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PERIODIC BOUNDARY VALUE PROBLEMS FOR SECOND ORDER DIFFERENTIAL EQUATIONS

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ABSTRACT. This paper presents sufficient conditions for the existence of solutions to periodic boundary-value problems of second order differential equations. Our results are obtained via a new fixed point theorem which is developed in the paper and lead to new existence principles.

1. Introduction. Let $J = [0, T]$ be a compact interval, and let $\|x\| = \max\{|x(t)| : t \in J\}$, $\|x\|_L = \int_0^T |x(t)|\,dt$ and $\|(x, a, b)\|_0 = \|x\| + |a| + |b|$ be the norms in Banach spaces $C(J)$, $L_1(J)$ and $C(J)\times\mathbf{R}^2$, respectively.

This paper is concerned with the existence of solutions for the periodic boundary value problem of second order differential equations (PBVP)

$$(1)\qquad \begin{cases} (x'(t) + g(t, x(t), x'(t)))' = f(t, x(t), x'(t)), \\ x(0) = x(T), \quad x'(0) = x'(T), \end{cases}$$

and the boundary value problem (BVP)

$$(2)\qquad \begin{cases} (x'(t) + g(t, x(t), x'(t)))' = f(t, x(t), x'(t)), \\ x(0) = A, x(T) = B, \end{cases}$$

where $g \in C(J \times \mathbf{R}^2)$, $g(0, x, y) = g(T, x, y)$ for $(x, y) \in \mathbf{R}^2$ and f satisfies the local Carathéodory conditions on $J \times \mathbf{R}^2$.

The solvability of periodic problems and boundary value problems (1), (2) have been widely investigated with $g = 0$ and one-sided growth restrictions to the third argument in the nonlinearity f. We refer, for instance, to [**3, 6, 7, 9**] and references therein. Reference [**8**] has considered periodic boundary value problems (1) and has shown that the existence of solutions for PBVP(1) is equivalent to the existence

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of fixed points of an operator of the form $U+V$ with U compact and V a strict contraction in the Banach space $C(J)\times \mathbf{R}^2$. In this paper, we give existence results of positive solutions on compact intervals for PBVP(1) and BVP(2).

The fundamental tool used in the proofs of our main results is essentially the fixed point theorem, see Lemma 1, based on Martelli [**6**]. However, the hypotheses imposed on the function f and methods of the proof in this paper are different from [**5**].

2. Preliminaries. Throughout the paper, we stipulate that $(x_1,a_1,b_1) \le (x_2,a_2,b_2)$ if and only if $x_1\le x_2$, $a_1\le a_2$, $b_1\le b_2$, where $(x_i,a_i,b_i)\in E\times \mathbf{R}^2$ with $i=1,2$ and $E=C(J)$ or $E=L_1(J)$. $x\le y$ if $x(t)\le y(t)$ for all $t\in J$, where $x,y\in C(J)$ $(x,y\in L_1(J))$.

Our results, based on Lemma 1, will be obtained through use of the following Leray-Schauder fixed point theorem:

Theorem. *Let E be a Banach space and $G:E\to E$ a condensing operator. If the set*

$$M := \{y\in E : y=\lambda Gy \text{ for some } \lambda\in(0,1)\}$$

is bounded, then G has a fixed point.

Lemma 1. *Let P be a closed and convex cone of a Banach space E. If the operators $D:E\to E$ and $B:P\to P$ satisfy, respectively,*

(a) *The operator equation $(I-D)x=y$ has a unique solution $x:=Sy\in P$ for each $y\in P$ such that the operator $S:P\to P$ is continuous. Here, as usual, I stands for the identity operator on E,*

(b) *B is completely continuous.*

(c) *The set $N=\{x\in P:$ there exists $\lambda\in(0,1)$ such that $x=\lambda D((1/\lambda)x)+\lambda Bx\}$ is bounded.*

Then there exists an $x\in P$ such that $x=Dx+Bx$.

Proof. For any $y\in P$, there exists an $x\in P$ such that $x-Dx=y$. This implies that S is a mapping from P into P. We have that

$S = (I - D)^{-1}$ by (a). Define a mapping $\mathbf{T}$ by

$$\mathbf{T}x = SBx.$$

Evidently, $\mathbf{T} : P \to P$ is completely continuous by conditions (a) and (b). Also, $x = \lambda \mathbf{T}x$ is equivalent to $x = \lambda D((1/\lambda)x) + \lambda Bx$ for each $\lambda \in (0,1)$ and $x \in P$. We first show that

(*). If $G : P \to P$ is a completely continuous operator and

(3) $$\alpha := \sup\{|x| : x \in P \text{ and there exists } \lambda \in (0,1) \text{ such that } x = \lambda Gx\} < \infty,$$

then G has a fixed point $x \in P$.

In fact, define the map $\widehat{G} : E \to P$ by

$$\widehat{G}(x) = \begin{cases} G(x) & \text{if } x \in P, \\ G(0) & \text{if } x \notin P. \end{cases}$$

Evidently, $\widehat{G}$ is completely continuous on E, therefore, $\widehat{G}$ is condensing. For any y belonging to M given in the above theorem, we have $y = \lambda \widehat{G}(y)$ for some $\lambda \in (0,1)$. If $y \in P$, then $|y| \leq \alpha$. Otherwise, $y = \lambda G(0)$, which yields that $|y| \leq |G(0)|$. Hence, M is bounded. By the theorem $\widehat{G}$ has a fixed point x. It is obvious that $x \in P$.

Therefore, the conclusion of the theorem immediately follows by an application of (*), if we show that the operator $\mathbf{T}$ satisfies expression (3). In order to do this, let $x = \lambda \mathbf{T}x$, i.e., $x = \lambda(I - D)^{-1}Bx$. This guarantees that $x = \lambda D((1/\lambda)x) + \lambda Bx$, i.e., $x \in N$. From the boundedness of N, it follows that there exists a positive constant c such that $|x| \leq c$ for all $x \in N$. By the arbitrariness of x, we have that $\mathbf{T}$ fulfills (3). The proof is completed. □

Remark 1. If D is a contraction operator, i.e., there exists a constant μ with $0 < \mu < 1$ such that

$$|Dx - Dy| \leq \mu |x - y|$$

for all $x, y \in E$, then $(I - D)^{-1}$ exists on E in view of Boyd and Wong [**1**].

Let us assume that the functions g and f in (1) and (2) satisfy some of the following assumptions:

(H1) For any $y \in C_+(J) := \{x \in C(J) : x \geq 0\}$, the equation $x + g(t, x, x') = y$ has a unique solution $x := Sy \geq 0$ such that the operator $S : C_+(J) \to C_+(J)$ is continuous.

(H2) There exists a constant $0 < k < 1$ such that

$$|g(t, x, y)| \leq k|y|.$$

(H3) (i) $f : J \times \mathbf{R}_+ \times \mathbf{R} \to \mathbf{R}_+$;

(ii) There exist a function $\alpha \in L_\infty(J, \mathbf{R}_+)$ (the norm is denoted by $\|\cdot\|_\infty$) and a continuous nondecreasing function $\beta : [0, \infty) \to [0, \infty)$ with $\lim_{t\to\infty} \int_0^t (1/\beta(s))\, ds = +\infty$ such that

$$|f(t, x, y)| \leq \alpha(t)\beta(|y|).$$

(H$'$2) There exists a constant $0 < k < 1$ such that

$$|g(t, x, y)| \leq k(|y - (B - A)/T|).$$

(H$'$3) (i) $f : J \times \mathbf{R}_+ \times \mathbf{R} \to \mathbf{R}_+$;

(ii) There exist a function $\alpha \in L_\infty(J, \mathbf{R}_+)$ (the norm is denoted by $\|\cdot\|_\infty$) and a continuous nondecreasing function $\beta : [0, \infty) \to [0, \infty)$ with $\lim_{t\to\infty} \int_0^t (1/\beta(s))\, ds = +\infty$ such that

$$|f(t, x, y)| \leq \alpha(t)\beta(|y - (B - A)/T|).$$

Here A and B are given in BVP(2), $L_\infty(J)$ is the space of all measurable function $x : J \to \mathbf{R}$ which are essentially bounded, i.e.,

$$\operatorname*{ess\,sup}_{t\in J} |x(t)| := \inf\{c \geq 0 : |x(t)| \leq c \text{ a.e. on } J\} < \infty,$$

and $\|x\|_\infty = \operatorname{ess\,sup}_{t\in J} |x(t)|$.

3. Periodic boundary value problems. In this section we consider the existence of positive solutions for PBVP(1). The function

$x \in C^1(J)$ is said to be a solution of PBVP (1) if $x'(t)+g(t,x(t),x'(t))$ is absolutely continuous on J, $(x'(t)+g(t,x(t),x'(t)))' = f(t,x(t),x'(t))$ is satisfied almost everywhere on J and $x(0) = x(T)$, $x'(0) = x'(T)$.

Theorem 1. *Let assumptions* (H1)–(H3) *be satisfied. Then* PBVP(1) *has at least one nonnegative solution.*

Proof. The proof will be given in three steps.

Step 1. Let $E = \{x \in C^1(J) : x(0) = x(T)\}$, $E_+ = \{x \in E : x(t) \geq 0$ for $t \in J\}$ and $\mathbf{R}_+^2 = \{(a,b) \in \mathbf{R}^2 : a \geq 0, b \geq 0\}$. Define the norm in E by $\|x\|$ for each $x \in E$. It is clear that $(E, \|\cdot\|)$ is a Banach space and E_+ is a closed convex cone of E. Let us introduce the operators $W, Z : E_+ \times \mathbf{R}_+^2 \to L_1(J) \times \mathbf{R}^2$ by

$$W(x,a,b) = \left(a + \int_0^t f(s, \varphi(x,a)(s), x(s))ds, a, b + \int_0^T x(s)\, ds\right),$$

and

$$Z(x,a,b) = (-g(t, \varphi(x,a)(t), x(t)), 0, 0)$$

with

$$\varphi(x,a)(t) = \int_0^t x(s)\, ds + a.$$

It is easy to prove $\varphi'(t) = x(t)$ and $\varphi(t)$ is a solution of PBVP(1) if and only if (x,a,b) is a fixed point of the operator $Z+W$.

Step 2. We shall show that $Z+W$ has fixed points in E_+ by applying Lemma 1. For this purpose, we check conditions (a)–(c) of Lemma 1.

First, by (H1), there exists a unique $x \in C_+(J)$ such that $x + g(t,x,x') = y$ for any $y \in C_+(J)$. It follows from this that $(I-Z)u = v$ and that there exists a unique solution $u = (x,a,b) \in C_+(J) \times \mathbf{R}_+^2$ for any $v = (y,a,b) \in C_+(J) \times \mathbf{R}_+^2$, that is, (a) is satisfied.

Next, (H3) (i) shows that $W(x,a,b) \geq 0$ when $(x,a,b) \in E_+ \times {\mathbf{R}_+}^2$. Part (ii) shows that W is bounded. By means of Lebesgue's dominated convergence theorem we have that W is continuous. Let $M \subset E_+$ be a bounded set. For any $t_1, t_2 \in J$, $x \in M$ and $a, b \in \mathbf{R}_+$, denote $z = W(x, a, b)$. From (H3) it follows that

$$\begin{aligned}
&\|z(t_2) - z(t_1)\|_0 \\
&\quad = \left| \int_0^{t_2} f(s, \varphi(x, a)(s), x(s))\, ds - \int_0^{t_1} f(s, \varphi(x, a)(s), x(s))\, ds \right| \\
&\quad \le \int_{t_1}^{t_2} |f(s, \varphi(x, a)(s), x(s))|\, ds \le \int_{t_1}^{t_2} \alpha(s)\beta(\|x\|)\, ds.
\end{aligned}$$

Obviously, β is bounded on M and $\alpha \in L_\infty(J)$. This implies that $W(M)$ is equicontinuous on J. By virtue of the Ascoli-Arzela theorem, we can conclude that $W(M)$ is a relatively compact subset in $E \times \mathbf{R}_+^2$; therefore, W is completely continuous. Thus (b) holds.

Finally, we shall prove that condition (c) is fulfilled. It is sufficient to prove this for $a = 0$ and fixed $b \in \mathbf{R}_+$. Suppose that this is not the case. Then there exist $(\lambda_n, x_n) \in (0, 1) \times E_+$ such that $u_n = \lambda_n Z((1/\lambda_n)u_n) + \lambda_n W(u_n)$ with $\mu_n = \|u_n\|_0 \ge n$ for $n = 1, 2, \dots$, which implies that $\|x_n\| \ge n - b$, where $u_n = (x_n, 0, b)$. Note that

$$\begin{aligned}
x_n(t) &= \lambda_n \int_0^t f(s, \varphi(x_n, 0)(s), x_n(s))\, ds \\
&\qquad - \lambda_n g\left(t, \varphi\left(\frac{1}{\lambda_n} x_n, 0\right)(t), \frac{1}{\lambda_n} x_n(t)\right) \\
&= \lambda_n \int_0^t f(s, \varphi(x_n, 0)(s), x_n(s))\, ds \\
&\qquad - \lambda_n g\left(t, \frac{1}{\lambda_n} \varphi(x_n, 0)(t), \frac{1}{\lambda_n} x_n(t)\right).
\end{aligned}$$

Denote $w_n(t) = |x_n(t)|$ for $n = 1, 2, \dots$. Then by (H2) and (H3) we have

$$\begin{aligned}
w_n(t) &\le \lambda_n \int_0^t |f(s, \varphi(x_n, 0)(s), x_n(s))|\, ds \\
&\qquad + \lambda_n \left| g\left(t, \frac{1}{\lambda_n} \varphi(x_n, 0)(t), \frac{1}{\lambda_n} x_n(t)\right) \right| \\
&\le \lambda_n \int_0^t \alpha(s)\beta(w_n(s))\, ds + k w_n(t).
\end{aligned}$$

So

$$w_n(t) \leq \frac{\lambda_n \|\alpha\|_\infty}{1-k} \int_0^t \beta(w_n(s))\, ds.$$

Let $h_n(t) = (\lambda_n \|\alpha\|_\infty)/(1-k) \int_0^t \beta(w_n(s))\, ds$. Then $h_n(0) = 0$ and $w_n(t) \leq h_n(t)$ for $t \in J$. This implies by differentiation that

$$h_n'(t) = \frac{\lambda_n \|\alpha\|_\infty}{1-k} \beta(w_n(t)) \leq \frac{\lambda_n \|\alpha\|_\infty}{1-k} \beta(h_n(t)),$$

or

$$\frac{h_n'(t)}{\beta(h_n(t))} \leq \frac{\lambda_n \|\alpha\|_\infty}{1-k}.$$

On integration of this inequality with respect to t from 0 to t yields

$$\int_0^t \frac{h_n'(s)}{\beta(h_n(s))}\, ds \leq \int_0^t \frac{\lambda_n \|\alpha\|_\infty}{1-k}\, ds \leq T \frac{\lambda_n \|\alpha\|_\infty}{1-k}.$$

Now a change of the variable in this inequality yields, for any $t \in J$,

$$\int_0^{h_n(t)} \frac{ds}{\beta(s)} \leq T \frac{\lambda_n \|\alpha\|_\infty}{1-k}.$$

From the arbitrariness of t it follows that

$$\int_0^{\|h_n\|} \frac{ds}{\beta(s)} \leq T \frac{\lambda_n \|\alpha\|_\infty}{1-k}.$$

Obviously, $n - b \leq \|x_n\| \leq \|h_n\|$ for $n = 1, 2, \ldots$. This contradicts (H3) for n approaching infinity, which completes the proof of (c).

Step 3. Now, by Lemma 1, there exists a fixed point of the operator $W(\cdot, 0, b) + Z(\cdot, 0, b)$ with b given in Step 2, say $(x, 0, b)$. Set

$$u(t) = \int_0^t x(s)\, ds + b, \quad \text{for} \quad t \in J.$$

Then $u(t)$ is a solution of PBVP(1). By Lemma 1, $u \in E_+$, i.e., $u \geq 0$.

Corollary 1. *Let condition* (H3) *hold.* PBVP(1) *has at least a nonnegative solution when the following hypotheses hold*:

(i) $g(t, x, 0) = 0$;

(ii) $|g(t, x, y_1) - g(t, x, y_2)| \leq k|y_1 - y_2|$ *for some constant* $0 < k < 1$;

(iii) $g(t, x, y) \leq -x$ *when* $x < 0$.

Proof. By Theorem 1, to prove the result it is thus sufficient to see that conditions (H1) and (H2) are satisfied. Conditions (i) and (ii) guarantee that (H2) is satisfied. From (ii) and Remark 1 it follows that $(I - Z)^{-1}$ exists on E_+. Condition (iii) shows that $(I - Z)^{-1}y \in E_+$ if $y \in E_+$. Hence, (H1) is true.

4. Boundary value problems. In this section, we consider the existence of positive solutions for BVP(2). The function $x \in C^1(J)$ is said to be a solution of BVP(2) if $x'(t) + g(t, x(t), x'(t))$ is absolutely continuous on J, $(x'(t) + g(t, x(t), x'(t)))' = f(t, x(t), x'(t))$ is satisfied almost everywhere on J and $x(0) = A, x(T) = B$.

Theorem 2. *Let assumptions* (H1), (H′2) *and* (H′ 3) *be satisfied. Then* BVP(2) *has at least one nonnegative solution.*

Proof. Set $\delta(t) = A + (t/T)(B - A), \eta = (B - A)/T$ and $y = x - \delta$; then BVP(2) is transformed into

$$\begin{cases} (y' + g(t, y(t) + \delta(t), y'(t) + \eta))' = f(t, y(t) + \delta(t), y'(t) + \eta), & t \in J, \\ y(0) = y(T) = 0. \end{cases}$$

Let $\hat{f}(t, y(t), y'(y)) = f(t, y(t) + \delta(t), y'(t) + \eta)$, $\hat{g}(t, y(t), y'(y)) = g(t, y(t) + \delta(t), y'(t) + \eta)$; then conditions (H′2) and (H′3) are transformed into (H2) and (H3), respectively. So, for the sake of convenience, we assume that $A = B = 0$ in BVP(2), which shows that $\delta(t) = 0$, $\eta = 0$.

Let $E = \{x \in C^1(J) : x(0) = x(T) = 0\}$ with norm $\|x\|$, $E_+ = \{x \in E : x(t) \geq 0 \text{ for } t \in J\}$. It is clear that $(E, \|\cdot\|)$ is a Banach space and E_+ is a closed convex cone of E. Let us introduce the operators $\widehat{W}, \widehat{Z} : AC(J) \to L_1(J)$ by

$$\widehat{W}(x) = \int_0^t f(s, \varphi(x)(s), x(s))\, ds,$$

and

$$\widehat{Z}(x) = -g(t, \varphi(x)(t), x(t))$$

with $\varphi(x)(t) = \int_0^t x(s)\,ds$. Similar to the proof of Theorem 1, we can check that the operator $\widehat{W} + \widehat{Z}$ satisfies the conditions in Lemma 1; therefore, there exists an $x \in E_+$ such that $x = \widehat{W}x + \widehat{Z}x$. Set $u(t) = \int_0^t x(s)\,ds$. Then u is a positive solution of BVP(2).

Example 1. Consider the differential equation

(4)

$$\begin{cases} (x' + \alpha(t)x')' = f_1(t, x, x')x^m + f_2(t, x, x') + f_3(t, x, x')(x')^n, \\ x(0) = x(T), \quad x'(0) = x'(T), \end{cases}$$

where $\alpha \in C(J)$, $\alpha(0) = \alpha(T)$, $\|\alpha\| < 1$, $m, n \in \mathbf{N}$ and $f_i \in J \times \mathbf{R}_+ \times \mathbf{R} \to \mathbf{R}_+$, $i = 1, 2, 3$, satisfy the local Carathéodory conditions. In addition, on the set $J \times \mathbf{R}^2$ the inequalities

$$|f_j(t, x, y)| \le \alpha_j(t)\beta_j(|y|), \ |f_3(t, x, y)| \le \alpha_3(t)$$

are satisfied, where α_j, $\alpha_3 \in L_\infty(J)$ and continuous nondecreasing function β_j satisfy

$$\lim_{t\to\infty} \int_0^t \frac{1}{\beta_j(s)}\,ds = \infty \quad \text{for} \quad j = 1, 2.$$

We see that the function

$$f(t, x, y) = f_1(t, x, y)x^m + f_2(t, x, y) + f_3(t, x, y)y^n$$

satisfies assumption (H3) with

$$\beta(t) = \max\{\beta_1(t), \ \beta_2(t), \ |t|^n\}.$$

Clearly, (H2) holds. For any $y \in E_+$, the equation $x + \alpha(t)x' = y$ has a unique continuous solution $u \in E_+$, see [**9**]. This implies that (H1) is fulfilled. Consequently, PBVP(4) has at least one solution $x \ge 0$.

Example 2. Consider the differential equation

(5)

$$\begin{cases} (x' + \alpha(t)x')' = f_1(t, x, x')x^m + f_2(t, x, x') + f_3(t, x, x')(x')^n, \\ x(0) = A, \ x(T) = B \end{cases}$$

where $\alpha(t)$ and m, n are given in Example 1, $A > B$ and f_i satisfy the local Carathéodory conditions. In addition, on the set $J \times \mathbf{R}^2$ the inequalities

$$|f_j(t,x,y)| \leq \alpha_j(t)\beta_j(|y-(B-A)/T|),\ |f_3(t,x,y)| \leq \alpha_3(t)$$

with α_i, β_i are given in Example 1, $i = 1, 2, 3$.

Similar to Example 1, we see that (H1) and (H$'$3) are fulfilled. Obviously, $|\alpha(t)y| \leq \|\alpha\||y| \leq \|\alpha\|(|y| - (B-A)/T) \leq \|\alpha\|(|y - (B-A)/T|)$, that is, (H$'$2) is true. This implies that the conditions of Theorem 2 are fulfilled. Consequently, BVP(5) has at least one solution $x \geq 0$.

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