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EXISTENCE OF COMPLEMENTED SUBSPACES ISOMORPHIC TO l^q IN QUASI BANACH INTERPOLATION SPACES

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ABSTRACT. Let (E_0, E_1) be a compatible couple of quasi Banach spaces and $0 < \theta < 1$, $0 < q < \infty$. We present a sufficient condition in order that the quasi Banach interpolation space $(E_0, E_1)_{\theta,q}$ has complemented subspaces isomorphic to l^q, extending in this way Levy's theorem. As an application we show that every space $(l^{p_0}(\mu), l^{p_1}(\mu))_{\theta,q}$, $0 < p_0 < p_1 \leq \infty$, has complemented subspaces isomorphic to l^q except in the case that $0 < p_\theta < q < 1$, $1/(p_\theta) := (1-\theta)/p_0 + \theta/p_1$ and $0 < \alpha \leq \mu(\{n\}) \leq \beta < \infty$, $n \in \mathbf{N}$ for some α, β in $\mathbf{R}$.

1. Introduction.

Let (E_0, E_1) be a compatible couple of Banach spaces. Levy has proved in [**10**] that every interpolated space $(E_0, E_1)_{\theta,q}$, $1 \leq q < \infty$, $0 < \theta < 1$ (real interpolation method) such that $E_0 \cap E_1$ is not closed in $E_0 + E_1$ has complemented subspaces isomorphic to l^q. In [**4**], Brudnyi and Krugljak extend the scope of Levy's result to more general interpolation functors.

The theory of real interpolation of Banach spaces has been extended to quasi Banach spaces by several authors, see for example, Krée [**9**], Holmstedt [**6**], Peetre [**15**] and Sagher [**16**]. Essentially with the same hypothesis of Levy, although with a slightly different presentation, we have shown in [**12**] that such interpolation spaces $(E_0, E_1)_{\theta,q}$, $0 < q < \infty$ of quasi Banach spaces E_0, E_1 have subspaces isomorphic to l^q. In this paper we prove that, *under suitable hypotheses*, these spaces have indeed *complemented subspaces isomorphic to* l^q too.

In general, our notation is standard. We recall that a quasi Banach space is a vector space E over the field $\mathbf{K}$ of real or complex numbers which is complete under the metric $d(x,y) = \|x - y\|$ where $\|.\| : E \to [0, \infty[$ is a quasi-norm, i.e., a function with properties

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1) $\|x\| = 0$ if and only if $x = 0$.

2) For all $x \in E$, for all $\lambda \in \mathbf{K}$ $\|\lambda x\| = |\lambda|\, \|x\|$.

3) There is a $K_E \geq 1$ such that for all $x, y \in E$ $\|x + y\| \leq K_E(\|x\| + \|y\|)$.

$B(E)$ will denote the closed unit ball in the quasi Banach space E. We say that a sequence $\{x_n\}_{n=1}^{\infty} \subset E$ is a *semi-normalized* sequence if there are numbers $0 < \alpha < \beta$ such that $\alpha \leq \|x_n\| \leq \beta$ for every $n \in \mathbf{N}$.

The paper is organized as follows. Section 2 gives definitions and results about the real method of interpolation of quasi Banach spaces, which are the foundation of our main argumentations in subsequent sections. Likely some of them are well known but dispersed in the current literature. Since we do not know concrete written references for them and since they involve delicate constructions, we prefer to give a detailed account for all necessary facts.

Section 3 is our main general result. In spite of its quite involved hypothesis, this result can be applied to many important concrete examples. That is done in Section 4, where we prove that the interpolation quasi Banach spaces $(l^{p_0}(\mu), l^{p_1}(\mu))_{\theta,q}$, $0 < p_0 < p_1 \leq \infty$, $0 < q < \infty$, μ an arbitrary measure in $\mathbf{N}$, have complemented subspaces isomorphic to l^q with the unique exception of a measure μ such that there exists $0 < \alpha < \beta < \infty$ such that $\alpha \leq \mu(\{n\}) \leq \beta$ for every $n \in \mathbf{N}$ and $0 < p_\theta < q < 1$, where $1/p_\theta := (1-\theta)/p_0 + (\theta/p_1)$.

2. On interpolation of quasi Banach spaces. Suppose (E_0, E_1) is a compatible couple of quasi Banach spaces (this means that E_0 and E_1 are continuously embedded in a same larger Hausdorff topological vector space). The spaces $E_0 + E_1$ and $E_0 \cap E_1$ will always be endowed with canonical quasi-norms

$$\|x\|_{E_0+E_1} = \inf\left\{\|a\|_{E_0} + \|b\|_{E_1} \;\middle|\; x = a + b, a \in E_0,\ b \in E_1\right\}$$

and

$$\|x\|_{E_0 \cap E_1} = \max\left\{\|x\|_{E_0}, \|x\|_{E_1}\right\},$$

respectively.

For every $h \in \mathbf{Z}$, let $\|.\|_h$ be the Minkowski functional of the set $e^{-\theta h}B(E_0) + e^{(1-\theta)h}B(E_1) \subset E_0 + E_1$. $\|.\|_h$ is a quasi-norm on $E_0 + E_1$

by the classical Aoki-Rolewicz theorem of equivalence between quasi-norms and r-norms, $0 < r \leq 1$, (see [**8**] for details) and some facts proved in [**7**, page 105]. As in the classical Banach space case, $\|.\|_h$ is equivalent to the initial quasi-norm of $E_0 + E_1$ and, hence, for every $n \in \mathbf{N}$, $(\sum_{|h|\leq n} \|x\|_h^q)^{1/q}$ is another equivalent quasi-norm in $E_0 + E_1$ (the proof is the same as the one given in [**1**] in the normed case).

As in the Banach case, we consider the $K(t, x, E_0, E_1)$-functional defined for every $t \geq 0$ and every $x \in E_0 + E_1$ by

$$\begin{aligned} &K(t, x, E_0, E_1) \\ &\quad := \inf\left\{\|x_0\|_{E_0} + t\|x_1\|_{E_1} \;\middle|\; x = x_0 + x_1,\ x_0 \in E_0,\ x_1 \in E_1\right\}. \end{aligned}$$

It is easy to see that

$$\begin{aligned} K(t, x+y, E_0, E_1) \leq \max&\{K_{E_0}, K_{E_1}\} \\ &\times (K(t, x, E_0, E_1) + K(t, y, E_0, E_1)). \end{aligned} \tag{1}$$

Then, given $0 < \theta < 1$ and $0 < q \leq \infty$, the interpolation space $(E_0, E_1)_{\theta,q}$ is defined as

$$(E_0, E_1)_{\theta,q} := \left\{x \in E_0 + E_1 \;\middle|\; \frac{K(t, x, E_0, E_1)}{t^\theta} \in L^q\left([0, \infty[\,, \frac{dt}{t}\right)\right\},$$

which becomes a quasi Banach space provided with the quasi norm

$$\|x\|_{\theta,q} = \left\|\frac{K(t, x, E_0, E_1)}{t^\theta}\right\|_{L^q\left([0,\infty[, \frac{dt}{t}\right)}$$

(see [**15**] for details). However, for our purposes it will be more convenient to use on $(E_0, E_1)_{\theta,q}$ some equivalent alternative quasi-norms. Since, for every $h \in \mathbf{Z}$, we obtain

$$K(e^h, x, E_0, E_1) \leq e\, K(e^{h-1}, x, E_0, E_1),$$

we have for $0 < q < \infty$,

$$\begin{aligned}\bigg(\int_0^\infty \frac{\big(\sum_{h\in\mathbf{Z}} K(e^h,x,E_0,E_1)\chi_{[e^h,e^{h+1}[}\big)^q}{t^{\theta q+1}}\,dt\bigg)^{1/q}&\\
\le \bigg(\int_0^\infty \frac{K(t,x,E_0,E_1)^q}{t^{\theta q+1}}\,dt\bigg)^{1/q}&\\
= \bigg(\sum_{h\in\mathbf{Z}}\int_{e^h}^{e^{h+1}} \frac{K(t,x,E_0,E_1)^q}{t^{\theta q+1}}\,dt\bigg)^{1/q}&\\
\le \bigg(\sum_{h\in\mathbf{Z}}\int_{e^h}^{e^{h+1}} \frac{K(e^{h+1},x,E_0,E_1)^q}{t^{\theta q+1}}\,dt\bigg)^{1/q}&\\
\le e\,\bigg(\sum_{h\in\mathbf{Z}}\int_{e^h}^{e^{h+1}} \frac{K(e^{h},x,E_0,E_1)^q}{t^{\theta q+1}}\,dt\bigg)^{1/q}&\\
= e\,\bigg(\int_0^\infty \frac{\big(\sum_{h\in\mathbf{Z}} K(e^h,x,E_0,E_1)\chi_{[e^h,e^{h+1}[}\big)^q}{t^{\theta q+1}}\,dt\bigg)^{1/q}&,\end{aligned}$$

and, with obvious changes, we get a similar result for the case $q=\infty$. As a consequence, the first function of the chain of inequalities defines an equivalent quasi-norm on $(E_0,E_1)_{\theta,q}$ for $0<q\le\infty$. Hence, using the definition of $K(t,x,E_0,E_1)$, it is easy to show that another equivalent quasi-norm on $(E_0,E_1)_{\theta,q}$ is, when $0<q<\infty$,

$$|||x|||_{\theta,q}=\inf\bigg\{\max_{j=0,1}\bigg(\sum_{h\in\mathbf{Z}} e^{(j-\theta)hq}\|x_h^j\|_{E_j}^q\bigg)^{1/q}\;\bigg|\; x=x_h^0+x_h^1,\, x_h^0\in E_0,\, x_h^1\in E_1 \quad\text{for all}\quad h\in\mathbf{Z}\bigg\}.$$

With the usual changes, we obtain a similar formula for $q=\infty$. From this we recover for the quasi Banach case one of the discrete classical alternative descriptions of $(E_0,E_1)_{\theta,q}$ in the normed case (see for instance [**1**]). This allows us to check that the proof given in [**1**, Chapter I, Section 4, Proposition 4] in the case of Banach spaces can be repeated in the quasi Banach case using the quasi-norm $|||.|||_{\theta,q}$ in order to show that the initial quasi-norm of $(E_0,E_1)_{\theta,q}$ is equivalent to the new quasi-norm, for all $x\in(E_0,E_1)_{\theta,q}$,

$$|||x|||_q=\bigg(\sum_{h\in\mathbf{Z}}\|x\|_h^q\bigg)^{1/q}. \tag{2}$$

From now on, we shall be concerned *exclusively* with the case $0 < q < \infty$. We define for later use $q' := \infty$ if $0 < q \leq 1$, $q' := q/(q-1)$ if $1 < q$, $\overline{q} = 1$ if $0 < q \leq 1$ and $\overline{q} = q$ if $q > 1$. Clearly, $q \leq \overline{q}$ always holds. We remark that the equalities

$$(E_0, E_1)_{\theta,q} = \overline{(E_0 \cap E_1)}^{\,(E_0,E_1)_{\theta,q}} \tag{3}$$

and

$$(E_0, E_1)_{\theta,q} = (\overline{(E_0 \cap E_1)}^{E_0}, \overline{(E_0 \cap E_1)}^{E_1})_{\theta,q} \tag{4}$$

hold as in the normed case (for instance, [**1**, Chapter II, Proof of Proposition 1] in the Banach case can be repeated in our setting).

In the rest of the section *we always assume that* $E_0 \cap E_1$ *is dense in every* E_i, $i = 0, 1$ *and* $E_0 + E_1$ *has a separating dual* $(E_0 + E_1)'$. Let

$$I_{\cap\cup} : E_0 \cap E_1 \longrightarrow E_0 + E_1, \quad I_{\cap i} : E_0 \cap E_1 \longrightarrow E_i, \qquad i = 0, 1,$$

denote the canonical inclusion maps. Then each E_i, $i = 0, 1$, $(E_0, E_1)_{\theta,q}$ and $E_0 \cap E_1$ also have nontrivial separating duals and the adjoint maps $I'_{\cap\cup} : (E_0 + E_1)'_i \to (E_0 \cap E_1)'$ and $I'_{\cap i} : E'_i \to (E_0 \cap E_1)'$, $i = 0, 1$, must be injective. Hence, every E'_i can be looked at as a linear subspace of $(E_0 \cap E_1)'$ and $E_0 + E_1$ as a linear subspace of $(E_0 + E_1)''$. Moreover, we can write $(E_0 + E_1)' = E'_0 \cap E'_1$ as in the Banach case, see [**3**].

Let B_h be the closed unit ball of the quasi Banach space $(E_0 + E_1, \|.\|_h)$. We denote by $\|.\|^*_h$ the dual norm of $\|.\|_h$ in the dual space $(E_0 + E_1)'$ and by B^*_h its closed unit ball. $((E_0 + E_1)', \|.\|^*_h)$ is a nontrivial Banach space. Let $\|.\|^{**}_h$ be the dual norm of $\|.\|^*_h$ in $(E_0 + E_1)''$, and let B^{**}_h be its closed unit ball. Clearly, we have, for all $x \in (E_0, E_1)_{\theta,q}$ and for all $h \in \mathbf{Z}$,

$$\|x\|^{**}_h \leq \|x\|_h. \tag{5}$$

We denote by F_{E_i}, $i = 0, 1$ (or F_i, $i = 0, 1$ in short if there is no risk of confusion) the topological subspace of E'_i

$$F_{E_i} := F_i := \overline{E'_0 \cap E'_1}^{E'_i}, \quad i = 0, 1.$$

Since each E_i' is a Banach space, F_i' is a quotient of E_i''. Let $K_i : E_i'' \to F_i'$ be the canonical quotient map. Clearly, $F_0 \cap F_1$ is dense in each F_i, $i = 0, 1$ and $F_0 \cap F_1 = E_0' \cap E_1'$. Hence, F_i', $i = 0, 1$, is a linear subspace of $(F_0 \cap F_1)'$. Moreover, every F_i being a Banach space, we have

$$F_0' + F_1' = (F_0 \cap F_1)' = (E_0' \cap E_1')' = (E_0 + E_1)''. \tag{6}$$

For every $h \in \mathbf{Z}$ we consider the *convex* set in $F_0' + F_1'$

$$\begin{aligned} W_h &:= e^{-\theta h} B(F_0') + e^{(1-\theta)h} B(F_1') \\ &= e^{-\theta\, h} K_0(B(E_0'')) + e^{(1-\theta)\, h} K_1(B(E_1'')). \end{aligned}$$

Let p_{W_h} be the Minkowski functional of W_h.

Lemma 1. *We have, for all* $x \in (E_0 + E_1)''$,

$$p_{W_h}(x) \le \|x\|_h^{**} \le 2\, p_{W_h}(x). \tag{7}$$

Proof. As every $B(F_i') = K_i(B(E_i''))$, $i = 0, 1$, is $\sigma(F_i', F_i)$-compact (Alaoglu Bourbaki theorem), they are $\sigma(F_0' + F_1', F_0 \cap F_1)$-compact too. Then W_h is $\sigma(F_0' + F_1', F_0 \cap F_1)$-closed. On the other hand, by the bipolar theorem, B_h^{**} is the $\sigma((E_0 + E_1)'', E_0' \cap E_1')$-closure of the absolutely convex cover $\Gamma(B_h)$ of B_h. If $0 \neq x \in (E_0 + E_1)''$, we have $x \in \|x\|_h^{**} B_h^{**}$. Then there is a directed set A and a net $\{x_\alpha, \alpha \in A\} \subset \Gamma(B_h)$ such that for every $n \in \mathbf{N}$ there are nets $\{x_\alpha^{in}, \alpha \in A\} \subset B_1(E_i)$, $i = 0, 1$, such that for all $\alpha \in A$,

$$x_\alpha = \left(1 + \frac{1}{n}\right) \sum_{\gamma=1}^{\gamma_\alpha} \eta_\gamma \Big(e^{-\theta h} x_\alpha^{0n\gamma} + e^{(1-\theta)h} x_\alpha^{1n\gamma} \Big),$$

with $x^{in\gamma} \in B(E_i)$ for every $i = 0, 1$, $1 \le \gamma \le \gamma_\alpha$, $\alpha \in A$, $\sum_{\gamma=1}^{\gamma_\alpha} |\eta_\gamma| \le 1$ for each $\alpha \in A$, and moreover, $x = \lim_{\alpha \in A} \|x\|_h^{**}\, x_\alpha$ in the topology $\sigma((E_0 + E_1)'', E_0' \cap E_1')$. As every $B(F_i')$, $i = 0, 1$, is absolutely convex, we have for all $\alpha \in A$,

$$\begin{aligned} K(x_\alpha) &= \left(1+\frac{1}{n}\right)(e^{-\theta h}K(x_\alpha^{0n\gamma}) + e^{(1-\theta)h}K(x_\alpha^{1n\gamma})) \\ &= \left(1+\frac{1}{n}\right)\left(e^{-\theta h}\sum_{\gamma=1}^{\gamma_\alpha}\eta_\gamma K_0(x_\alpha^{0n\gamma}) + e^{(1-\theta)h}\sum_{\gamma=1}^{\gamma_\alpha}\eta_\gamma K_1(x_\alpha^{1n\gamma})\right) \\ &\in \left(1+\frac{1}{n}\right) W_h, \end{aligned}$$

and so we obtain

$$x = \lim_{\alpha\in A}\lim_{n\to\infty}\left(1+\frac{1}{n}\right)\|x\|_h^{**}\, x_\alpha \in \left(1+\frac{1}{n}\right)\|x\|_h^{**}\, W_h.$$

As a consequence, n being arbitrary, $p_{W_h}(x) \le \|x\|_h^{**}$.

Conversely, $x \in (p_{W_h}(x)+(1/n))W_h$ for every $n\in\mathbf{N}$ and hence there is an $f_i^h \in B(E_i'')$, $i=0,1$, such that

$$\frac{x}{p_{W_h}(x)+(1/n)} = e^{-\theta\, h}K_0(f_0^h) + e^{(1-\theta)\, h}K_1(f_1^h).$$

But $B(E_i'') = B(E_i)^{\circ\circ}$, $i=0,1$. By the bipolar theorem, there are nets $\{x_i^{\alpha_i},\ \alpha_i\in A_i\}$ in the absolutely convex cover $\Gamma(B(E_i))$ of $B(E_i)$ which are $\sigma(E_i'',E_i')$-convergent to f_i^h, $i=0,1$. Since $B_h^*\subset E_0'\cap E_1'$ and $e^{(i-\theta)\, h}B(E_i)\subset B_h$ for every $i=0,1$, we have

$$\begin{aligned} \left\|\frac{x}{p_{W_h}(x)+(1/n)}\right\|_h^{**} &= \sup_{\varphi\in B_h^*}\left|\left\langle\frac{x}{p_{W_h}(x)+(1/n)},\varphi\right\rangle\right| \\ &= \sup_{\varphi\in B_h^*}\lim_{\alpha_0\in A_0}\lim_{\alpha_1\in A_1}\left|\left\langle e^{-\theta\, h}K_0(x_0^{\alpha_0}) + e^{(1-\theta)\, h}K_1(x_1^{\alpha_1}),\varphi\right\rangle\right| \le 2 \end{aligned}$$

getting $\|x\|_h^{**} \le 2\ p_{W_h}(x) + (1/n)$ for every $n\in\mathbf{N}$. In consequence $\|x\|_h^{**}\le 2\, p_{W_h}(x)$, and the lemma follows. □

Let $\overline{F_0\cap F_1}$ be the closure of $F_0\cap F_1$ in the interpolation Banach space $(F_0,F_1)_{\theta,q'}$. By the remark following [**3**, Theorem 3.7.1], we have

(8) $$(\overline{F_0\cap F_1})' = (F_0',F_1')_{\theta,\overline{q}}.$$

Directly from (2) and Lemma 1, we get

Corollary 2. *The canonical norm in* $(F_0', F_1')_{\theta,\overline{q}}$ *is equivalent to the norm, for all* $x \in (F_0', F_1')_{\theta,\overline{q}}$,

$$|||x|||^{**} = \Big(\sum_{h\in\mathbf{Z}} (\|x\|_h^{**})^{\overline{q}} \Big)^{1/\overline{q}}.$$

Lemma 3. *For every* $h \in \mathbf{Z}$, *the normed space* $((E_0, E_1)_{\theta,q}, \|.\|_h^{**})$ *is a topological subspace of the normed space* $((F_0', F_1')_{\theta,\overline{q}}, \|.\|_h^{**})$, *and hence,* $(E_0, E_1)_{\theta,q}$ *can be looked at as an algebraic linear subspace of* $(F_0', F_1')_{\theta,\overline{q}}$.

Proof. With suitable natural changes in the proof given in the Banach case in [**1**, Chapter IV], and using (4), we obtain the continuous inclusions

$$\overline{F_0 \cap F_1} \subset (F_0, F_1)_{\theta,q'} = (E_0', E_1')_{\theta,q'} \subset ((E_0, E_1)_{\theta,q})', \tag{9}$$

and taking transposed maps, by (6) we get the chain of *natural mappings*

$$(E_0, E_1)_{\theta,q} \longrightarrow ((E_0, E_1)_{\theta,q})'' \longrightarrow ((F_0, F_1)_{\theta,q'})' \\ = (F_0', F_1')_{\theta,\overline{q}} \subset (E_0 + E_1)''.$$

Let $H : (E_0, E_1)_{\theta,q} \to (F_0', F_1')_{\theta,\overline{q}}$ denote the composition of the corresponding maps of this chain. As a consequence, for all $x \in (E_0, E_1)_{\theta,q}$ and for all $y \in (F_0, F_1)_{\theta,q'} \supset F_0 \cap F_1 \supset E_0' \cap E_1'$, $\langle H(x), y\rangle = \langle x, y\rangle$, and hence, for all $x \in (E_0, E_1)_{\theta,q}$ and for all $h \in \mathbf{Z}$,

(10)

$$\begin{aligned}\|H(x)\|_h^{**} &= \sup\Big\{ |\langle H(x), y\rangle|, \ \ y \in (E_0 + E_1)', \|y\|_h^* \le 1 \Big\} \\ &= \sup\Big\{ |\langle x, y\rangle|, \ \ y \in (E_0 + E_1)', \|y\|_h^* \le 1 \Big\} = \|x\|_h^{**}. \qquad \square\end{aligned}$$

3. Main result.

Theorem 4. *Let $0 < q < \infty$, and let (E_0, E_1) be a compatible couple of quasi Banach spaces such that E_0+E_1 has a separating dual. Assume that there is a semi-normalized sequence $\{x_n\}_{n=1}^{\infty}$ in $(E_0, E_1)_{\theta,q}$, which is semi-normalized too in $(F'_{E_0}, F'_{E_1})_{\theta,\overline{q}}$, but $\lim_{n\to\infty} \|x_n\|_{E_0+E_1} = 0$. Then, $(E_0, E_1)_{\theta,q}$ has a complemented subspace isomorphic to l^q.*

Proof. By (3) and (4) we can suppose without loss of generality that $E_0 \cap E_1$ is dense in every E_i, $i = 0, 1$. By hypothesis and Corollary 2, there are a bounded sequence $\{x_n\}_{n=1}^{\infty} \subset (E_0, E_1)_{\theta,q}$ and numbers $\varepsilon > 0$ and $0 < \alpha < \beta$ such that

$$\lim_{n\to\infty} \|x_n\|_{E_0+E_1} = 0, \qquad \inf_{n\in\mathbf{N}} |||x_n|||_q \geq \varepsilon \tag{11}$$

and, for all $n \in \mathbf{N}$,

$$0 < \alpha < |||x_n|||^{**} < \beta. \tag{12}$$

By [**12**], and switching to a suitable subsequence, it can be assumed that the closed linear span G in $(E_0, E_1)_{\theta,q}$ of $\{x_n\}_{n=1}^{\infty}$ is isomorphic to l^q. That means that there is a $K > 0$ such that for every $(a_n) \in \mathbf{K}^{\mathbf{N}}$

$$\frac{1}{K}\bigg(\sum_{n=1}^{\infty} |a_n|^q\bigg)^{1/q} \leq \bigg|\bigg|\bigg|\sum_{n=1}^{\infty} a_n x_n\bigg|\bigg|\bigg|_q \leq K\bigg(\sum_{n=1}^{\infty} |a_n|^q\bigg)^{1/q}. \tag{13}$$

Define, for all $n \in \mathbf{N}$,

$$u_n := \frac{x_n}{|||x_n|||^{**}} \in (E_0, E_1)_{\theta,q}.$$

The closed linear span of the sequence $\{u_n\}_{n=1}^{\infty}$ is again isomorphic to l^q since by (13) and (12), for every $(a_n) \in \mathbf{K}^{\mathbf{N}}$, we have

$$\begin{aligned} \frac{1}{K\beta}\bigg(\sum_{n=1}^{\infty} |a_n|^q\bigg)^{1/q} &\leq \frac{1}{K}\bigg(\sum_{n=1}^{\infty} \bigg|\frac{a_n}{|||x_n|||^{**}}\bigg|^q\bigg)^{1/q} \\ &\leq \bigg|\bigg|\bigg|\sum_{n=1}^{\infty} a_n u_n\bigg|\bigg|\bigg|_q \leq K\bigg(\sum_{n=1}^{\infty} \bigg|\frac{a_n}{|||x_n|||^{**}}\bigg|^q\bigg)^{1/q} \\ &\leq \frac{K}{\alpha}\bigg(\sum_{n=1}^{\infty} |a_n|^q\bigg)^{1/q}. \end{aligned} \tag{14}$$

Let $C \geq 1$ be the quasi-norm constant corresponding to $|||.|||_q$ and fix a number $0 < \delta$ such that $(\delta C K \beta)/\alpha < 1$. We consider separately the cases $q \leq 1$ and $q > 1$.

Case $q \leq 1$**.** Put $h_1 = 1$. Suppose we have to find strictly increasing finite sequences $(m_i)_{i=0}^n$ and $(h_i)_{i=1}^n$ in $\mathbf{N}$ with $m_0 = 0$, $h_1 = 1$, such that for all $1 \leq j \leq n$ and for all $1 \leq s \leq j$,

$$\sum_{|i|>m_j} \|u_{h_s}\|_i^{**} \leq \left(\frac{\delta^q}{2^{j+2+q} C^q K^q}\right)^{1/q}, \tag{15}$$

for all $1 \leq k \leq n$,

$$1 - \sum_{|i|=m_{k-1}+1}^{m_k} \|u_{h_k}\|_i^{**} \leq \frac{\delta}{2^{(k+1)/q} C K}, \tag{16}$$

and for all $j = 2, 3, \dots, n$,

$$\sum_{|i|\leq m_{j-1}} \|u_{h_j}\|_i^{**} \leq \left(\frac{\delta^q}{2^{j+1+q} C^q K^q}\right)^{1/q}. \tag{17}$$

Remark that for all $n \in \mathbf{N}$,

$$|||u_n|||^{**} = \sum_{m\in\mathbf{Z}} \|u_n\|_m^{**} = 1 \tag{18}$$

and that, by (5) and (11), for all $h \in \mathbf{N}$,

$$\begin{aligned}\lim_{n\to\infty} \sum_{|j|\leq h} \|u_n\|_j^{**} &\leq \lim_{n\to\infty} \left(\sum_{|j|\leq h} (\|u_n\|_j^{**})^q\right)^{1/q} \\ &\leq \lim_{n\to\infty} \left(\sum_{|j|\leq h} \|u_n\|_j^q\right)^{1/q} = 0,\end{aligned} \tag{19}$$

having in mind that $(\sum_{|j|\leq h} \|x\|_j^q)^{1/q}$ is a quasi-norm in $E_0 + E_1$ equivalent to its canonical quasi-norm. Then we can find $h_{n+1} > h_n$ such that (17) holds for $j = n+1$. Now, by (18), there is an $m_{n+1} > m_n$

such that (15) and (16) hold for $j = n+1$ and the process can be repeated indefinitely.

Let X be the closed linear span of $\{u_{h_n}\}_{n=1}^{\infty}$ in $(E_0, E_1)_{\theta,q}$. Clearly, X is again isomorphic to l^q. For every $n \in \mathbf{Z}$, let H_n be the Banach envelope of the quasi-normed space $(E_0 + E_1, \|.\|_n)$. Clearly, H_n is a topological subspace of $((E_0+E_1)'', \|.\|_n^{**})$. Since $H_n' = (E_0+E_1)'$, for every $n \in \mathbf{Z}$ there is a $\varphi_n \in ((E_0+E_1)', \|.\|_n^*)$ such that for all $n \in \mathbf{N}$ and for all $m_{n-1}+1 \le |i| \le m_n$,

$$\|\varphi_i\|_i^* \le 1 \quad \text{and} \quad |\langle \varphi_i, u_{h_n}\rangle| = \|u_{h_n}\|_i^{**}. \tag{20}$$

We define $T : (E_0, E_1)_{\theta,q} \to X$ by, for all $x \in (E_0, E_1)_{\theta,q}$,

$$T(x) = \sum_{n=1}^{\infty} \Bigg(\sum_{|i|=m_{n-1}+1}^{m_n} \langle \varphi_i, x\rangle \Bigg) u_{h_n}. \tag{21}$$

T is well defined and continuous since, by (14) and (20) we have

$$\begin{aligned} |||T(x)|||_q &\le \frac{K}{\alpha} \Bigg(\sum_{n=1}^{\infty} \Bigg| \Big\langle \sum_{|i|=m_{n-1}+1}^{m_n} \varphi_i, x \Big\rangle \Bigg|^q \Bigg)^{1/q} \\ &\le \frac{K}{\alpha} \Bigg(\sum_{n=1}^{\infty} \sum_{|i|=m_{n-1}+1}^{m_n} (\|\varphi_i\|_i^*)^q \, \|x\|_i^q \Bigg)^{1/q} \\ &\le \frac{K}{\alpha} \, |||x|||_q. \end{aligned}$$

Let $W : X \to X$ be the restriction of T to X. If we show that the inverse operator $V := W^{-1}$ exists, the proof will be finished since the composition VT will be a continuous projection onto X.

In order to show the existence of V, it will be enough to see that $\|I_X - W\| < (\delta K \beta)/\alpha$ (where I_X is the identity map on X) since in such a case, by consecutive application of the additive property of a quasi-norm, we will obtain for all $k, h \in \mathbf{N}$,

$$\Bigg\| \sum_{n=k}^{k+h} (W - I_X)^n \Bigg\| \le \sum_{n=1}^{h+1} C^n \|W - I_X\|^{k+n-1} \le \frac{C((\delta K\beta)/\alpha)^k}{1 - C\|W - I_X\|},$$

which is arbitrarily small letting $k \to \infty$ since $C\|W - I_X\| \leq (\delta C K \beta)/\alpha < 1$. This means that the series of operators $V := \sum_{n=0}^{\infty}(I_X - W)^n$ will be convergent and, since

$$\bigg(\sum_{n=0}^{\infty}(I_X - W)^n\bigg)(I_X - (I_X - W)) = I_X,$$

the operator W will be invertible with inverse V.

Hence, let us see that $\|W - I_X\| < (\delta K \beta)/\alpha$. Let

$$x = \sum_{n=1}^{\infty} a_n u_{h_n} \in X.$$

Using (14), we get

$$\begin{aligned}
|||T(x) - x|||_q &= \bigg|\bigg|\bigg| \sum_{n=1}^{\infty} \bigg(\Big\langle \sum_{|j|=m_{n-1}+1}^{m_n} \varphi_j, x\Big\rangle - a_n\bigg) u_{h_n} \bigg|\bigg|\bigg|_q \\
&= \bigg|\bigg|\bigg| \sum_{n=1}^{\infty} \bigg(\bigg(\sum_{t=1}^{\infty} a_t \Big\langle \sum_{|j|=m_{n-1}+1}^{m_n} \varphi_j, u_{h_t}\Big\rangle\bigg) - a_n\bigg) u_{h_n} \bigg|\bigg|\bigg|_q \\
&\leq C \bigg|\bigg|\bigg| \sum_{n=1}^{\infty} \bigg(\Big\langle \sum_{|j|=m_{n-1}+1}^{m_n} \varphi_j, u_{h_n}\Big\rangle - 1\bigg) a_n u_{h_n} \bigg|\bigg|\bigg|_q \\
&\quad + C \bigg|\bigg|\bigg| \sum_{n=1}^{\infty} \bigg(\sum_{t \neq n} a_t \Big\langle \sum_{|j|=m_{n-1}+1}^{m_n} \varphi_j, u_{h_t}\Big\rangle\bigg) u_{h_n} \bigg|\bigg|\bigg|_q \\
&\leq C \frac{K}{\alpha} \bigg(\sum_{n=1}^{\infty} \bigg| \bigg(\Big\langle \sum_{|j|=m_{n-1}+1}^{m_n} \varphi_j, u_{h_n}\Big\rangle - 1\bigg) a_n \bigg|^q \bigg)^{1/q} \\
&\quad + C \frac{K}{\alpha} \bigg(\sum_{n=1}^{\infty} \bigg| \sum_{t \neq n} a_t \Big\langle \sum_{|j|=m_{n-1}+1}^{m_n} \varphi_j, u_{h_t}\Big\rangle \bigg|^q \bigg)^{1/q} \qquad (22)
\end{aligned}$$

and by (20) and (16)

$$\leq \frac{\delta}{2\alpha} \bigg(\sum_{i=1}^{\infty} |a_i|^q \bigg)^{1/q}$$

$$
\begin{aligned}
&+ C\frac{K}{\alpha}\left(\sum_{n=1}^{\infty}\sum_{t\neq n}|a_t|^q\left(\sum_{|j|=m_{n-1}+1}^{m_n}\|u_{h_t}\|_j^{**}\right)^q\right)^{1/q}\\
&\le \frac{\delta}{2\alpha}\left(\sum_{i=1}^{\infty}|a_i|^q\right)^{1/q}\\
&\quad + C\frac{K}{\alpha}\left(\sum_{n=1}^{\infty}\left(\sum_{t>n}|a_t|^q\left(\sum_{|j|\le m_{t-1}}\|u_{h_t}\|_j^{**}\right)^q\right.\right.\\
&\quad \left.\left. + \sum_{t<n}|a_t|^q\left(\sum_{|j|>m_{n-1}}\|u_{h_t}\|_j^{**}\right)^q\right)\right)^{1/q}
\end{aligned}
$$

and by (17) and (15)

$$
\begin{aligned}
&\le \frac{\delta}{2\alpha}\left(\sum_{i=1}^{\infty}|a_i|^q\right)^{1/q}\\
&\quad + C\frac{K}{\alpha}\left(\sum_{n=1}^{\infty}\left(\sum_{t>n}|a_t|^q\frac{\delta^q}{2^{t+1+q}C^qK^q}\right.\right.\\
&\quad \left.\left. + \sum_{t<n}|a_t|^q\frac{\delta^q}{2^{n+1+q}C^qK^q}\right)\right)^{1/q}\\
&\le \frac{\delta}{2\alpha}\left(\sum_{i=1}^{\infty}|a_i|^q\right)^{1/q}\\
&\quad + C\frac{K}{\alpha}\left(\sum_{n=1}^{\infty}\left(\frac{\delta^q}{2^{n+1+q}C^qK^q}\sum_{t>n}|a_t|^q\right.\right.\\
&\quad \left.\left. + \frac{\delta^q}{2^{n+1+q}C^qK^q}\sum_{t<n}|a_t|^q\right)\right)^{1/q}\\
&\le \frac{\delta}{2\alpha}\left(\sum_{i=1}^{\infty}|a_i|^q\right)^{1/q}\\
&\quad + \frac{1}{2\alpha}\left(\sum_{k=1}^{\infty}|a_k|^q\right)^{1/q}\left(\sum_{n=1}^{\infty}\frac{\delta^q}{2^n}\right)^{1/q}
\end{aligned}
$$

and by (14)

$$\leq \left(\frac{\delta}{2\alpha}+\frac{\delta}{2\alpha}\right)\left(\sum_{k=1}^{\infty}|a_k|^q\right)^{1/q} \leq \frac{\delta K\beta}{\alpha}|||x|||_q,$$

and hence,

$$\|W-I_X\| < \frac{\delta K\beta}{\alpha}.$$

Case $q>1$**.** The pattern of the proof is the same but with different technical details. As a first step, with a similar argument to the one used for (15)–(17) and (19), we find a subsequence $\{u_{h_n}\}_{n=1}^{\infty}$ such that for every $n\in \mathbf{N}$, we have for all $1\leq j\leq n$ and for all $1\leq s\leq j$,

$$\sum_{|i|>m_j}(\|u_{h_s}\|_i^{**})^q \leq \frac{\delta^q}{2^{1+(j+1)q}C^qK^q}, \tag{23}$$

for all $1\leq k\leq n$,

$$1-\left(\sum_{|i|=m_{k-1}+1}^{m_k}(\|u_{h_k}\|_i^{**})^q\right)^{1/q} \leq \frac{\delta}{2^{k+1}CK} \tag{24}$$

and for all $j=2,3,\dots,n$,

$$\sum_{|i|\leq m_{j-1}}(\|u_{h_j}\|_i^{**})^q \leq \frac{\delta^q}{2^{1+(j+1)q}C^qK^q}. \tag{25}$$

The spaces X and H_n, $n\in\mathbf{Z}$, are defined as above. By (14), for every $n\in\mathbf{N}$, for all $(x_i)_{|i|=m_{n-1}+1}^{m_n}\in\prod_{|i|=m_{n-1}+1}^{m_n}H_i$,

$$O_n((x_i)) = \left(\sum_{|i|=m_{n-1}+1}^{m_n}(\|x_i\|_i^{**})^q\right)^{1/q}$$

defines a norm on $\Pi_{|i|=m_{n-1}+1}^{m_n}H_i$ getting a Banach space with a nontrivial dual. Hence, there are $\varphi_i\in(E_0+E_1)'$, $m_{n-1}+1\leq|i|\leq m_n$, such that

$$\left(\sum_{|i|=m_{n-1}+1}^{m_n}(\|\varphi_i\|_i^{*})^{q/(q-1)}\right)^{(q-1)/q} \leq 1 \tag{26}$$

and

$$\sum_{|i|=m_{n-1}+1}^{m_n} \langle \varphi_i, u_{h_n} \rangle = \Bigg(\sum_{|i|=m_{n-1}+1}^{m_n} (\|u_{h_n}\|_i^{**})^q \Bigg)^{1/q}. \tag{27}$$

The map T is defined as in (21). By (14), Hölder's inequality, (26) and (5) we obtain for all $x \in (E_0, E_1)_{\theta,q}$,

$$\begin{aligned}
|||T(x)|||_q &\leq \frac{K}{\alpha} \Bigg(\sum_{n=1}^{\infty} \Bigg| \Bigg\langle \sum_{|i|=m_{n-1}+1}^{m_n} \varphi_i, x \Bigg\rangle \Bigg|^q \Bigg)^{1/q} \\
&\leq \frac{K}{\alpha} \Bigg(\sum_{n=1}^{\infty} \Bigg(\sum_{|i|=m_{n-1}+1}^{m_n} \|\varphi_i\|_i^* \ \|x\|_i^{**} \Bigg)^q \Bigg)^{1/q} \\
&\leq \frac{K}{\alpha} \Bigg(\sum_{n=1}^{\infty} \Bigg(\sum_{|i|=m_{n-1}+1}^{m_n} (\|\varphi_i\|_i^*)^{q/(q-1)} \Bigg)^{q-1} \\
&\qquad \times \Bigg(\sum_{|i|=m_{n-1}+1}^{m_n} (\|x\|_i^{**})^q \Bigg) \Bigg)^{1/q} \leq \frac{K}{\alpha} \ |||x|||_q,
\end{aligned}$$

and T is also continuous.

Now W and V have the same meaning as previously. To show the existence of V, from (26) and (24) we obtain

$$C\frac{K}{\alpha} \Bigg(\sum_{n=1}^{\infty} \Bigg| \Bigg(\Bigg\langle \sum_{|j|=m_{n-1}+1}^{m_n} \varphi_j, u_{h_n} \Bigg\rangle - 1 \Bigg) a_n \Bigg|^q \Bigg)^{1/q} \leq \frac{\delta}{2\alpha} \Bigg(\sum_{k=1}^{\infty} |a_k|^q \Bigg)^{1/q}.$$

On the other hand, applying Minkowski's inequality in (22), we get

$$\begin{aligned}
&C\frac{K}{\alpha} \Bigg(\sum_{n=1}^{\infty} \Bigg| \sum_{t \neq n} a_t \Bigg\langle \sum_{|j|=m_{n-1}+1}^{m_n} \varphi_j, u_{h_t} \Bigg\rangle \Bigg|^q \Bigg)^{1/q} \\
&\quad \leq C\frac{K}{\alpha} \sum_{t=1}^{\infty} |a_t| \Bigg(\sum_{n \neq t} \Bigg| \Bigg\langle \sum_{|j|=m_{n-1}+1}^{m_n} \varphi_j, u_{h_t} \Bigg\rangle \Bigg|^q \Bigg)^{1/q}
\end{aligned}$$

$$\leq C\frac{K}{\alpha}\sum_{t=1}^{\infty}|a_t|\bigg(\sum_{n\neq t}\bigg(\sum_{|j|=m_{n-1}+1}^{m_n}\|\varphi_j\|_j^*\;\|u_{h_t}\|_j^{**}\bigg)^q\bigg)^{1/q}$$

and by Hölder's inequality

$$\leq C\frac{K}{\alpha}\sum_{t=1}^{\infty}|a_t|\bigg(\sum_{n\neq t}\bigg(\sum_{|j|=m_{n-1}+1}^{m_n}(\|\varphi_j\|_j^*)^{q/(q-1)}\bigg)^{q-1}$$
$$\times\bigg(\sum_{|j|=m_{n-1}+1}^{m_n}(\|u_{h_t}\|_j^{**})^q\bigg)\bigg)^{1/q}$$

and by (26)

$$\leq C\frac{K}{\alpha}\sum_{t=1}^{\infty}|a_t|\bigg(\sum_{n\neq t}\sum_{|j|=m_{n-1}+1}^{m_n}(\|u_{h_t}\|_j^{**})^q\bigg)^{1/q}$$
$$= C\frac{K}{\alpha}\sum_{t=1}^{\infty}|a_t|\bigg(\sum_{n<t}\sum_{|j|=m_{n-1}+1}^{m_n}(\|u_{h_t}\|_j^{**})^q$$
$$+\sum_{n>t}\sum_{|j|=m_{n-1}+1}^{m_n}(\|u_{h_t}\|_j^{**})^q\bigg)^{1/q}$$
$$\leq C\frac{K}{\alpha}\sum_{t=1}^{\infty}|a_t|\bigg(\sum_{|j|\leq m_{t-1}}(\|u_{h_t}\|_j^{**})^q+\sum_{|j|>m_t}(\|u_{h_t}\|_j^{**})^q\bigg)^{1/q}$$

and by (25) and (23)

$$\leq C\frac{K}{\alpha}\sup_{k\in\mathbf{N}}|a_k|\sum_{t=1}^{\infty}\bigg(\frac{\delta^q}{2^{1+(t+1)q}C^qK^q}+\frac{\delta^q}{2^{1+(t+1)q}C^qK^q}\bigg)^{1/q}$$
$$\leq\frac{\delta}{2\alpha}\bigg(\sum_{k=1}^{\infty}|a_k|^q\bigg)^{1/q},$$

and the proof is finished as in the former case. □

4. Applications. We wish to apply the previous theorem to the particular case of interpolation spaces $(l^{p_0}(\mu), l^{p_1}(\mu))_{\theta,q}$ where

$0 < p_0 < p_1 < \infty$, $0 < q < \infty$ and μ is an arbitrary measure in $\mathbf{N}$. As a technical tool we shall use Lorentz sequence spaces $l^{p,q}(\mu)$. We recall the most relevant aspects for our purposes and refer the reader to [**2, 11**] for more details.

Given a measure μ in $\mathbf{N}$ we put $\mu_n := \mu(\{n\})$ for every $n \in \mathbf{N}$. For a sequence $a := (a_n)_{n=1}^{\infty} \in \mathbf{R}^{\mathbf{N}}$, the distribution function μ_a of a with respect to μ is the real function $\mu_a : [0, \infty[\to \mathbf{R}$ given by, for all $\lambda \geq 0$,

$$\mu_a(\lambda) := \mu\left\{n \in \mathbf{N} \mid |a_n| > \lambda\right\}.$$

The decreasing rearrangement of a is the function a_μ^* defined on $[0, \infty[$ by, for all $t \geq 0$,

$$a_\mu^*(t) := \inf\left\{\lambda \geq 0 \mid \mu_a(\lambda) \leq t\right\}.$$

In particular, the decreasing rearrangement of a sequence $a = (a_n) \in c_0$ is the function $a^*(t)$ given by

$$a^*(t) = |a_{\sigma(1)}| \quad \text{if} \quad t \in \big[0, \mu_{\sigma(1)}\big[\tag{28}$$

and, for all $i > 1$,

$$a^*(t) = |a_{\sigma(i)}| \quad \text{if} \quad t \in \left[\sum_{j=1}^{i-1} \mu_{\sigma(j)}, \sum_{j=1}^{i} \mu_{\sigma(j)}\right[\tag{29}$$

where $\sigma : \mathbf{N} \to \mathbf{N}$ is any map verifying

$$|a_{\sigma(1)}| = \max\Big\{|a_n| \mid n \in \mathbf{N}\Big\},$$

and, if $i > 1$,

$$|a_{\sigma(i)}| = \max\Big\{|a_n| \mid n \notin \{\sigma(1), \sigma(2), \dots, \sigma(i-1)\}\Big\}.$$

Suppose now $1 < p < \infty$, $1 \leq q < \infty$. Then the Lorentz space $l^{p,q}(\mu)$ consists of all $(a_n) \in$ in $\mathbf{R}^{\mathbf{N}}$ such that

$$\|f\|_{p,q} := \left(\int_0^\infty \Big(t^{1/p} a^*(t)\Big)^q \frac{dt}{t}\right)^{1/q} < \infty. \tag{30}$$

It can be proved that $\|.\|_{\mu,p,q}$ is a quasi-norm in $l^{p,q}(\mu)$ (see, for instance, [**2**, Chapter 4, Theorem 4.3]). However, in concrete computations with elements of $l^{p,q}(\mu)\cap c_0$, it will be more easy to use a quasi-norm equivalent to $\|.\|_{p,q}$, see [**12**], defined, for all $a=(a_n)\in l^{p,q}(\mu)\cap c_0$, as

$$(31)\qquad |||a|||_{p,q} := \left(\sum_{i=1}^{\infty} |a_{\sigma(i)}|^q \mu_{\sigma(i)}\left(\sum_{j=1}^{i}\mu_{\sigma(j)}\right)^{(q/p)-1}\right)^{1/q}.$$

Let $\rho=\{\rho_i\}_{i=1}^{\infty}$ be a sequence of strictly positive numbers. In some instances we shall need weighted spaces

$$l^{p,q}(\rho,\mu) := \{(x_i)\mid (x_i\rho_i)\in l^{p,q}(\mu)\}.$$

In such cases, the canonical quasi-norm of $(x_i)\in l^{p,q}(\rho,\mu)$ is

$$\|(x_i\rho_i)\|_{l^{p,q}(\mu)}.$$

Given $\alpha>0$, the symbol μ^{α} will denote the sequence (μ_i^{α}). Analogously, we define $\rho/\mu := (\rho_i/\mu_i)$, $1/\rho := (1/\rho_i)$ and $\rho\mu := (\rho_i\mu_i)$. For every $i\in\mathbf{N}$, the sequence $(0,0,\dots,1,0,0,\dots)$ with 1 in the ith place will always be denoted by the symbol $\mathbf{e}_i$.

The next result of Freitag [**5**] will be important for us:

Theorem 5. 1) *If $0<p<\infty$, $0<\theta<1$, $0<q<\infty$, the isomorphism*

$$(l^p(w_0,\mu), l^{\infty}(w_1,\mu))_{\theta,q} \approx l^{p/(1-\theta),q}\left(w_1, \left(\frac{w_0}{w_1}\right)^{p_0}\mu\right)$$

holds by means of the identity map.

2) *If $0<p_0<p_1<\infty$, $0<\theta<1$, $1/p := (1-\theta)/p_0+\theta/p_1$ and $0<q<\infty$, the identity map gives us the isomorphism*

$$(l^{p_0}(w_0,\mu), l^{p_1}(w_1,\mu))_{\theta,q} \approx l^{p,q}\left(\left(\frac{w_1^{p_1}}{w_0^{p_0}}\right)^{1/(p_1-p_0)}, \left(\frac{w_0}{w_1}\right)^{p_0p_1/(p_1-p_0)}\mu\right).$$

In order to study the existence of complemented subspaces isomorphic to l^q in interpolated spaces of type $\lambda := (l^{p_0}(\mu), l^{p_1}(\mu))_{\theta,q}$, we remark that, after projection onto a sectional subspace, we can assume the measure space $(\mathbf{N},\mu)$ enjoys one of the following possibilities:

1) There are $0<\alpha<\beta$ in $\mathbf{R}$ such that, for all $n\in\mathbf{N}$,

$$\alpha \le \mu_n \le \beta. \tag{32}$$

In this case we have $(l^q(\mu), l^\infty(\mu))_{\theta,q} \approx (l^q, l^\infty)_{\theta,q}$ indeed.

2) The sequence $\{\mu_n\}_{n=1}^\infty$ is strictly increasing and $\lim_{n\to\infty}\mu_n=\infty$.

3) The sequence $\{\mu_n\}_{n=1}^\infty$ is strictly decreasing, $\lim_{n\to\infty}\mu_n=0$ and $\mu(\mathbf{N})=\sum_{n=1}^\infty \mu_n<\infty$.

Given λ as above, define p_θ such that $1/p_\theta=(1-\theta)/p_0+\theta/p_1$.

Theorem 6. *Let μ be a measure on $\mathbf{N}$ of types* 1), 2) *or* 3). *Let $0<p_0<p_1<\infty$ and $0<q<\infty$. Then $(l^{p_0}(\mu), l^{p_1}(\mu))_{\theta,q}$ has complemented subspaces isomorphic to l^q except when $0<p_\theta<q<1$ holds in the case of type* 1).

Proof. By Theorem 5 we have the isomorphisms

$$(l^{p_0}(\mu), l^{p_1}(\mu))_{\theta,q} \approx l^{p_\theta,q}(\mu) \approx (l^{(1-\theta)\,p_\theta}(\mu), l^\infty(\mu)_{\theta,q}.$$

Hence, having in mind Levy's theorem, it is enough to consider the case of non Banach spaces of type $\lambda := (l^p(\mu), l^\infty(\mu)_{\theta,q} = l^{p/(1-\theta),q}(\mu)$.

Case 1) Assume (32) holds. If $q=p/(1-\theta)$ we obtain $\lambda=l^q$ and there is nothing to prove. We only need to consider two subcases.

Case 1a) Assume moreover $q<p/(1-\theta)$. To avoid the Banach space case we have $q<1$ necessarily. If $p\le q$, take $0<\nu<1$ such that $(q(1-\theta)/p)<1-\nu$. By Freitag's result, we have

$$l^{p/(1-\theta),q} \approx (l^p, l^\infty)_{\theta,q} \approx (l^{p(1-\nu)/(1-\theta)}, l^\infty)_{\nu,q}$$

and $q<(p(1-\nu))/(1-\theta)$. Hence, we can assume $q<p<p/(1-\theta)$ and $0<q<1$.

Subcase 1a1) Suppose moreover $p<1$. Put $k_0:=0$ and suppose $\{k_j\}_{j=1}^h$ and $\{z_j\}_{j=1}^h$ are defined. The sequence $\{\log n/(\log(k_h+n))\}_{n=2}^\infty$

is strictly increasing with limit 1. Since $0 < 1-(q\,(1-\theta))/p < 1$, choose $\beta > 0$ such that

$$0 < \beta < \frac{q\,(1-\theta)\,(p-q)}{p\,(1-q)} < \frac{q\,(1-\theta)}{p} \tag{33}$$

and $\alpha_{h+1} < 1$ small enough in order that

$$\alpha_{h+1} + \beta + 1 - \frac{q\,(1-\theta)}{p} < 1, \tag{34}$$

$$\frac{(1-q)\alpha_{h+1}}{1-(q/p)} < \frac{\log 2}{\log(k_h+2)} < \frac{\log n}{\log(k_h+n)}, \quad n \geq 2, \tag{35}$$

and

$$(\alpha_{h+1} + \beta) < \frac{q\,(1-\theta)\,(p-q)}{p\,(1-q)}. \tag{36}$$

Then, for all $n \geq 2$,

$$(k_j+n)^{\alpha_{j+1}} \leq (k_j+n)^{q\,\alpha_j}\, n^{1-(q/p)}. \tag{37}$$

The series

$$\sum_{n=1}^{\infty} \frac{1}{(k_h+n)^{\alpha_{h+1}+\beta} n^{1-(q\,(1-\theta))/p}}$$

is divergent. Then let $s_{h+1} \in \mathbf{N}$ be the least natural number such that

$$1 \leq \sum_{n=1}^{s_{h+1}} \frac{1}{(k_h+n)^{\alpha_{h+1}+\beta}\, n^{1-(q\,(1-\theta))/p}} < 2. \tag{38}$$

We define $k_{h+1} := k_h + s_{h+1}$, and

$$z_{h+1} := \sum_{n=1}^{s_{h+1}} \frac{1}{(k_h+n)^{(\alpha_{h+1}+\beta)/q}}\, \mathbf{e}_{k_h+n}.$$

Then by (31) and (38)

$$1 < \|z_{h+1}\|_\lambda = \left(\sum_{n=1}^{s_{h+1}} \frac{1}{(k_h+n)^{\alpha_{h+1}+\beta} n^{1-((q\,(1-\theta))/p)}} \right)^{1/q} \leq 2^{1/q}.$$

On the other hand, $\lambda = (l^p, l^\infty)_{\theta,q} = (l^p, c_0)_{\theta,q}$ and $\overline{l^\infty \cap l^1}^{l^\infty} = c_0$, and hence $(F'_{E_0}, F'_{E_1})_{\theta,\overline{q}} = (l^1, l^\infty)_{\theta,1} \approx l^{1/(1-\theta),1}$. By (31) and (38) we obtain

$$\begin{aligned}\|z_{h+1}\|_{l^{1/(1-\theta),1}} &= \sum_{n=1}^{s_{h+1}} \frac{1}{(k_h+n)^{(\alpha_{h+1}+\beta)/q} n^\theta} \\ &\geq \sum_{n=1}^{s_{h+1}} \frac{1}{(k_h+n)^{\alpha_{h+1}+\beta}\ n^{1-(q\,(1-\theta))/p}} \geq 1.\end{aligned}$$

Since $l^p + l^\infty = l^\infty$ and

$$\|z_{h+1}\|_{l^\infty} \leq \frac{1}{k_h^{\alpha_{h+1}+\beta}} \leq \frac{1}{k_h^\beta},$$

the proof is finished.

Subcase 1a2) Suppose that $p \geq 1$. Put $k_0 := 0$, and define inductively $\{k_j\}_{j=1}^\infty$ noting that, once $\{k_j\}_{j=1}^h$ are defined, since $(q\,(1-\theta))/p < 1$, the series

$$\sum_{n=1}^{\infty} \frac{1}{(k_h+n)^{q\,(1-\theta)/p}\ n^{1-(q\,(1-\theta))/p}}$$

is divergent, we can define k_{h+1} as the greatest natural number such that

$$1 < \sum_{n=1}^{k_{h+1}} \frac{1}{(k_h+n)^{q\,(1-\theta)/p}\ n^{1-(q\,(1-\theta))/p}} \leq 2. \tag{39}$$

Let us see that $\{k_h\}_{h=1}^\infty$ is not decreasing. Assume we have shown that $k_1 \leq k_2 \leq \cdots \leq k_h$ and that $k_{h+1} < k_h$ holds. By definition of k_{h+1} and k_h, we have

$$\begin{aligned}1 &< \sum_{n=1}^{k_{h+1}} \frac{1}{(k_h+n)^{q\,(1-\theta)/p}\ n^{1-(q\,(1-\theta))/p}} \\ &< \sum_{n=1}^{k_h} \frac{1}{(k_h+n)^{q\,(1-\theta)/p}\ n^{1-(q\,(1-\theta))/p}} \\ &\leq \sum_{n=1}^{k_h} \frac{1}{(k_{h-1}+n)^{q\,(1-\theta)/p}\ n^{1-(q\,(1-\theta))/p}} \leq 2,\end{aligned}$$

and hence, again by the definition of k_{h+1} there would be $k_h \le k_{h+1}$ in contradiction with the hypothesis. Hence, $k_h \le k_{h+1}$.

Now put $s_h := k_1 + k_2 + \cdots + k_h$, and define for all $h \in \mathbf{N}$,

$$z_{h+1} = \sum_{n=1}^{k_{h+1}} \frac{1}{(k_h + n)^{(1-\theta)/p}} \, \mathbf{e}_{s_h+n}.$$

Then by (39),

$$\|z_{h+1}\|_\lambda = \left(\sum_{n=1}^{k_{h+1}} \frac{1}{(k_h+n)^{q(1-\theta)/p} \, n^{1-(q\,(1-\theta))/p}} \right)^{1/q} \le 2^{1/q}.$$

From (4) we have $(l^\infty, l^1)_{\theta,q'} = (c_0, l^1)_{\theta,q'}$, and hence, in every case $(F'_{E_0}, F'_{E_1})_{\theta,\overline{q}} = (l^p, l^\infty)_{\theta,1} \approx l^{p/(1-\theta),1}$. We finish by noting that

$$\begin{aligned}
\|z_{h+1}\|_{l^{p/(1-\theta),1}} &= \sum_{n=1}^{k_{h+1}} \frac{1}{(k_h+n)^{(1-\theta)/p} n^{1-((1-\theta)/p)}} \\
&\ge \sum_{n=1}^{k_h} \frac{1}{(k_h+n)^{(1-\theta)/p} \, n^{1-(1/(1-\theta))}} \\
&\ge \frac{k_h}{(2\,k_h)^{(1-\theta)/p} \, k_h^{1-((1-\theta)/p)}} = \frac{1}{2^{(1-\theta)/p}}
\end{aligned}$$

and

$$\lim_{h\to\infty} \|z_h\|_{l^\infty} = \lim_{h\to\infty} \frac{1}{(k_h+1)^{\frac{1-\theta}{p}}} = 0.$$

Case 1b) Assume $p/(1-\theta) < q$. To avoid the Banach case we can suppose moreover that $p/(1-\theta) < 1$. Let $r \in \mathbf{N}$ be such that $1 + ((r-1)/q) < (1-\theta)/p \le 1 + (r/q)$. Hence,

$$0 \le q + r - \frac{q(1-\theta)}{p} < q + r - (q + r - 1) = 1. \tag{40}$$

Subcase 1b1) Suppose furthermore that $1 < q$. Choose $0 < \nu < 1$ such that $1/(1-\nu) < q$. By Freitag's result we have

$$l^{p/(1-\theta),q} \approx (l^p, l^\infty)_{\theta,q} \approx (l^{p(1-\nu)/(1-\theta)}, l^\infty)_{\nu,q} = (l^{p_0/(1-\nu)}, l^\infty)_{\nu,q},$$

where $p_0 := p(1-\nu)^2/(1-\theta)$. Then

$$\frac{p_0}{1-\nu} = \frac{p(1-\nu)}{1-\theta} < \frac{p}{1-\theta} < q$$

and $q(1-\nu) > 1$. Hence, we can assume moreover that $q(1-\theta) > 1$.

For every $h \in \mathbf{N}$, let $r_h \in \mathbf{N}$ be such that

$$1 < \sum_{j=1}^{r_h} \frac{1}{h^q\, j^q} < 2. \tag{41}$$

Suppose we have defined $\{k_{h,i}\}_{i=1}^{j-1} \in \mathbf{N}$ for some $1 \leq j \leq r_h$. By (40) the series

$$\sum_{n=1}^{\infty} \frac{(k_{h,1} + k_{h,2} + \cdots + k_{h,j-1} + n)^{(q(1-\theta)/p)-1}}{n^{q+r-1}}$$

is divergent. Hence, let $k_{h,j}$ be such that

$$1 < \sum_{n=1}^{k_{h,j}} \frac{(k_{h,1} + k_{h,2} + \cdots + k_{h,j-1} + n)^{(q(1-\theta)/p)-1}}{n^{q+r-1}} < 2. \tag{42}$$

Put $s_h = k_{h,1} + k_{h,2} + \cdots + k_{h,r_h}$, and define

$$z_h := \sum_{j=1}^{r} \sum_{n=1}^{k_{h,j}} \frac{1}{h\, j\, n^{(q+r-1)/q}}\ \mathbf{e}_{s_{h-1}+n}.$$

We have

$$\|z_h\|_\lambda = \left(\sum_{j=1}^{r_h} \sum_{n=1}^{k_{h,j}} \frac{(k_{h,1} + k_{h,2} + \cdots + k_{h,j-1} + n)^{(q(1-\theta)/p)-1}}{h^q\, j^q\, n^{q+r-1}} \right)^{1/q},$$

and, since $q > 1$, by (42) and (41),

$$\leq \left(\sum_{j=1}^{r_h} \frac{2}{h^q\, j^q} \right)^{1/q} \leq 4^{1/q}.$$

Now $(F'_{E_0}, F'_{E_1})_{\theta,\overline{q}} = (l^1, l^\infty)_{\theta,q} = l^{1/(1-\theta),q}$. We have

$$\begin{aligned}\|z_h\|_{l^{(1/(1-\theta)),q}} &= \bigg(\sum_{j=1}^{r_h}\sum_{n=1}^{k_{h,j}} \frac{(k_{h,1}+k_{h,2}+\cdots+k_{h,j-1}+n)^{q(1-\theta)-1}}{h^q\, j^q\, n^{q+r-1}}\bigg)^{1/q} \\ &\geq \bigg(\sum_{j=1}^{r_h}\sum_{n=1}^{k_{h,j}} \frac{1}{h^q\, j^q\, n^{q+r-1}}\bigg)^{1/q} \geq \bigg(\sum_{j=1}^{r_h} \frac{1}{h^q\, j^q}\bigg)^{1/q} \geq 1.\end{aligned}$$

Finally,

$$\lim_{h\to\infty} \|z_h\|_{l^\infty} = \lim_{h\to\infty} \frac{1}{h} = 0.$$

Subcase 1b2) Assume $p/(1-\theta) < q < 1$. We show that in this case *there is no complemented subspace isomorphic to l^q in $\lambda = l^{p/(1-\theta),q}$.* In fact, since $(l^s)' = l^\infty$ isometrically for every $0 < s \leq 1$ and $l^{p_0} \subset l^{p/(1-\theta),q} \subset l^{p_1}$ for every $p_0 < p/(1-\theta) < p_1 < 1$, we obtain easily that $\lambda' = (l^{p/(1-\theta),q})' \approx l^\infty$. Moreover, $\lambda \subset l^1$ with dense image. By a result of Mendez and Mitrea, see [**14**], the Banach envelope of λ is isomorphic to l^1.

Assume now that there would be an isomorphism $S : l^q \to X \subset \lambda$ onto a subspace X of λ and a continuous projection $P : \lambda \to X$ from λ onto X. Let $P_n : l^q \to \mathbf{K}$ be the continuous projection onto the nth axis of l^q, and put $z_n := S(\mathbf{e}_n)$ for every $n \in \mathbf{N}$. Then $\varphi_n = P_i S^{-1} P \in \lambda' = l^\infty$ and $\|\varphi_n\| \leq \|P\|\,\|S^{-1}\|$. Clearly $\{z_n\}_{n=1}^\infty$ is a semi-normalized sequence in λ and, for every $n \in \mathbf{N}$, we have

$$\|z_n\|_{l^1} \leq \|S\| \|z_n\|_\lambda \leq \|S\|\,\|S^{-1}\|$$

and

$$\begin{aligned}\|z_n\|_{l^1} &\geq \frac{1}{\|P\|\,\|S^{-1}\|}\,|\langle z_n, \varphi_n\rangle| \\ &= \frac{1}{\|P\|\,\|S^{-1}\|}\,|\langle S^{-1}(z_n), P_n\rangle| \\ &= \frac{1}{\|P\|\,\|S^{-1}\|}\,|\langle \mathbf{e}_n, \mathbf{e}_n\rangle| \\ &= \frac{1}{\|P\|\,\|S^{-1}\|},\end{aligned}$$

which means that $\{z_n\}_{n=1}^{\infty}$ is a semi-normalized sequence in l^1 too. Then $\{z_n\}_{n=1}^{\infty}$ would be equivalent to the standard bases of l^q and l^1 simultaneously, which is absurd.

Case 2) Assume now that there is a subsequence $\{k_n\}_{n=1}^{\infty}$ such that $\lim_{n\to\infty} \mu_{k_n} = \infty$. Projecting onto a suitable sectional subspace we can suppose that $\lim_{n\to\infty} \mu_n = \infty$.

Subcase 2a) Suppose $p/(1-\theta) \geq q$. As above we assume moreover $q < 1$ to avoid the Banach case. We define, for all $h \in \mathbf{N}$,

$$z_h = \frac{1}{\mu_h^{(1-\theta)/p}}\ \mathbf{e}_h.$$

Then, by (31)

$$\|z_h\|_\lambda = \left(\frac{\mu_h}{\mu_h^{(q\,(1-\theta)/p)+1-(q\,(1-\theta)/p)}}\right)^{1/q} = 1.$$

On the other hand, if $p < 1$, using the fact that $(l^p)' = l^\infty$, it is easy to check that the dual $(l^p(\mu))'$ is the weighted space $l^\infty(\mu^{1-(1/p)}, \mu)$ under the duality formula for all $(x_i) \in l^p(\mu)$ and

$$\Big\langle (x_i), (\beta_i) \Big\rangle = \sum_{i=1}^{\infty} x_i\, \beta_i\, \mu_i$$

for all $(\beta_i) \in l^\infty(\mu^{1-(1/p)}, \mu)$. Now we have $F_{E_0} = c_0(\mu^{1-(1/p)}, \mu)$, $F_{E_1} = l^1(\mu)$ and

$$(F'_{E_0}, F'_{E_1})_{\theta,\overline{q}} = (l^1(\mu^{(1/p)-1}, \mu), l^\infty(\mu))_{\theta,1} = l^{1/(1-\theta),1}(\mu^{1/p}).$$

We obtain

$$\|z_h\|_{l^{1/(1-\theta),1}(\mu^{1/p})} = \frac{\mu_h^{1/p}}{\mu_h^{(1-\theta)/p}\,\mu_{k_{h+1}+i}^{\theta/p}} = 1$$

and

$$\lim_{h\to\infty} \|z_h\|_{l^\infty(\mu)} = \lim_{h\to\infty} \frac{1}{\mu_h^{(1-\theta)/p}} = 0.$$

If $1 \leq p$, we obtain analogously $(F'_{E_0}, F'_{E_1})_{\theta,\overline{q}} = l^{p/(1-\theta),1}(\mu)$ and

$$\|z_{k_{h+1}}\|_{l^{(p/(1-\theta)),1}(\mu)} = \sum_{i=1}^{s} \frac{\mu_{k_{h+1}+i}}{i^{\alpha/q} \mu_{k_{h+1}+i}^{(1-\theta)/p} \mu_{k_{h+1}+i}^{1-((1-\theta)/p)}} = \sum_{i=1}^{s} \frac{1}{i^{\alpha/q}},$$

and we finish as above.

Subcase 2b) Now suppose $p/(1-\theta) < q$. The quasi Banach case is present only if $p < (p/(1-\theta)) < 1$. We define as above, for all $h \in \mathbf{N}$,

$$z_h := \frac{1}{\mu_h^{(1-\theta)/p}} \, \mathbf{e}_h. \tag{43}$$

Then

$$\|z_h\|_{\lambda(\mu)} = \left(\frac{\mu_h \, \mu_h^{(q\,(1-\theta)/p)-1}}{\mu_h^{q\,(1-\theta)/p}} \right)^{1/q} = 1. \tag{44}$$

As in Subcase 2a), if $q < 1$, we get $(F'_{E_0}, F'_{E_1})_{\theta,\overline{q}} = l^{1/(1-\theta),1}(\mu^{1/p})$. Hence,

$$\|z_h\|_{l^{1/(1-\theta),1}(\mu^{1/p})} = \frac{\mu_h^{1/p}}{\mu_h^{(1-\theta)/p} \, \mu_h^{\theta/p}} = 1, \tag{45}$$

and

$$\lim_{h\to\infty} \|z_h\|_{l^{\infty}(\mu)} = \lim_{h\to\infty} \frac{1}{\mu_h^{(1-\theta)/p}} = 0.$$

If $q \geq 1$ we obtain $(F'_{E_0}, F'_{E_1})_{\theta,\overline{q}} = l^{1/(1-\theta),q}(\mu^{1/p})$, and, since $q(1-\theta) < 1$,

$$\|z_h\|_{l^{1/(1-\theta),q}(\mu^{1/p})} = \left(\frac{\mu_h^{1/p}}{\mu_h^{q\,(1-\theta)/p} \, \mu_h^{(1-q(1-\theta))/p}} \right)^{1/q} = 1, \tag{46}$$

and we follow as above.

Case 3) Assume $\mu(\mathbf{N}) = \sum_{j=1}^{\infty} \mu_j < \infty$. The proof goes along the same lines of Case 2, choosing the same z_h, $h \in \mathbf{N}$, the space

$(F'_{E_0}, F'_{E_1})_{\theta,\overline{q}}$ as above in every alternative $p < 1$ or $p \geq 1$, but noting that $l^p(\mu) + l^\infty(\mu) = l^p(\mu)$ because $\mu(\mathbf{N}) < \infty$, and hence

$$\lim_{h\to\infty} \|z_h\|_{l^p(\mu)} = \lim_{h\to\infty} \left(\frac{\mu_h}{\mu_h^{1-\theta}}\right)^{1/p} = \lim_{h\to\infty} \mu_h^{\theta/p} = 0. \qquad \square$$

Remark. In [**13**, Lemma 3.1] it is asserted that if $q \leq v$, there are noncompact maps from $l^{p,q}(\nu)$ into $l^{u,v}(\mu)$ for every measure ν and μ in n, whatever p and u be. Due to the conclusion of case 1) when $0 < p/(1-\theta) < q < 1$, the proof given in [**13**] is actually wrong under these circumstances. However, the conclusion of Lemma 3.1 in [**13**] is correct still. In fact, note that in that case we have $p < q$. If $p \leq u$, the inclusion $l^{p,q} \subset l^{u,v}$ is not compact. If $p > u$, we have the chain of continuous inclusions $l^{p,q} \subset l^q \subset l^v$ whose composition is not compact. Since l^v is always *a subspace* of $l^{u,v}$, the proof is complete.

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