$\lambda^X_n(x_n)\lambda^A_n(a_n) := \lambda^X_n(x_n a_n),\ \ \langle \lambda^X_n(x^n_1), \lambda^X_n(x^n_2)\rangle := \lambda^A_n(\langle x^n_1, x^n_2\rangle),$$

for $a_n \in A_n$, $x_n, x^n_i \in X_n$, $i = 1, 2$. (Therefore, $(\lambda^X_n, \lambda^A_n) : (X_n, A_n) \to (X_\infty, A_\infty)$ is a homomorphism of right Hilbert C^*-modules for all n.)

Let M be a right Hilbert C^*-module over a C^*-algebra B and, for each $n \in \mathbf{N}$, let $(\mu^X_n, \mu^A_n) : (X_n, A_n) \to (M, B)$ be a homomorphism of right Hilbert C^*-modules, such that the diagrams

$$\begin{array}{ccc} X_n & \xrightarrow{\phi_n^X} & X_{n+1} \\ {\scriptstyle \mu_n^X}\downarrow & \swarrow {\scriptstyle \mu_{n+1}^X} & \\ M & & \end{array}$$

$$\begin{array}{ccc} A_n & \xrightarrow{\phi_n^A} & A_{n+1} \\ {\scriptstyle \mu_n^A}\downarrow & \swarrow {\scriptstyle \mu_{n+1}^A} & \\ B & & \end{array}$$

commute. If $\mu^X : X_\infty \to M$, $\mu^A : A_\infty \to B$ *are the canonical maps yielded by the universal property of the direct limit, then* (μ^X, μ^A) *is a homomorphism of Hilbert* C^**-modules.*

Besides, the norm on X_∞ *induced by its structure of right* A_∞*-Hilbert* C^**-module agrees with the original norm.*

Proof. We first check that the definition of the action on the right makes sense. Assume that $\lambda_k^A(a_k) = \lambda_n^A(a_n)$ and that $\lambda_k^X(x_k) = \lambda_n^X(x_n)$, for some $a_n \in A_n$, $a_k \in A_k$, $x_n \in X_n$ and $x_k \in X_k$.

Given $\varepsilon > 0$, choose $j \in \mathbf{N}$, $j \geq k$, $j \geq n$, and large enough to have $\|\phi_{n,j}^X(x_n) - \phi_{k,j}^X(x_k)\| < \varepsilon$ and $\|\phi_{n,j}^A(a_n) - \phi_{k,j}^A(a_k)\| < \varepsilon$. Then

$$\begin{aligned} \|\lambda_n^X(x_n a_n) - \lambda_k^X(x_k a_k)\| &\leq \|\phi_{n,j}^X(x_n a_n) - \phi_{k,j}^X(x_k a_k)\| \\ &= \|\phi_{n,j}^X(x_n)\phi_{n,j}^A(a_n) - \phi_{k,j}^X(x_k)\phi_{k,j}^A(a_k)\| \\ &\leq \|\phi_{n,j}^X(x_n)(\phi_{n,j}^A(a_n) - \phi_{k,j}^A(a_k))\| \\ &\quad + \|(\phi_{n,j}^X(x_n) - \phi_{k,j}^X(x_k))\phi_{k,j}^A(a_k)\| \\ &\leq (\|x_n\| + \|a_k\|)\varepsilon. \end{aligned}$$

Besides,

$$\begin{aligned} \|\lambda_n^X(x_n a_n)\| &= \lim_m \|\phi_{n,m}^X(x_n a_n)\| \\ &= \lim_m \|\phi_{n,m}^X(x_n)\phi_{n,m}^A(a_n)\| \\ &\leq \left(\lim_m \|\phi_{n,m}^X(x_n)\|\right)\left(\lim_m \|\phi_{n,m}^A(a_n)\|\right) \\ &= \|x\|\|a\|, \end{aligned}$$

which shows that the right action of $\cup_n \lambda_n^A(A_n)$ on $\cup_n \lambda_n^X(X_n)$ extends by continuity to a right action of A_∞ on X_∞.

As for the definition of the right inner product, it makes sense because if $\lambda_n^X(x_i^n) = \lambda_k^X(x_i^k)$ for some $x_i^n \in X_n$, $x_i^k \in X_k$, $i = 1,2$, then for any $\varepsilon > 0$ we can choose $j \in \mathbf{N}$ such that $j \geq k$, $j \geq n$, and $\|\phi_{n,j}^X(x_i^n) - \phi_{k,j}^X(x_i^k)\| < \varepsilon$ for $i = 1,2$. Then:

$$\begin{aligned}
\|\lambda_n^A(\langle x_1^n, x_2^n\rangle) - \lambda_k^A(\langle x_1^k, x_2^k\rangle)\| \\
&\leq \|\phi_{n,j}^A(\langle x_1^n, x_2^n\rangle) - \phi_{k,j}^A(\langle x_1^k, x_2^k\rangle)\| \\
&= \|\langle \phi_{n,j}^X(x_1^n), \phi_{n,j}^X(x_2^n)\rangle - \langle \phi_{k,j}^X(x_1^k), \phi_{k,j}^X(x_2^k)\rangle\| \\
&\leq \|\langle \phi_{n,j}^X(x_1^n) - \phi_{k,j}^X(x_1^k), \phi_{n,j}^X(x_2^n)\rangle\| \\
&\quad + \|\langle \phi_{k,j}^X(x_1^k), \phi_{n,j}^X(x_2^n) - \phi_{k,j}^X(x_2^k)\rangle\| \\
&< \varepsilon\big(\|x_2^n\| + \|x_1^k\|\big).
\end{aligned}$$

Also note that, for $x_n \in X_n$, we have

$$\begin{aligned}
\|\langle \lambda_n^X(x_n), \lambda_n^X(x_n)\rangle\|_{A_\infty} &= \|\lambda_n^A\big(\langle x_n, x_n\rangle\big)\|_{A_\infty} \\
&= \lim_m \|\phi_{n,m}^A\big(\langle x_n, x_n\rangle\big)\|_{A_m} \\
&= \lim_m \|\langle \phi_{n,m}^X(x_n), \phi_{n,m}^X(x_n)\rangle\|_{A_m} \\
&= \lim_m \|\phi_{n,m}^X(x_n)\|_{X_m}^2 \\
&= \|\lambda_n^X(x_n)\|_{X_\infty}^2,
\end{aligned}$$

which shows that the two norms on X_∞ agree.

The remaining properties and statements are apparent from the definitions. □

Example 2.4. The following example will be of importance in this work. Given a correspondence (X, ϕ_X) over a C^*-algebra A, let $X(A)$ denote the C^*-subalgebra $X(A) = \mathcal{K}(X) + \operatorname{Im}\phi_X$. Note that X is an $X(A) - A$ Hilbert C^*-bimodule.

Given a right A-Hilbert C^*-module M, we define the right $X(A)$-Hilbert C^*-module $X(A)$ by $X(M) := M \otimes_{\phi_X} X(A)$, where $X(A)$ is viewed as an $\operatorname{Im}\phi_X - X(A)$ correspondence in the obvious way.

Note that

$$\Big\langle \sum_{i=1}^{n} m_i \otimes \phi_X(a_i), \sum_{j=1}^{m} p_j \otimes \phi_X(b_j) \Big\rangle = \sum_{i,j} \langle m_i \otimes \phi_X(a_i), p_j \otimes \phi_X(b_j)\rangle$$
$$= \sum_{i,j} \phi_X(a_i)^* \phi_X(\langle m_i, p_j\rangle)\phi_X(b_j)$$
$$= \sum_{i,j} \phi_X(\langle m_i a_i, p_j b_j\rangle)$$
$$= \phi_X\Big(\Big\langle \sum_i m_i a_i, \sum_j p_j b_j \Big\rangle\Big),$$

for $m_i, p_j \in M$, $a_i, b_j \in A$, and $i = 1, 2, \ldots, n$, $j = 1, 2, \ldots, m$.

In particular, $\| \sum_{i=1}^{n} m_i \otimes \phi_X(a_i)\|^2 = \|\phi_X(|\sum_i m_i a_i|)\|^2 \leq \| \sum_i m_i a_i\|^2$. This shows that one can define a map $\psi_M^X : M \to X(M)$ by $\psi_M^X(ma) = m \otimes \phi_X(a)$ so that (ψ_M^X, ϕ_X) is a homomorphism of right Hilbert C^*-modules. When $M = X$ the map ψ_X^X will be denoted by ψ_X. In this case (ψ_X, ϕ_X) is a homomorphism of correspondences.

Now, since $X(X)$ and $X(M)$ are, respectively, a correspondence and a right Hilbert C^*-module over $X(A)$, the construction above can be iterated to get a sequence $\{A_n^X\}_{n\geq 0}$ of C^*-algebras and, for each $n \geq 0$, a correspondence X_n over A_n^X and a right A_n^X-Hilbert C^*-module M_n^X by setting $A_0^X = A$, $X_0 = X$, $M_0^X = M$, and, for $n \geq 0$:

$$A_{n+1}^X = X_n(A_n^X), \quad X_{n+1} = X_n(X_n), \quad \text{and} \quad M_{n+1}^X = X_n(M_n^X).$$

We also get right Hilbert C^*-module homomorphisms

$$(\phi_n^{M,X}, \phi_n^{A,X}) : (M_n^X, A_n^X) \longrightarrow (M_{n+1}^X, A_{n+1}^X) \quad \text{for all } n \geq 0,$$

given by $\phi_n^{A,X} = \phi_{X_n}$ and $\phi_n^{M,X} = \psi_{M_n}^{X_n}$, that is, $\phi_n^{A,X}(a) = a \otimes \mathrm{id}_{A_n^X}$ for all $n \geq 1$, and $\phi_n^{M,X}(ma) = m \otimes \phi_n^{A,X}(a)$, for $m \in M_n^X$ and $a \in A_n^X$.

When $M = X$ we write ϕ_n^X in place of $\phi_n^{M,X}$. In that case $(\phi_n^{A,X}, \phi_n^X, \phi_n^{A,X})$ and $(\phi_{n+1}^{A,X}, \phi_n^X, \phi_n^{A,X})$ are, respectively, homomorphisms of correspondences and Hilbert C^*-bimodules:

$$\big(\theta_{\phi_n^X(xa), \phi_n^X(yb)}\big)(z \otimes c) = x \otimes \phi_n^{A,X}(a)\langle y \otimes \phi_n^{A,X}(b), z \otimes c\rangle$$

$$\begin{aligned} &= x \otimes \phi_n^{A,X}(a)\phi_n^{A,X}(b^*\langle y, z\rangle)c \\ &= xa\langle yb, z\rangle \otimes c \\ &= \big(\phi_{n+1}^{A,X}(\theta_{xa,xb})\big)(z \otimes c), \end{aligned}$$

for all $x, y, z \in X_n$, $a, b \in A_n$ and $c \in A_{n+1}$.

Let $(X_\infty, A_\infty^X, \{\lambda_n^X\}, \{\lambda_n^A\})$ and $(M_\infty^X, A_\infty^X, \{\lambda_n^M\}, \{\lambda_n^A\})$ denote the direct limits of the sequences $\{(X_n, A_n^X, \phi_n^X, \phi_n^{A,X})\}$ and $\{(M_n^X, A_n^X, \phi_n^{M,X}, \phi_n^{A,X})\}$, respectively. By Proposition 2.3, both X_∞ and M_∞^X are right Hilbert C^*-modules over A_∞^X.

Remark 2.5. In fact, X_∞ is a Hilbert C^*-bimodule over A_∞^X: since X_n is an $A_{n+1}^X - A_n^X$ Hilbert C^*-bimodule for all $n \in \mathbf{N}$, the proof of Proposition 2.3 carries over to the left structure of X_∞, and the compatibility between the left and the right structures on X_∞ is easily checked.

Proposition 2.6. *Let $\{(X_n, A_n, \phi_n^X, \phi_n^A)\}$ be a directed sequence of right Hilbert C^*-modules with direct limit $(X_\infty, A_\infty, \{\lambda_n^X\}, \{\lambda_n^A\})$.*

Then $(\mathcal{K}(X_\infty), \{(\lambda_n^X)_*\})$ is the direct limit of $\{(\mathcal{K}(X_n), (\phi_n^X)_*)\}$, where $(\lambda_n^X)_*$ and $(\phi_n^X)_*$ are defined as in Lemma 1.2.

Proof. It is well known that for any integer k, $(M_k(A_\infty), \{(\lambda_n^A)^{(k)}\})$ is the direct limit of $\{M_k(A_n), (\phi_n^A)^{(k)}\}$, which in particular implies that

$$\|(\lambda_n^A)^{(k)}(T)\|_{M_k(A_\infty)} = \lim_m \|\big(\phi_{n,m}^A\big)^{(k)}(T)\| \quad \text{for all } T \in M_k(A_n).$$

Now, the commuting diagram

$$\begin{array}{ccc} X_n & \xrightarrow{\phi_n^X} & X_{n+1} \\ \lambda_n^X \downarrow & \swarrow \lambda_{n+1}^X & \\ X_\infty & & \end{array}$$

yields the commuting diagram

$$\begin{array}{ccc} \mathcal{K}(X_n) & \xrightarrow{(\phi_n^X)_*} & \mathcal{K}(X_{n+1}) \\ {\scriptstyle (\lambda_n^X)_*}\big\downarrow & \swarrow {\scriptstyle (\lambda_{n+1}^X)_*} & \\ \mathcal{K}(X_\infty) & & \end{array}$$

which in turn yields a map $H : \varinjlim \mathcal{K}(X_n) \to \mathcal{K}(X_\infty)$, defined by $H(l_n(T)) = (\lambda_n^X)_*(T)$ for $T \in \mathcal{K}(X_n)$, where $l_n : \mathcal{K}(X_n) \to \varinjlim \mathcal{K}(X_n)$ is the canonical map.

Note that $\{\theta_{r,s} : r, s \in \cup\lambda_n^X(X_n)\}$ is dense in $\mathcal{K}(X_\infty)$ because $X_\infty = \overline{\cup\lambda_n^X(X_n)}$. It follows from that fact that H is onto, since $\theta_{\lambda_n^X(x),\lambda_n^X(y)} = H(l_n(\theta_{x,y}))$, for $x, y \in X_n$.

The map H is also isometric: let $T \in \mathcal{K}(X_n)$, $T = \sum_{i=1}^k \theta_{x_i,y_i}$, where $x_i, y_i \in X_n$. Then

$$\begin{aligned} \|H(l_n(T))\| = \|(\lambda_n^X)_*(T)\| &= \Big\| \sum_1^k \theta_{\lambda_n^X(x_i),\lambda_n^X(y_i)} \Big\| \\ &= \|(\lambda_n^A)^{(k)}(X^{1/2}Y^{1/2})\|, \end{aligned}$$

where [**6**, 2.1] $X_{ij} = \langle x_i, x_j \rangle$ and $Y_{ij} = \langle y_i, y_j \rangle$.

Therefore, by applying [**6**, 2.1] again,

$$\begin{aligned} \|H(l_n(T))\| = \|(\lambda_n^A)^{(k)}(X^{1/2}Y^{1/2})\| &= \lim_m \|(\phi_{n,m}^A)^{(k)}(X^{1/2}Y^{1/2})\| \\ &= \lim_m \|(\phi_{n,m}^X)_*(T)\| = \|l_n(T)\|. \qquad \square \end{aligned}$$

3. Cuntz-Pimsner C^*-algebras and crossed-products by Hilbert C^*-bimodules. In this section we show that the pair (A_∞, X_∞) obtained in Example 2.4 is such that $A_\infty \rtimes X_\infty$ is isomorphic to $\widetilde{\mathcal{O}}_X$. We begin by recalling some well-known facts about adjointable operators on the direct sum of Hilbert C^*-modules.

Given a sequence $\{X_n\}$ of right Hilbert C^*-modules over a C^*-algebra A, let $E = \oplus_0^\infty X_n$. If $K_0, K_1 \subset \mathbf{N}$, we identify $\mathcal{L}(\oplus_{n\in K_0} X_n, \oplus_{n\in K_1} X_n)$ with a subspace of $\mathcal{L}(E)$ by extending $\widetilde{T} \in \mathcal{L}(\oplus_{n\in K_0} X_n, \oplus_{n\in K_1} X_n)$ to $T \in \mathcal{L}(E)$ so that $T|_{X_n} = 0$ for $n \notin K_0$.

Let $J = \overline{\cup_m \mathcal{L}(\oplus_0^m X_n)} \subset \mathcal{L}(E)$, and let M denote the idealizer of J in $\mathcal{L}(E)$, that is, $M = \{T \in \mathcal{L}(E) : TS, ST \in J \text{ for all } S \in J\}$.

For an integer k, let

$$\Delta_k = \{T \in \mathcal{L}(E) : T(X_n) \subset X_{n+k} \\ \text{if } n \geq \max\{0, -k\},\ T|_{X_n} = 0 \text{ otherwise}\}.$$

Given $T \in \Delta_k$, we denote by T_n the map $T_n \in \mathcal{L}(X_n, X_{n+k})$ obtained by restricting T to X_n. Then $T = \oplus_0^\infty T_n$ and $\|T\| = \sup_n \|T_n\|$. Note that $\Delta_k \subset M$ for all $k \in \mathbf{Z}$.

Lemma 3.1. *If $T \in \Delta_k$, then $\|T + J\|_{M/J} = \limsup_n \|T_n\|$.*

Proof. We can assume that $k = 0$, since $T^*T \in \Delta_0$ and $(T^*T)_n = (T_n)^* T_n$ for all $T \in \Delta_k$. Let L denote $\limsup_n \|T_n\|$. Given $\varepsilon > 0$, let n_0 be such that $\|T_n\| < L + \varepsilon$ for all $n \geq n_0$.

Then $\|T + J\|_{M/J} \leq \| \oplus_{n_0}^\infty T_n\| = \sup_{n \geq n_0} \|T_n\| \leq L + \varepsilon$, which shows that $\|T + J\|_{M/J} \leq L$. On the other hand, if $l < L$ and $S \in \mathcal{L}(\oplus_0^m X_n) \subset J$, then

$$\|T - S\| = \left\| \left[\bigoplus_0^m T_n - S \right] \oplus \bigoplus_{m+1}^\infty T_n \right\| \geq \left\| \bigoplus_{m+1}^\infty T_n \right\| \\ = \sup_n \{\|T_n\| : n > m\|\} > l.$$

Therefore, $\|T + J\|_{M/J} \geq l$ for all $l < L$, which ends the proof. □

We next recall the definitions of the Cuntz-Pimsner algebras $\mathcal{O}_X$ and $\widetilde{\mathcal{O}}_X$ given in [**9**]. Given a correspondence X over a C^*-algebra A, let $X_n = X^{\otimes n}$, where $X^{\otimes 0} = A$, and let $E = \oplus_0^\infty X_n$.

If $x \in X^{\otimes k}$, we denote by T_x the map $T_x \in \Delta_k \subset \mathcal{L}(E)$ given by $T_x(y) = x \otimes y$ if $k > 0$ and by $T_a(y) = ay$, if $a \in A$, where $x \otimes a$ is identified with xa, for $x \in X^{\otimes k}$, $k \geq 0$, and $a \in A$.

For M and J defined as above, let $\pi : M \to M/J$ be the canonical projection, and set $S_x = \pi(T_x)$, for $x \in X_k$, $k \geq 0$. The Cuntz-Pimsner C^*-algebra $\mathcal{O}_X$ and the augmented Cuntz-Pimsner C^*-algebra $\widetilde{\mathcal{O}}_X$ are the C^*-subalgebras of M/J generated by $\{S_x : x \in X\}$ and by $\{S_t : t \in X \cup A\}$, respectively. Notice that $S_{x_1 \otimes x_2 \otimes \cdots \otimes x_k} = S_{x_1} S_{x_2} \cdots S_{x_k}$, which implies that $S_x \in \mathcal{O}_X$ for all $x \in X^{\otimes k}$, $k \geq 1$.

Remark 3.2. Let $x \in X^{\otimes m}$, for $m \geq 0$. Since $\|(T_x)_n\| = \|(T_x)_{n-1} \otimes \mathrm{id}_X\| \leq \|(T_x)_{n-1}\|$ for all $n \geq 1$, we have by Lemma 3.1,

$$\|S_x\| = \limsup_n \|(T_x)_n\| = \inf_{n \geq 1} \|(T_x)_k \otimes \mathrm{id}_{X^{\otimes n}}\|,$$

for all $k \geq 1$.

Lemma 3.3. *Let (X_i, ϕ_i) be $A - B_i$ correspondences for $i = 1, 2$, and let Y be a right Hilbert C^*-module over A. If $\ker \phi_1 \subset \ker \phi_2$, then $\|T \otimes \mathrm{id}_{X_1}\| \geq \|T \otimes \mathrm{id}_{X_2}\|$ for all $T \in \mathcal{L}(Y)$.*

Proof. Let $T \in \mathcal{L}(Y)$. Then $T \otimes \mathrm{id}_{X_i} = 0$ if and only if $0 = \|Ty \otimes x\|^2 = \|\langle x, \langle Ty, Ty \rangle \cdot x \rangle\|$ for all $x \in X_i$, $y \in Y$. That is, $T \otimes \mathrm{id}_{X_i} = 0$ if and only if $\langle Ty, Ty \rangle \in \ker \phi_i$ for all $y \in Y$. We can thus define a map $T \otimes \mathrm{id}_{X_1} \mapsto T \otimes \mathrm{id}_{X_2}$, which is a (norm-decreasing) *-homomorphism between the C^*-algebras $\{T \otimes \mathrm{id}_{X_1} : T \in \mathcal{L}(Y)\}$ and $\{T \otimes \mathrm{id}_{X_2} : T \in \mathcal{L}(Y)\}$. □

Corollary 3.4. *Let (X, ϕ_X) and Y be, respectively, a correspondence and a right Hilbert C^*-module over a C^*-algebra A. Let $X(A)$ be as in Example* 2.4. *Then, for any $T \in \mathcal{L}(Y)$ we have*

$$\|T \otimes \mathrm{id}_X\| = \|T \otimes \mathrm{id}_{X(A)}\|.$$

Proof. It suffices to notice that an element a of A acts on $X(A)$ by left multiplication by $\phi_X(a)$. Since $\mathrm{Im}\, \phi_X \subset X(A)$, we conclude that $a \cdot X(A) = 0$ if and only if $\phi_X(a) = 0$. Then the previous lemma applies in both directions and the equality holds. □

Lemma 3.5. *Given a correspondence (X, ϕ_X) over a C^*-algebra A, let $X(X)$, $X(A)$ and $\psi_X : X \to X(X)$ be as in Example* 2.4.

For $n \geq 1$, let $\beta_n : X(X)^{\otimes n} \to X^{\otimes n} \otimes_{\phi_X} X(A)$ be the isomorphism of $X(A)$-correspondences given by $\beta_n = \mathrm{id}_X \otimes L_{X(A),X}^{\otimes n-1} \otimes \mathrm{id}_{X(A)}$, where $L_{X(A),X}$ is as in Notation 1.1. *Then*:

(1) $\beta_n \circ \psi_X^{\otimes n} = id_{X^{\otimes n-1}} \otimes \psi_X$, *for all* $n \geq 1$.

(2) $\beta_{n+m}(\theta_{\psi_X^{\otimes n}(z),\psi_X^{\otimes n}(w)} \otimes \mathrm{id}_{X(X)^{\otimes m}})\beta_{n+m}^* = \theta_{z,w} \otimes \mathrm{id}_{(X^{\otimes m} \otimes X(A))}$, *for* $z, w \in X^{\otimes n}$, $n \geq 1$, $m \geq 0$.

(3) $\beta_{m+1}(\phi_X(a) \otimes \mathrm{id}_{X(A) \otimes X(X)^{\otimes m}})\beta_{m+1}^* = \phi_X(a) \otimes \mathrm{id}_{(X^{\otimes m} \otimes X(A))}$, *for all* $m \geq 0$.

Proof. (1) Let $x_i \in X$, $a_i \in A$, for $i = 1, 2, \ldots, n$. Then

$$\begin{aligned}
&[\beta_n \circ \psi_X^{\otimes n}](x_1 a_1 \otimes x_2 a_2 \otimes \cdots \otimes x_n a_n) \\
&\quad = \beta_n(x_1 \otimes \phi_X(a_1) \otimes x_2 \otimes \phi_X(a_2) \otimes \cdots \otimes x_n \otimes \phi_X(a_n)) \\
&\quad = x_1 \otimes \phi_X(a_1) x_2 \otimes \phi_X(a_2) x_3 \otimes \cdots \otimes \phi_X(a_{n-1}) x_n \otimes \phi_X(a_n) \\
&\quad = x_1 a_1 \otimes x_2 a_2 \otimes \cdots \otimes x_{n-1} a_{n-1} \otimes x_n \otimes \phi_X(a_n) \\
&\quad = (\mathrm{id}_{X^{\otimes n-1}} \otimes \psi_X)(x_1 a_1 \otimes x_2 a_2 \otimes \cdots \otimes x_n a_n).
\end{aligned}$$

(2) We first prove the statement for $m = 0$. We assume, without loss of generality, that $z = z_0 \otimes xa$, $w = w_0 b$, for $x \in X$, $z_0 \in X^{\otimes n-1}$, $w_0 \in X^{\otimes n}$, and $a, b \in A$. Let $u \in X^{\otimes n}$, $r \in X(A)$. Then, by (1):

$$\begin{aligned}
&\left(\beta_n \theta_{\psi_X^{\otimes n}(z),\psi_X^{\otimes n}(w)} \beta_n^*\right)(u \otimes r) \\
&\qquad = \left(\theta_{(\mathrm{id}_{X^{\otimes n-1}} \otimes \psi_X)(z_0 \otimes xa),(\mathrm{id}_{X^{\otimes n-1}} \otimes \psi_X)(w_0 b)}\right)(u \otimes r) \\
&\qquad = z_0 \otimes x \otimes \phi_X(a)\langle w_0 \otimes \phi_X(b), u \otimes r\rangle \\
&\qquad = z_0 \otimes xa \otimes \phi_X(b^*\langle w_0, u\rangle) r \\
&\qquad = z_0 \otimes xa\langle w_0 b, u\rangle \otimes r = z\langle w, u\rangle \otimes r \\
&\qquad = (\theta_{z,w} \otimes \mathrm{id}_{X(A)})(u \otimes r).
\end{aligned}$$

Let us denote by L the map $L_{X(A),X}$ defined in Notation 1.1. For $m \geq 1$ we have

$$\beta_{n+m} = \left(\mathrm{id}_{X^{\otimes n}} \otimes L^{\otimes m} \otimes \mathrm{id}_{X(A)}\right)\left(\beta_n \otimes \mathrm{id}_{X(X)^{\otimes m}}\right).$$

Therefore,

$$\begin{aligned}
&\beta_{n+m}\left(\theta_{\psi_X^{\otimes n}(z),\psi_X^{\otimes n}(w)} \otimes \mathrm{id}_{X(X)^{\otimes m}}\right)\beta_{n+m}^* \\
&\quad = \left(\mathrm{id}_{X^{\otimes n}} \otimes L^{\otimes m} \otimes \mathrm{id}_{X(A)}\right)\left(\theta_{z,w} \otimes \mathrm{id}_{X(A)} \otimes \mathrm{id}_{X(X)^{\otimes m}}\right) \\
&\quad\quad \times \left(\mathrm{id}_{X^{\otimes n}} \otimes (L^{\otimes m})^* \otimes \mathrm{id}_{X(A)}\right) \\
&\quad = \theta_{z,w} \otimes \mathrm{id}_{X^{\otimes m} \otimes X(A)}.
\end{aligned}$$

(3)

$$\begin{aligned}\beta_{m+1}\left(\phi_X(a)\otimes \mathrm{id}_{X(A)\otimes X(X)^{\otimes m}}\right)\beta_{m+1}^*\\ &= (\mathrm{id}_X\otimes L^{\otimes m}\otimes \mathrm{id}_{X(A)})(\phi_X(a)\otimes \mathrm{id}_{(X(A)\otimes X)^{\otimes m}}\otimes \mathrm{id}_{X(A)})\\ &\quad \times (\mathrm{id}_X\otimes (L^{\otimes m})^*\otimes \mathrm{id}_{X(A)})\\ &= \phi_X(a)\otimes \mathrm{id}_{X^{\otimes m}\otimes X(A)}. \qquad \square\end{aligned}$$

Remark 3.6. As discussed in [**9**, Remark 1.2, (2)], the automorphism of X given by $x\mapsto \lambda x$ for $\lambda\in S^1$ yields an automorphism γ_λ of $\widetilde{\mathcal{O}}_X$, determined by $\gamma_\lambda(S_x) = \lambda^k S_x$ for $x\in X^{\otimes k}$, $k\geq 0$. In fact, this automorphism of X extends to an automorphism $\dot{\gamma}_\lambda$ of E defined by $(\dot{\gamma}_\lambda(\eta))(k) = \lambda^k\eta(k)$, for $\eta\in E$. Conjugation by $\dot{\gamma}_\lambda$ is an automorphism of $\mathcal{L}(E)$ that maps T_x into $\lambda^k T_x$ for $x\in X^{\otimes k}$, $k\geq 0$, and it leaves J invariant.

Thus, one gets an action γ of S^1 on $\widetilde{\mathcal{O}}_X$ that is easily checked to be strongly continuous. The fixed-point subalgebra of this action is $E_0(\widetilde{\mathcal{O}}_X) = \overline{\mathrm{span}}\{S_xS_y^* : x,y\in X^{\otimes n},\ n\geq 0\}$, and its first spectral subspace $E_1(\widetilde{\mathcal{O}}_X) = \overline{\mathrm{span}}\{S_xS_y^* : x\in X^{\otimes n+1},\ y\in X^{\otimes n},\ n\geq 0\} = \overline{\mathrm{span}}\{S_xe : x\in X,\ e\in E_0(\widetilde{\mathcal{O}}_X)\}$. This last statement is shown by means of the usual argument, since $\mathrm{span}\{S_xS_y^* : x\in X^{\otimes n}, y\in X^{\otimes m},\ n,m\geq 0\}$ is dense in $\widetilde{\mathcal{O}}_X$, and the maps $P_i : \widetilde{\mathcal{O}}_X\to E_i(\widetilde{\mathcal{O}}_X)$ given by

$$P_i(u) = \int_{S^1} z^{-i}\gamma_z(u)\,dz \quad \text{for} \quad u\in\widetilde{\mathcal{O}}_X$$

are surjective contractions, see [**4**] for details, and $\gamma_\lambda(S_xS_y^*) = \lambda^{n-m}S_x S_y^*$, for $x\in X^{\otimes n}$, $y\in X^{\otimes m}$, and $n,m\geq 0$, $i=0,1$.

Now, since $\widetilde{\mathcal{O}}_X$ is generated as a C^*-algebra by $E_0(\widetilde{\mathcal{O}}_X)$ and $E_1(\widetilde{\mathcal{O}}_X)$, Theorem 3.1 in [**1**] applies, and $\widetilde{\mathcal{O}}_X$ is isomorphic to the crossed-product $E_0(\widetilde{\mathcal{O}}_X)\rtimes E_1(\widetilde{\mathcal{O}}_X)$.

Proposition 3.7. *Let (X,ϕ_X) be a correspondence over a C^*-algebra A, and let $X(A)$, $X(X)$ and ψ_X be as in Example* 2.4. *Then there is an isomorphism of Hilbert C^*-bimodules $(\eta_1,\eta_0) : (E_1(\widetilde{\mathcal{O}}_X), E_0(\widetilde{\mathcal{O}}_X))\to (E_1(\widetilde{\mathcal{O}}_{X(X)}), E_0(\widetilde{\mathcal{O}}_{X(X)}))$ carrying S_x and S_a to $S_{\psi_X(x)}$ and $S_{\phi_X(a)}$, respectively, for $x\in X$ and $a\in A$.*

Besides, if $(i_X, i_A) : (X, A) \to \widetilde{\mathcal{O}}_X$ *is given by* $i_X(x) = S_x$ *and* $i_A(a) = S_a$ *and similarly for* $(X(X), X(A))$*, then*

$$\begin{array}{ccc} (X,A) & \xrightarrow{(\psi_X,\phi_X)} & (X(X),X(A)) \\ {\scriptstyle (i_X,i_A)}\downarrow & & \downarrow{\scriptstyle (i_{X(X)},i_{X(A)})} \\ \big(E_1(\widetilde{\mathcal{O}}_X),E_0(\widetilde{\mathcal{O}}_X)\big) & \xrightarrow{(\eta_1,\eta_0)} & \big(E_1(\widetilde{\mathcal{O}}_{X(X)}),E_0(\widetilde{\mathcal{O}}_{X(X)})\big) \end{array}$$

is a commuting diagram of homomorphisms of correspondences.

Proof. We would like to define $\eta_0 : E_0(\widetilde{\mathcal{O}}_X) \to E_0(\widetilde{\mathcal{O}}_{X(X)})$ by

$$\eta_0\left(S_a + \sum_{i=1}^{k} S_{x_i}S_{y_i}^*\right) := S_{\phi_X(a)} + \sum_{i=1}^{k} S_{\psi_X^{\otimes n_i}(x_i)}S^*_{\psi_X^{\otimes n_i}(y_i)},$$

where $a \in A$, $x_i, y_i \in X^{\otimes n_i}$, and $n_i > 0$ for all $i = 1, \dots, k$.

We first show that the definition above makes sense. Let a, x_i, y_i be as above, and let $m = \max\{n_i : i = 1, \dots, k\}$. Then (see the beginning of Section 1 in [**9**] for the first equality)

$$\begin{aligned} T_a + \sum_{i=1}^{k} T_{x_i}T_{y_i}^* &= \bigoplus_{n=m}^{\infty} (T_a)_n + \left(\sum_i \theta_{x_i,y_i} \otimes \mathrm{id}_{X^{\otimes n-n_i}}\right) \pmod J \\ &= \bigoplus_{n=m}^{\infty} \tau(\{a, x_i, y_i\}) \otimes \mathrm{id}_{X^{\otimes n-m}}, \end{aligned}$$

where $\tau(\{a, x_i, y_i\}) = \phi_X(a) \otimes \mathrm{id}_{X^{\otimes m-1}} + \sum_i \theta_{x_i,y_i} \otimes \mathrm{id}_{X^{\otimes m-n_i}}$.

Now, by parts (2) and (3) of Lemma 3.5:

$$\begin{aligned} &\tau(\{a, x_i, y_i\}) \otimes \mathrm{id}_{X^{\otimes n} \otimes X(A)} \\ &\quad = \phi_X(a) \otimes \mathrm{id}_{X^{\otimes m-1+n}} \otimes \mathrm{id}_{X(A)} \\ &\qquad + \sum_i \theta_{x_i,y_i} \otimes \mathrm{id}_{X^{\otimes m-n_i+n}} \otimes \mathrm{id}_{X(A)} \\ &\quad = \beta_{n+m}\big(\phi_X(a) \otimes \mathrm{id}_{X(A)} \otimes \mathrm{id}_{X(X)^{\otimes m-1+n}} \\ &\qquad + \sum_i \theta_{\psi_X^{\otimes n_i}(x_i),\psi_X^{\otimes n_i}(y_i)} \otimes \mathrm{id}_{(X(X))^{\otimes m-n_i+n}}\big)\beta_{n+m}^* \\ &\quad = \beta_{n+m}\big(\tau(\{\phi_X(a), \psi_X^{\otimes n_i}(x_i), \psi_X^{\otimes n_i}(y_i)\}) \otimes \mathrm{id}_{X(X)^{\otimes n}}\big)\beta_{n+m}^*. \end{aligned}$$

Therefore, by Lemma 3.1 and Corollary 3.4,

$$\begin{aligned}\|S_a + \sum_i S_{x_i} S^*_{y_i}\| &= \lim_n \|\tau(\{a, x_i, y_i\}) \otimes \mathrm{id}_{X^{\otimes n}}\| \\ &= \lim_n \|\tau(\{a, x_i, y_i\}) \otimes \mathrm{id}_{X^{\otimes n}} \otimes \mathrm{id}_{X(A)}\| \\ &= \lim_n \|\tau(\{\phi_X(a), \psi_X^{\otimes n_i}(x_i), \psi_X^{\otimes n_i}(y_i)\}) \otimes \mathrm{id}_{X(X)^{\otimes n}}\| \\ &= \|\eta_0\big(S_a + \sum_i S_{x_i} S^*_{y_i}\big)\|.\end{aligned}$$

This shows that η_0 can be extended to an isometry $\eta_0 : E_0(\widetilde{\mathcal{O}}_X) \to E_0(\widetilde{\mathcal{O}}_{X(X)})$ which is easily checked to be an isometric *-homomorphism, in view of the properties listed in [**9**, Proposition 1.3].

We next show that η_0 is onto. First note that $S_{\psi_X(x)} S^*_{\psi_X(y)} = S_{\theta_{x,y}}$ for all $x, y \in X$. Let $\pi : M \to M/J \supset \widetilde{\mathcal{O}}_{X(X)}$ be as in the beginning of this section. By [**9**, 1.3]:

$$S_{\psi_X(x)} S^*_{\psi_X(y)} = \pi\left(\oplus_{n=0}^{\infty} \theta_{\psi_X(x), \psi_X(y)} \otimes \mathrm{id}_{X(X)^{\otimes n}}\right) = S_{\theta_{x,y}}$$

because

$$\theta_{\psi_X(x), \psi_X(y)} = \theta_{x,y} \otimes \mathrm{id}_{X(A)} = \phi_{X(X)}\big(\theta_{x,y}\big).$$

Also, by [**9**, 1.3],

$$S_{xa \otimes \theta_{y,z}} = S_{[x \otimes \phi(a)]\theta_{y,z}} = S_{\psi_X(xa)} S_{\theta_{y,z}} = S_{\psi_X(xa)} S_{\psi_X(y)} S^*_{\psi_X(z)}.$$

Since $S_{\phi_X(a)} = \eta_0(S_a)$, $S_{\theta_{x,y}} = \eta_0(S_x S^*_y)$, and $S_{u_1 \otimes u_2 \otimes \cdots \otimes u_n} = S_{u_1} S_{u_2} \cdots S_{u_n}$, it only remains to show that $S_{u_1} S_{u_2} \cdots S_{u_n} S^*_{v_n} \cdots S^*_{v_2} S^*_{v_1} \in \operatorname{Im} \eta_0$ for all $u_i, v_i \in X(X)$ and $n \geq 1$. We proceed by induction on n. The case $n = 1$ follows from the fact that, by the identities above:

$$\begin{aligned}&S_{x \otimes (\phi_X(a) + \theta_{y,z})} S^*_{x' \otimes (\phi_X(a') + \theta_{y',z'})} \\ &= \Big(S_{\psi_X(xa)} + S_{\psi_X(x)} S_{\psi_X(y)} S^*_{\psi_X(z)}\Big) \\ &\quad \times \Big(S_{\psi_X(x'a')} + S_{\psi_X(x')} S_{\psi_X(y')} S^*_{\psi_X(z')}\Big)^* \\ &= \eta_0\left(S_{xa} S^*_{x'a'} + S_{xa} S_{z'} S^*_{y'} S^*_{x'} + S_x S_y S^*_z S^*_{x'a'} + S_x S_{y\langle z, z'\rangle} S^*_{y'} S^*_{x'}\right)\end{aligned}$$

for all $a, a' \in A$ and $x, x', y, y', z, z' \in X$. The induction step follows from the fact that for all $a, a' \in A$ and $x, x', y, y', z, z' \in X$

$$S_{x\otimes(\phi_X(a)+\theta_{y,z})}\,(\mathrm{Im}\,\eta_0)\,S^*_{x'\otimes(\phi_X(a')+\theta_{y',z'})} \subset \mathrm{Im}\,\eta_0,$$

which is checked by applying the action γ of Remark 3.6 (or by direct computation).

We now define

$$\eta_1 : E_1(\widetilde{\mathcal{O}}_X) \longrightarrow E_1(\widetilde{\mathcal{O}}_{X(X)}) \text{ by } \eta_1\Big(\sum_i S_{x_i} e_i\Big) = \sum_i S_{\psi_X(x_i)}\eta_0(e_i),$$

for $x_i \in X$ and $e_i \in E_0(\widetilde{\mathcal{O}}_X)$. To check that the map η_1 thus defined makes sense and extends to an isometric map on $E_1(\widetilde{\mathcal{O}}_X)$, notice that

$$\begin{aligned}\Big\|\sum_i S_{\psi_X(x_i)}\eta_0(e_i)\Big\|^2 &= \Big\|\sum_{i,j}\eta_0(e_i)^* S^*_{\psi_X(x_i)}S_{\psi_X(x_j)}\eta_0(e_j)\Big\| \\ &= \Big\|\sum_{i,j}\eta_0\left(e_i^* S^*_{x_i} S_{x_j} e_j\right)\Big\| \\ &= \Big\|\sum_i S_{x_i} e_i\Big\|^2.\end{aligned}$$

Straightforward computations show that (η_0, η_1) is a Hilbert C^*-bimodule homomorphism. Besides, the map η_1 is onto because so is η_0. It is clear from the definitions that the diagram commutes. Finally, it follows from [**9**, Proposition 1.3] that (i_X, i_A) is a homomorphism of correspondences and it was shown in Example 2.4 that so is (ψ_X, ϕ_X). □

By considering the composition of the inclusion with the maps (ψ_X, ϕ_X) in the proposition below, one gets, in the terminology of [**5**], a Cuntz-Pimsner covariant representation of (X, A) on the C^*-algebra $\widetilde{\mathcal{O}}_{X(X)}$, and the existence of (η_0, η_1) follows from [**5**, 1.3]. However, we still need to go through the computations in the proof of that proposition in order to show that (η_0, η_1) is an isomorphism because, since the representation might not be faithful, [**5**, 4.1] does not apply.

Corollary 3.8. *Let (X, ϕ_X) be a correspondence over a C^*-algebra A, and let $X(A)$ and $X(X)$ be as in Example* 2.4. *Then $\widetilde{\mathcal{O}}_X$ and $\widetilde{\mathcal{O}}_{X(X)}$ are isomorphic.*

Proof. The isomorphism of Hilbert C^*-bimodules (η_1, η_0) obtained in Proposition 3.7 induces an isomorphism from $E_0(\widetilde{\mathcal{O}}_X) \rtimes E_1(\widetilde{\mathcal{O}}_X)$ to $E_0(\widetilde{\mathcal{O}}_{X(X)}) \rtimes E_1(\widetilde{\mathcal{O}}_{X(X)})$. The statement now follows from Remark 3.6. □

Theorem 3.9. *Let X be a correspondence over a C^*-algebra A, and let (X_∞, A^X_∞) be as in Example* 2.4. *Then $\widetilde{\mathcal{O}}_X \cong A^X_\infty \rtimes X_\infty$.*

Proof. As in Example 2.4, let $(X_\infty, A^X_\infty, \{\lambda^X_n\}, \{\lambda^A_n\})$ be the direct limit of the directed sequence $\{(X_n, A^X_n, \phi^X_n, \phi^{A,X}_n)\}$, and let

$$(i_{X_n}, i_{A^X_n}) : (X_n, A^X_n) \longrightarrow \Big(E_1(\widetilde{\mathcal{O}}_{X_n}), E_0(\widetilde{\mathcal{O}}_{X_n})\Big)$$

and

$$(\eta^n_1, \eta^n_0) : \Big(E_1(\widetilde{\mathcal{O}}_{X_n}), E_0(\widetilde{\mathcal{O}}_{X_n})\Big) \longrightarrow \Big(E_1(\widetilde{\mathcal{O}}_{X_{n+1}}), E_0(\widetilde{\mathcal{O}}_{X_{n+1}})\Big)$$

be as in Proposition 3.7.

Let $\Upsilon^n_i : E_i(\widetilde{\mathcal{O}}_{X_n}) \to E_i(\widetilde{\mathcal{O}}_X)$ be the isomorphism of Hilbert C^*-bimodules defined by $\Upsilon^n_i = (\eta^0_i \eta^1_i \cdots \eta^{n-1}_i)^{-1}$ for all $n \geq 1$, $i = 0, 1$.

By Propositions 2.3 and 3.7 and Remark 2.5, there are homomorphisms of Hilbert C^*-bimodules (i^X_∞, i^A_∞) making Diagram 1 commute.

Since by Proposition 2.3 the pair $(i^X_\infty, i^A_\infty) : (X_\infty, A^X_\infty) \to \widetilde{\mathcal{O}}_X$ is covariant in the sense of [**1**, 2.1], it induces, by the universal property of the crossed product, a *-homomorphism $i : A^X_\infty \rtimes X_\infty \to \widetilde{\mathcal{O}}_X$, which is onto because its image contains $\{S_x : x \in X \cup A\}$. It only remains to check that i^A_∞ is injective, since this would imply by [**4**, 2.9] that so is i, i being covariant for the dual action, [**1**, 3], on the crossed product and the action γ discussed in Remark 3.6 on $\widetilde{\mathcal{O}}_X$.

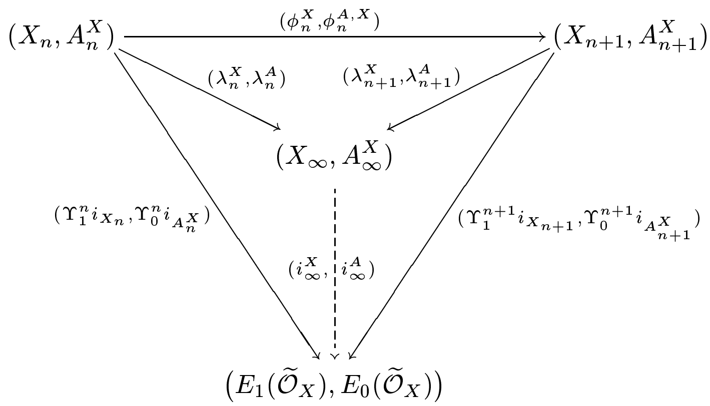

DIAGRAM 1.

First notice that

$$\|\phi_n^X(a_n) \otimes \mathrm{id}_{X_n^{\otimes k}}\| = \|a_n \otimes \mathrm{id}_{X_{n-1}^{\otimes k+1}}\|,$$

for $n \geq 1$, $k \geq 0$, $a_n \in A_n^X \subset \mathcal{L}(X_{n-1})$. In fact, the unitary $\mathrm{id}_{X_{n-1}} \otimes L_{A_n^X, X_{n-1}}^{\otimes k} \otimes \mathrm{id}_{A_n^X}$, for $L_{A_n^X, X_{n-1}}$ as in Notation 1.1, intertwines $\phi_n^X(a_n) \otimes \mathrm{id}_{X_n^{\otimes k}}$ and $a_n \otimes \mathrm{id}_{X_{n-1}^{\otimes k}} \otimes \mathrm{id}_{A_n}$. Now, by Corollary 3.4:

$$\|\phi_n^X(a_n) \otimes \mathrm{id}_{X_n^{\otimes k}}\| = \|a_n \otimes \mathrm{id}_{X_{n-1}^{\otimes k}} \otimes \mathrm{id}_{A_n^X}\| = \|a_n \otimes \mathrm{id}_{X_{n-1}^{\otimes k+1}}\|.$$

It now follows by induction on $m - n$ that

$$\|\phi_{n,m}^X(a_n)\| = \|a_n \otimes \mathrm{id}_{X_{n-1}^{\otimes m-n}}\|,$$

for $m \geq n \geq 1$ and $a_n \in A_n^X$.

We next show that i_∞^A is injective by showing that its restriction to $\lambda_n^A(A_n^X)$ is isometric for all $n \geq 1$. Take $a_n \in A_n^X$ for $n \geq 1$. Then:

$$\begin{aligned}
\|\lambda_n^A(a_n)\| &= \lim_m \|\phi_{n,m}^X(a_n)\| \\
&= \lim_m \|a_n \otimes \mathrm{id}_{X_{n-1}^{\otimes m-n}}\| \\
&= \lim_m \|a_n \otimes \mathrm{id}_{X_{n-1}^{\otimes m}}\| \\
&= \|S_{a_n}\| \\
&= \|i_{A_n^X}(a_n)\| \\
&= \|i_\infty^A(\lambda_n^A(a_n))\|. \qquad \square
\end{aligned}$$

4. Morita equivalence for Cuntz-Pimsner C^*-algebras. We establish in this section a sufficient condition for the Morita equivalence of two augmented Cuntz-Pimsner C^*-algebras. In order to do so, we view these algebras as crossed products by Hilbert C^*-bimodules as in Theorem 3.9, and then we use the condition for the Morita equivalence of crossed products given in [**1, 2, 4**]. Within this section we will be making extensive use of the construction described in Example 2.4.

Lemma 4.1. *Let (Y, ϕ_Y) be a correspondence over a C^*-algebra B, and let M be a right Hilbert C^*-module over B. For $Y(B)$ and $Y(M)$ as in Example 2.4, there is an isometric $*$-homomorphism $O_1 : \mathcal{K}(M \otimes Y) \to \mathcal{K}(Y(M))$ such that*

$$[O_1(\theta_{m_1 \otimes y_1, m_2 \otimes y_2})](m \otimes r) = m_1 \otimes \theta_{y_1, y_2} \phi_Y(\langle m_2, m \rangle) r,$$

for all $m, m_1, m_2 \in M$, $y_1, y_2 \in Y$, and $r \in Y(B)$, where $Y(M)$ is viewed as a $Y(B)$-right Hilbert C^* module.

Besides, $O_1(\theta_{m_1 \otimes y_1 \langle y_2, z_2 \rangle, m_2 \otimes z_1}) = \theta_{m_1 \otimes \theta_{y_1, y_2}, m_2 \otimes \theta_{z_1, z_2}}$.

Proof. It was shown in [**9**, 2.2] that $\mathcal{K}(M \otimes Y)$ and $\mathcal{K}(M \otimes \mathcal{K}(Y))$ are isomorphic. Now $\mathcal{K}(M \otimes \mathcal{K}(Y))$ can be viewed as contained in $\mathcal{K}(Y(M))$, since $M \otimes \mathcal{K}(Y)$ is a closed $Y(B)$-right Hilbert C^*-submodule of $Y(M)$: In fact, if $x_i, y_i \in M \otimes \mathcal{K}(Y)$ for $i = 1, \dots, n$, we have by [**6**, 2.1]:

$$\left\| \sum_{i=1}^{n} \theta_{x_i, y_i}^{Y(M)} \right\| = \|A^{1/2} C^{1/2}\|_{M_n(Y(B))} = \left\| \sum \theta_{x_i, y_i}^{M \otimes \mathcal{K}(Y)} \right\|,$$

where $A_{ij} = \langle x_i, x_j \rangle^{Y(M)} = \langle x_i, x_j \rangle^{M \otimes \mathcal{K}(Y)}$, and analogously for C.

In this way we can obtain an isometric *-homomorphism $I : \mathcal{K}(M \otimes \mathcal{K}(Y)) \hookrightarrow \mathcal{K}(Y(M))$, defined by $I(\theta_{x,y}^{Y(M)}) = \theta_{x,y}^{M \otimes \mathcal{K}(Y)}$. The map O_1 is now defined to be the composition of the isomorphism P in [**9**, 2.2] with I. By keeping track of the proof in [**9**, 2.2], we get the formulas in the statement. In fact, let us identify $w_1 \otimes \widetilde{w}_2$ with θ_{w_1, w_2}, for $w_i \in Y$, $i = 1, 2$.

Then, according to [**9**, 2.2], $[P(\theta_{m_1 \otimes y_1, m_2 \otimes y_2})](m \otimes \theta_{w_1, w_2})]$ gets identified with

$$(\theta_{m_1\otimes y_1, m_2\otimes y_2}(m\otimes w_1))\otimes \widetilde{w}_2 = m_1\otimes y_1\langle y_2, \phi_Y(\langle m_2, m\rangle)w_1\rangle\otimes \widetilde{w}_2,$$

which gets identified with $m_1\otimes \theta_{y_1,y_2}\phi_Y(\langle m_2, m\rangle)\theta_{w_1,w_2}$.

Straightforward computations now show that

$$[P(\theta_{m_1\otimes y_1\langle y_2,z_2\rangle, m_2\otimes z_1})](\xi) = \theta_{m_1\otimes\theta_{y_1,y_2}, m_2\otimes\theta_{z_1,z_2}}(\xi)$$

when $\xi\in M\otimes\mathcal{K}(Y)$. Then, by applying the map I we get:

$$O_1(\theta_{m_1\otimes y_1\langle y_2,z_2\rangle, m_2\otimes z_1}) = \theta_{m_1\otimes\theta_{y_1,y_2}, m_2\otimes\theta_{z_1,z_2}},$$

which yields the formulas in the statement. □

Proposition 4.2. *Let (Y,ϕ_Y) be a correspondence over a C^*-algebra B, and let M be a right Hilbert C^*-module over B. Let L_1 and L_2 be the C^*-subalgebras of $\mathcal{L}(M\otimes Y)$ defined by $L_1=\mathcal{K}(M\otimes Y)$ and $L_2=\{T\otimes id_Y : T\in\mathcal{K}(M)\}$, and let $L=L_1+L_2$ be the C^*-subalgebra of $\mathcal{L}(M\otimes Y)$ generated by $L_1\cup L_2$. Then there is an isomorphism $O: L\to\mathcal{K}(Y(M))$.*

Proof. We first set $O_i: L_i\to\mathcal{K}(Y(M))$, for $i=1,2$, as follows: O_1 is the *-homomorphism defined in Lemma 4.1 and, in view of Corollary 3.4, we set $O_2(T\otimes \mathrm{id}_Y) = T\otimes\mathrm{id}_{Y(B)}$, for $T\in\mathcal{K}(M)$.

Our aim is to define $O(T_1+T_2) = O_1(T_1)+O_2(T_2)$ for $T_i\in L_i$, $i=1,2$. To make sense of this, first note that

$$O_i(T)\otimes\mathrm{id}_Y = (\mathrm{id}_M\otimes L_{Y(B),Y})^{-1}T(\mathrm{id}_M\otimes L_{Y(B),Y}),$$

for $T\in L_i$, $i=1,2$, and $L_{Y(B),Y}$ as in Notation 1.1.

The equality is easily checked for $i=2$ whereas, if $T=\theta_{m_1\otimes y_1, m_2\otimes y_2}$, for $m_1,m_2\in M$, $y_1,y_2\in Y$, and $m\otimes r\otimes y\in M\otimes Y(B)\otimes Y$, then:

$$\begin{aligned}
[(\mathrm{id}_M\otimes L_{Y(B),Y})^{-1}&T(\mathrm{id}_M\otimes L_{Y(B),Y})](m\otimes r\otimes y)\\
&= m_1\otimes L_{Y(B),Y}^{-1}(y_1\langle m_2\otimes y_2, m\otimes ry\rangle)\\
&= m_1\otimes L_{Y(B),Y}^{-1}(y_1\langle y_2, \phi_Y(\langle m_2,m\rangle)ry\rangle)\\
&= m_1\otimes\theta_{y_1,y_2}\phi_Y(\langle m_2,m\rangle)r\otimes y\\
&= (O_1(T)\otimes\mathrm{id}_Y)(m\otimes r\otimes y).
\end{aligned}$$

On the other hand, it is straightforwardly verified that

$$O_i(T) \otimes \mathrm{id}_{Y(B)} = (\mathrm{id}_M \otimes L_{Y(B),Y(B)})^{-1} O_i(T)(\mathrm{id}_M \otimes L_{Y(B),Y(B)}),$$

for $T \in L_i$, $i = 1, 2$.

By virtue of Corollary 3.4 and the identities above, we have, for $T_i \in L_i$, $i = 1, 2$:

$$\begin{aligned}
\|O_1(T_1) + O_2(T_2)\| & \\
&= \|(\mathrm{id}_M \otimes L_{Y(B),Y(B)})[(O_1(T_1) + O_2(T_2)) \otimes \mathrm{id}_{Y(B)}] \\
&\qquad \times (\mathrm{id}_M \otimes L_{Y(B),Y(B)})^{-1}\| \\
&= \|(O_1(T_1) + O_2(T_2)) \otimes \mathrm{id}_{Y(B)}\| \\
&= \|(O_1(T_1) + O_2(T_2)) \otimes \mathrm{id}_Y\| \\
&= \|(\mathrm{id}_M \otimes L_{Y(B),Y})^{-1}(T_1 + T_2)(\mathrm{id}_M \otimes L_{Y(B),Y})\| \\
&= \|T_1 + T_2\|,
\end{aligned}$$

which shows that O can be defined as above, and it is an isometric linear map that preserves the involution.

Now, if $T_i \in L_i$, $T_1 = \theta_{m_1 \otimes y_1, m_2 \otimes y_2}$ and $T_2 = S \otimes \mathrm{id}_Y$, then

$$\begin{aligned}
O_1(T_2 T_1) &= O_1(\theta_{S m_1 \otimes y_1, m_2 \otimes y_2}) \\
&= (S \otimes \mathrm{id}_{Y(B)}) O_1(\theta_{m_1 \otimes y_1, m_2 \otimes y_2}) \\
&= O_2(T_2) O_1(T_1).
\end{aligned}$$

It follows that O is multiplicative from the preceding and from the fact that O_1 and O_2 are *-homomorphisms. It only remains to show that O is onto. This fact follows from the following identities that can be verified directly from the definitions:

- $\theta_{m_1 \otimes \phi_Y(b_1), m_2 \otimes \phi_Y(b_2)} = \theta_{m_1 b_1, m_2 b_2} \otimes \mathrm{id}_{Y(B)} = O(\theta_{m_1 b_1, m_2 b_2} \otimes \mathrm{id}_Y)$
- $\theta_{m_1 \otimes \theta_{y_1, y_2}, m_2 \otimes \phi_Y(b)} = O(\theta_{m_1 \otimes y_1, m_2 b \otimes y_2})$
- $\theta_{m_1 \otimes \theta_{y_1, y_2}, m_2 \otimes \theta_{z_1, z_2}} = O(\theta_{m_1 \otimes y_1 \langle y_2, z_2 \rangle, m_2 \otimes z_1})$,

where $m_1, m_2 \in M$, $y_1, y_2, z_1, z_2 \in Y$, and $b, b_1, b_2 \in B$. □

Remark 4.3. Notice that we have shown at the beginning of the proof of Proposition 4.2 the identity

$$(\mathrm{id}_M \otimes L_{Y(B),Y})^{-1} T (\mathrm{id}_M \otimes L_{Y(B),Y}) = O(T) \otimes \mathrm{id}_Y$$

for any T belonging to the C^*-subalgebra of $\mathcal{L}(M\otimes Y)$ generated by $\mathcal{K}(M\otimes Y)\cup\{T\otimes \mathrm{id}_Y : T\in\mathcal{K}(M)\}$.

Proposition 4.4. *Let X and Y be correspondences over C^*-algebras A and B, respectively, and let M be an $A-B$ Hilbert C^*-bimodule that is full on the left and such that there is an isomorphism $J: X\otimes M\to M\otimes Y$ of $A-B$ correspondences.*

Let $I: X(A)\to\mathcal{K}(Y(M))$ be given by $I(T)=O(J(T\otimes\mathrm{id}_M)J^{-1})$, for O as in Proposition 4.2. Then I is an isomorphism, and $I(\phi_X(a))=\phi_M(a)\otimes\mathrm{id}_{Y(B)}$ for all $a\in A$.

Proof. By Proposition 4.2, it suffices to show that $T\mapsto J(T\otimes\mathrm{id}_M)J^{-1}$ is an isomorphism from $X(A)$ to the C^*-subalgebra L of $\mathcal{L}(M\otimes Y)$ generated by $\mathcal{K}(M)\otimes\mathrm{id}_Y$ and $\mathcal{K}(M\otimes Y)$.

The image of $X(A)$ by the map $T\mapsto T\otimes\mathrm{id}_M$ is the C^*-subalgebra C of $\mathcal{L}(X\otimes M)$ generated by $\mathcal{K}(X\otimes M)\cup\{\phi_X(a)\otimes\mathrm{id}_M : a\in A\}$, since $\theta_{x_1\langle m_1,m_2\rangle_A,x_2}\otimes\mathrm{id}_M=\theta_{x_1\otimes m_1,x_2\otimes m_2}$ for all $x_1,x_2\in X$, $m_1,m_2\in M$.

Besides, if $T\otimes\mathrm{id}_M=0$ for some $T\in\mathcal{L}(X)$, then $0=\langle Tx\otimes m, Tx\otimes m\rangle=\langle m,\langle Tx,Tx\rangle m\rangle$, for all $m\in M$, $x\in X$. It follows that $T=0$ because A acts faithfully on M.

Notice now that conjugation by J carries C isomorphically into L because

$$J\left(\theta_{x_1\otimes m_1,x_2\otimes m_2}\right)J^{-1}=\theta_{J(x_1\otimes m_1),J(x_2\otimes m_2)},$$
$$J(\phi_X(a)\otimes\mathrm{id}_M))J^{-1}=\phi_M(a)\otimes\mathrm{id}_Y,$$

for all $x_1,x_2\in X$, $m_1,m_2\in M$, and $a\in A$. Besides, $\{\phi_M(a): a\in A\}=\mathcal{K}(M)$.

Finally, $I(\phi_X(a))=O(\phi_M(a)\otimes\mathrm{id}_Y)=\phi_M(a)\otimes\mathrm{id}_{Y(B)}$, for all $a\in A$. ◻

Proposition 4.5. *Let X, Y, and M be as in Proposition 4.4. Let $\{(X_n, A_n^X,\phi_n^X,\phi_n^A)\}$ and $\{(M_n^Y,B_n^Y,\phi_n^{M,Y},\phi_n^B)\}$ be the directed sequences defined in Example 2.4, and let $(X_\infty, A_\infty^X,\{\lambda_n^X\},\{\lambda_n^A\})$ and $(M_\infty^Y,B_\infty^Y,\{\mu_n^M\},\{\mu_n^B\})$, respectively, denote their direct limits.*

Then M_∞^Y is an $A_\infty^X-B_\infty^Y$ is a Hilbert C^*-bimodule that is full on the left.

Besides, the canonical maps $(\lambda_n^A, \mu_n^M, \mu_n^B) : (A_n^X, M_n^Y, B_n^Y) \to (A_\infty^X, M_\infty^Y, B_\infty^Y)$ *are homomorphisms of Hilbert* C^**-bimodules.*

If M *is also full on the right, and* Y *is left nondegenerate as a* B*-module, that is, if* $\phi_Y(B)Y = Y$*, then* M_∞^Y *is an* $A_\infty^X - B_\infty^Y$ *Morita equivalence bimodule.*

Proof. All the statements involving the right structure except for the last one, which we discuss at the end, were taken care of in Proposition 2.3, so we focus on the left structure. We have shown that M_1^Y is an $A_1^X - B_1^Y$ left full Hilbert C^*-bimodule by identifying A_1^X with $\mathcal{K}(M_1^Y)$ via the isomorphism I of Proposition 4.4.

Our aim is to show, in the notation of Example 2.4, that M_n^Y is an $A_n^X - B_n^Y$ Hilbert C^*-bimodule that is full on the left in a compatible way with the corresponding directed sequences, which will provide M_∞^Y with a structure of $A_\infty^X - B_\infty^Y$ Hilbert C^*-bimodule.

First notice that the map $J_1 : X_1 \otimes_{A_1^X} M_1^Y \to M_1^Y \otimes_{B_1^Y} Y_1$ given by

$$J_1 = (\mathrm{id}_M \otimes L^{-1}_{B_1^Y, Y} \otimes \mathrm{id}_{B_1^Y})(J \otimes \mathrm{id}_{B_1^Y})(\mathrm{id}_X \otimes L_{A_1^X, M_1^Y})$$

is an isomorphism of $A_1^X - B_1^Y$ correspondences. Note that J_1 preserves the left action of A_1^X because, by Remark 4.3:

$$\begin{aligned}(\mathrm{id}_M \otimes L^{-1}_{B_1^Y, Y} &\otimes \mathrm{id}_{B_1^Y})(J \otimes \mathrm{id}_{B_1^Y})(r \otimes \mathrm{id}_{M_1^Y}) \\ &= [(\mathrm{id}_M \otimes L^{-1}_{B_1^Y, Y}) J (r \otimes \mathrm{id}_M)] \otimes \mathrm{id}_{B_1^Y} \\ &= [(O(J(r \otimes \mathrm{id}_M)J^{-1}) \otimes \mathrm{id}_Y)(\mathrm{id}_M \otimes L^{-1}_{B_1^Y, Y})J] \otimes \mathrm{id}_{B_1^Y} \\ &= (I(r) \otimes \mathrm{id}_{Y_1})(\mathrm{id}_M \otimes L^{-1}_{B_1^Y, Y} \otimes \mathrm{id}_{B_1^Y})(J \otimes \mathrm{id}_{B_1^Y}),\end{aligned}$$

for $r \in A_1^X$. Besides, J_1 is onto and preserves the right action of B_1^Y and the B_1^Y-valued inner product because so do the maps composed to get J.

Notice also that

$$\begin{array}{ccc} X \otimes_A M & \xrightarrow{\quad J \quad} & M \otimes_B Y \\ {\scriptstyle \phi_0^X \otimes \phi_0^{M,Y}} \big\downarrow & & \big\downarrow {\scriptstyle \phi_0^{M,Y} \otimes \phi_0^Y} \\ X_1 \otimes_{A_1^X} M_1^Y & \xrightarrow{\quad J_1 \quad} & M_1^Y \otimes_{B_1^Y} Y_1 \end{array}$$

commutes because

$$
\begin{aligned}
&(\mathrm{id}_M \otimes L^{-1}_{B_1^Y,Y} \otimes \mathrm{id}_{B_1^Y})(J \otimes \mathrm{id}_{B_1^Y}) \\
&\qquad\qquad\qquad \times (\mathrm{id}_X \otimes L_{A_1^X, M_1^Y})(\phi_0^X \otimes \phi_0^{M,Y})(xa \otimes mb) \\
&\quad = (\mathrm{id}_M \otimes L^{-1}_{B_1^Y,Y} \otimes \mathrm{id}_{B_1^Y})(J \otimes \mathrm{id}_{B_1^Y}) \\
&\qquad \times (\mathrm{id}_X \otimes L_{A_1^X, M_1^Y})(x \otimes \phi_X(a) \otimes m \otimes \phi_Y(b)) \\
&\quad = (\mathrm{id}_M \otimes L^{-1}_{B_1^Y,Y} \otimes \mathrm{id}_{B_1^Y})(J \otimes \mathrm{id}_{B_1^Y}) \\
&\qquad \times [x \otimes I(\phi_X(a))(m \otimes \phi_Y(b))] \\
&\quad = (\mathrm{id}_M \otimes L^{-1}_{B_1^Y,Y} \otimes \mathrm{id}_{B_1^Y})(J \otimes \mathrm{id}_{B_1^Y})(x \otimes am \otimes \phi_Y(b)) \\
&\quad = (\mathrm{id}_M \otimes L^{-1}_{B_1^Y,Y} \otimes \mathrm{id}_{B_1^Y})J(xa \otimes m) \otimes \phi_Y(b) \\
&\quad = (\phi_0^{M,Y} \otimes \phi_0^Y)J(xa \otimes mb),
\end{aligned}
$$

for $x \in X$, $a \in A$, $m \in M$ and $b \in B$.

Now this yields, by Proposition 4.4, an isomorphism $I_2 : A_2^X \to \mathcal{K}(M_2^Y)$. Furthermore, the diagram

$$
\begin{array}{ccc}
A_1^X & \xrightarrow{\ I_1\ } & \mathcal{K}(M_1^Y) \\
{\scriptstyle \phi_1^A}\big\downarrow & & \big\downarrow{\scriptstyle (\phi_1^{M,Y})_*} \\
A_2^X & \xrightarrow{\ I_2\ } & \mathcal{K}(M_2^Y)
\end{array}
$$

commutes, since by Proposition 4.4 we have, for $r \in A_1^X$:

$$
I_2(\phi_1^{A,X}(r)) = \phi_{M_1}(r) \otimes \mathrm{id}_{B_2^Y} = I_1(r) \otimes \mathrm{id}_{B_2^Y} = [(\phi_1^{M,Y})_*](I_1(r)),
$$

the last equality being due to the fact that

$$
\theta_{\phi_1^{M,Y}(m_1b_1), \phi_1^{M,Y}(m_2b_2)} = \theta_{m_1 \otimes \phi_1^{B,Y}(b_1), m_2 \otimes \phi_1^{B,Y}(b_2)} = \theta_{m_1b_1, m_2b_2} \otimes \mathrm{id}_{B_2^Y},
$$

for $m_i \in M_1$, $b_i \in B_1$, and $i = 1, 2$.

It is clear now that, by iterating this construction, we get isomorphisms I_n such that the diagram

$$\begin{array}{ccc} A_n^X & \xrightarrow{\quad I_n \quad} & \mathcal{K}(M_n^Y) \\ {\scriptstyle \phi_n^{A,X}}\downarrow & & \downarrow{\scriptstyle (\phi_n^{M,Y})_*} \\ A_{n+1}^X & \xrightarrow{\quad I_{n+1} \quad} & \mathcal{K}(M_{n+1}^Y) \end{array}$$

commutes for all $n \geq 0$.

This shows that A_∞^X is isomorphic to the direct limit of $\{(\mathcal{K}(M_n^Y), (\phi_n^{M,Y})_*)\}$, which by Proposition 2.6 is $(\mathcal{K}(M_\infty^Y), (\mu_n^M)_*)$. Therefore, M_∞^Y is an $A_\infty^X - B_\infty^Y$ Hilbert C^*-bimodule that is full on the left, with left structure defined by

$$\begin{aligned} \lambda_n^A(a_n)\mu_n^M(m_n) := \mu_n^M(a_n m_n),\ \langle \mu_n^M(m_n), \mu_n^M(m_n')\rangle_{A_\infty^X} \\ := \lambda_n^A(\langle m_n, m_n'\rangle_{A_n^X}), \end{aligned}$$

for $m_n, m_n' \in M_n^Y$ and $a_n \in A_n^X$, where we write, as we will do from now on, $a_n m_n$ and $\langle m_n, m_n'\rangle_{A_n^X}$ instead of $[I_n(a_n)](m_n)$ and $I_n^{-1}(\theta_{m_n, m_n'})$, respectively.

Notice that the last equality shows that (λ_n^A, μ_n^M) is a homomorphism of left Hilbert C^*-modules.

If Y is nondegenerate on the left, and M is a Morita equivalence bimodule, then M_1^Y is an $A_1^X - B_1^Y$ Morita equivalence bimodule because $\langle m\otimes r, n\otimes s\rangle = r^*\phi_Y(\langle m,n\rangle_R)s$, for $m, n \in M$ and $r, s \in B_1^Y$. Therefore, as one sees by taking an approximate identity for B_1^Y, $\langle M_1^Y, M_1^Y\rangle_R$ contains $\operatorname{Im}\phi_Y$ and $\phi_Y(B)\mathcal{K}(Y)$. But nondegeneracy implies that $\phi_Y(B)\mathcal{K}(Y) = \mathcal{K}(Y)$ since, given $x, y \in Y$, then

$$\theta_{x,y} = \theta_{\phi_Y(b)x',y} = \phi_Y(b)\theta_{x',y},$$

for some $x' \in Y$ and $b \in B$. Thus we conclude that M_1 is full on the right as well.

It will follow by induction that M_n is full on the right for all $n \geq 0$ once we show that Y_n is always nondegenerate on the left as a B_n^Y-module. In fact:

$$\phi_{Y_n}(B_n^Y)Y_n = \phi_{Y_n}(B_n^Y)(Y_{n-1}\otimes B_n^Y) = B_n^Y Y_{n-1}\otimes B_n^Y = Y_{n-1}\otimes B_n^Y,$$

since $B_n^Y \supset \mathcal{K}(Y_{n-1})$.

Finally, we conclude that in that case M_∞^Y is full on the right because $\langle M_\infty^Y, M_\infty^Y\rangle$ contains $\mu_n^B(\langle M_n^Y, M_n^Y\rangle)$ for all $n \geq 0$. $\square$

Remark 4.6. Let (Y, ϕ_Y) be a correspondence over a C^*-algebra B, and let Y_n, B_n^Y and M_n^Y be as in Example 2.4. The proof of Proposition 4.5 shows that

(1) The B_n^Y-left module Y_n is nondegenerate for all $n \geq 1$. Of course, this might fail for $n = 0$.

(2) If X, Y and M are as in Proposition 4.5, M is full on the right, and Y is nondegenerate, then M_n^Y is an $A_n^X - B_n^Y$ Morita equivalence bimodule such that the $A_n^X - B_n^Y$ correspondences $X_n \otimes M_n^Y$ and $M_n^Y \otimes Y_n$ are isomorphic for all $n \geq 0$.

Theorem 4.7. *Let (X, ϕ_X) and (Y, ϕ_Y) be correspondences over the C^*-algebras A and B, respectively. If, in the notation of Example* 2.4, *there exists an $A_{n_0}^X - B_{m_0}^Y$ Morita equivalence bimodule M such that $X_{n_0} \otimes M$ and $M \otimes Y_{m_0}$ are isomorphic as $A_{n_0}^X - B_{m_0}^Y$ correspondences for some $n_0 \geq 0$, $m_0 \geq 1$, then the augmented Cuntz-Pimsner C^*-algebras $\widetilde{\mathcal{O}}_X$ and $\widetilde{\mathcal{O}}_Y$ are Morita equivalent.*

Proof. The bimodules X_∞ and Y_∞ and the C^*-algebras A_∞^X and B_∞^Y of Example 2.4 can be obtained as the limits of the corresponding directed sequences starting, respectively, at n_0 and m_0. Besides, the directed sequence $\{(M_n^Y, \phi_n^{M,Y})\}_{n \geq m_0}$ can be constructed as in Example 2.4. Our aim is to show that $X_\infty \otimes_{A_\infty^X} M_\infty^Y$ and $M_\infty^Y \otimes_{B_\infty^Y} Y_\infty$ are isomorphic as $A_\infty^X - B_\infty^Y$ Hilbert C^*-bimodules. It follows from the remarks above that we can assume that $n_0 = m_0 = 0$ and, in view of the last part of Remark 4.6, that Y is left nondegenerate over B. The result will then follow from Theorem 3.9, Proposition 4.5 and [**1**, 4.2].

As in Proposition 4.5 and Example 2.4 we have the commuting diagrams:

$$\begin{array}{ccc} X_n & \xrightarrow{\phi_n^X} & X_{n+1} \\ {\scriptstyle \lambda_n^X}\downarrow & \swarrow {\scriptstyle \lambda_{n+1}^X} & \\ X_\infty & & \end{array} \qquad \begin{array}{ccc} A_n^X & \xrightarrow{\phi_n^{A,X}} & A_{n+1}^X \\ {\scriptstyle \lambda_n^A}\downarrow & \swarrow {\scriptstyle \lambda_{n+1}^A} & \\ A_\infty^X & & \end{array}$$

$$\begin{array}{ccc} Y_n & \xrightarrow{\phi_n^Y} & Y_{n+1} \\ {\scriptstyle \mu_n^Y}\downarrow & \swarrow {\scriptstyle \mu_{n+1}^Y} & \\ Y_\infty & & \end{array} \qquad \begin{array}{ccc} B_n^Y & \xrightarrow{\phi_n^{B,Y}} & B_{n+1}^Y \\ {\scriptstyle \mu_n^B}\downarrow & \swarrow {\scriptstyle \mu_{n+1}^B} & \\ B_\infty^Y & & \end{array}$$

$$\begin{array}{ccc} M_n^Y & \xrightarrow{\phi_n^{M,Y}} & M_{n+1}^Y \\ {\scriptstyle \mu_n^M}\downarrow & \swarrow {\scriptstyle \mu_{n+1}^M} & \\ M_\infty^Y & & \end{array} \qquad \begin{array}{ccc} X_n \otimes_{A_n^X} M_n^Y & \xrightarrow{J_n} & M_n^Y \otimes_{B_n^Y} Y_n \\ {\scriptstyle \phi_n^X \otimes \phi_n^{M,Y}}\downarrow & & \downarrow {\scriptstyle \phi_n^{M,Y} \otimes \phi_n^Y} \\ X_{n+1} \otimes_{A_{n+1}^X} M_{n+1}^Y & \xrightarrow{J_{n+1}} & M_{n+1}^Y \otimes_{B_{n+1}^Y} Y_{n+1}. \end{array}$$

Notice that, if $m, m' \in M_n^Y$, $y, y' \in Y_n$, then by Propositions 2.3 and 4.5,

$$\begin{aligned} \langle \mu_n^M(m) \otimes \mu_n^Y(y), \mu_n^M(m') \otimes \mu_n^Y(y') \rangle &= \langle \mu_n^Y(y), \mu_n^B(\langle m, m' \rangle)\mu_n^Y(y') \rangle \\ &= \mu_n^B(\langle y, \langle m, m' \rangle y' \rangle) \\ &= \mu_n^B(\langle m \otimes y, m' \otimes y' \rangle) \end{aligned}$$

and

$$\begin{aligned} \langle \lambda_n^X(x) \otimes \mu_n^M(m), \lambda_n^X(x') \otimes \mu_n^M(m') \rangle &= \langle \mu_n^M(m), \lambda_n^A(\langle x, x' \rangle)\mu_n^M(m') \rangle \\ &= \langle \mu_n^M(m), \mu_n^M(\langle x, x' \rangle m') \rangle \\ &= \mu_n^B(\langle x \otimes m, x' \otimes m' \rangle) \end{aligned}$$

for $x, x' \in X_n$ and $m, m' \in M_n^Y$.

We now want to define $J_\infty : X_\infty \otimes_{A_\infty^X} M_\infty^Y \to M_\infty^Y \otimes_{B_\infty^Y} Y_\infty$ by

$$J_\infty((\lambda_n^X \otimes \mu_n^M)(x_n \otimes m_n)) := (\mu_n^M \otimes \mu_n^Y)J_n(x_n \otimes m_n).$$

Now,

$$\begin{aligned} &\langle (\mu_n^M \otimes \mu_n^Y)J_n(x_n \otimes m_n), (\mu_n^M \otimes \mu_n^Y)J_n(x_n' \otimes m_n') \rangle \\ &\qquad = \mu_n^B(\langle J_n(x_n \otimes m_n), J_n(x_n' \otimes m_n') \rangle) \\ &\qquad = \mu_n^B(\langle x_n \otimes m_n, x_n' \otimes m_n' \rangle) \\ &\qquad = \langle (\lambda_n^X \otimes \mu_n^M)(x_n \otimes m_n), (\lambda_n^X \otimes \mu_n^M)(x_n' \otimes m_n') \rangle. \end{aligned}$$

This shows that J_∞ as defined above extends to a right Hilbert C^*-module homomorphism that preserves the left action of A_∞. In fact, by Proposition 4.5, given $a_n \in A_n$, $x_n \in X_n$ and $m_n \in M_n$, we have

$$\begin{aligned}
&J_\infty[\lambda_n(a_n)\cdot(\lambda_n^X(x_n)\otimes\mu_n^M(m_n))]\\
&\qquad = J_\infty[\lambda_n^X(a_nx_n)\otimes\mu_n^M(m_n)]\\
&\qquad = (\mu_n^M\otimes\mu_n^Y)J_n(a_nx_n\otimes m_n)\\
&\qquad = (\mu_n^M\otimes\mu_n^Y)(\phi_{M_n}(a_n)\otimes \mathrm{id}_{Y_n})J_n(x_n\otimes m_n)\\
&\qquad = \lambda_n(a_n)\cdot J_\infty(x_n\otimes m_n).
\end{aligned}$$

Analogous computations show that J_∞ preserves the right action. Besides, J_∞ is onto because its image contains $\cup_n(\mu_n^M(M_n)\otimes\mu_n^Y(Y_n))$, which is dense in $M_\infty^Y\otimes Y_\infty$.

It remains to show that J_∞ preserves the left inner product. This follows as in [**2**, 1.2]: if $\xi_0,\xi_1,\xi_2\in X\otimes M$, then

$$\begin{aligned}
\langle J_\infty(\xi_0),J_\infty(\xi_1)\rangle_{A_\infty^X}J_\infty(\xi_2) &= J_\infty(\xi_0)\langle J_\infty(\xi_1),J_\infty(\xi_2)\rangle_{B_\infty^Y}\\
&= J_\infty(\xi_0)\langle\xi_1,\xi_2\rangle_{B_\infty^Y}\\
&= J_\infty\left(\xi_0\langle\xi_1,\xi_2\rangle_{B_\infty}\right)\\
&= J_\infty\left(\langle\xi_0,\xi_1\rangle_{A_\infty^X}\xi_2\right)\\
&= \langle\xi_0,\xi_1\rangle_{A_\infty^X}J_\infty(\xi_2). \qquad \square
\end{aligned}$$

Remark 4.8. A similar result was shown by Muhly and Solel [**8**] for Cuntz-Pimsner C^*-algebras $\mathcal{O}_X$ of correspondences (X,ϕ_X) such that ϕ_X is injective and X left nondegenerate. Our result for augmented Cuntz-Pimsner C^*-algebras does not require the faithfulness of ϕ. Nondegeneracy, however, might play a role, as the following corollary shows.

Corollary 4.9. *Let (X,ϕ_X) and (Y,ϕ_Y) be correspondences over the C^*-algebras A and B, and let M be a Morita equivalence $A-B$ bimodule such that $X\otimes M$ and $M\otimes Y$ are isomorphic as $A-B$ correspondences. If Y is left nondegenerate, then the augmented Cuntz-Pimsner C^*-algebras $\widetilde{\mathcal{O}}_X$ and $\widetilde{\mathcal{O}}_Y$ are Morita equivalent.*

Proof. By Remark 4.6 the conditions in Theorem 4.7 are then met. $\square$

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REFERENCES

1. B. Abadie, S. Eilers and R. Exel, *Morita equivalence for crossed products by Hilbert C^*-bimodules*, Trans. Amer. Math. Soc. **350** (1998), 3043–3054.

2. B. Abadie and R. Exel, *Hilbert C^*-bimodules over commutative C^*-algebras and an isomorphism condition for quantum Heisenberg manifolds*, Rev. Math. Phys. **9** (1997), 411–423.

3. L.G. Brown, J.A. Mingo and N. Shen, *Quasi-multipliers and embeddings of Hilbert C^*-bimodules*, Canad. J. Math. **46** (1994), 1150–1174.

4. R. Exel, *Circle actions on C^*-algebras, partial automorphisms, and a generalized Pimsner-Voiculescu exact sequence*, J. Funct. Anal. **122** (1994), 361–401.

5. N. Fowler, P. Muhly and I. Raeburn, *Representations of Cuntz-Pimsner algebras*, Indiana U. Math. J. **52** (2003), 569–603.

6. T. Kajiwara, C. Pinzari and Y. Watatani, *Ideal structure and simplicity of the C^*-algebras generated by Hilbert bimodules*, J. Funct. Anal. **159** (1998), 295–322.

7. C. Lance, *Hilbert C^*-modules. A toolkit for operator algebraists*, London Math. Soc. Lect. Notes Ser. **210**, Cambridge University Press, Cambridge, 1995.

8. P. Muhly and B. Solel, *On the Morita equivalence of tensor algebras*, Proc. London. Math. Soc. **81** (2000), 113–168.

9. M.V. Pimsner, *A class of C^*-algebras generalizing both Cuntz-Krieger algebras and crossed products by* **Z**, Fields Inst. Comm. **12**, Amer. Math. Soc., 1997.

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