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A NOTE ON SOME CHARACTERIZATIONS OF ARITHMETIC FUNCTIONS

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An arithmetic function is a mapping from positive integers into the field of complex numbers. We shall denote the set of arithmetic functions by $\mathcal{A}$. Various binary product operations dependent on the divisibility properties of the natural number n may be defined on the set $\mathcal{A}$. One such well-known product is the Dirichlet convolution

$$(f * g)(n) = \sum_{d|n} f(d) g\left(\frac{n}{d}\right), \tag{1}$$

where $f, g \in \mathcal{A}$. A large number of analogues and generalizations of Dirichlet's convolution have been studied in the literature, and for further information, the reader is referred to [**3**–**7**]. In this paper, we investigate the functions defined by

$$G_{(f\circ g)}(n, m) = \sum_{d|(n,m)} f(d) g\left(\frac{m}{d}\right) \tag{2}$$

and

$$G_{(h\circ f\circ g)}(n, m) = \sum_{d|(n,m)} h(d) f\left(\frac{n}{d}\right) g\left(\frac{m}{d}\right) \tag{3}$$

where $f, g, h \in \mathcal{A}$, and n, m are natural numbers with (n, m) as the gcd of n and m. It follows that $G_{(f\circ g)}(m, m) = (f * g)(m)$. These functions play a role in the study of arithmetic functions within the context of Dirichlet convolution as will be demonstrated in this note.

Definitions. (i) An arithmetic function f which is not identically zero is called multiplicative if $f(mn) = f(m)f(n)$ whenever $(m, n) = 1$,

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and it is called completely multiplicative if $f(mn) = f(m)f(n)$ for all natural numbers m and n.

(ii) An arithmetic function $G(n,m)$ of two variables is a mapping from $Z^+ \times Z^+ \to C$ the field of complex numbers.

(iii) $G(n,m)$ is said to be multiplicative in n,m if $G(n,m)G(n',m') = G(nn',mm')$ whenever $(nm, n'm') = 1$.

We shall denote the set of multiplicative functions by $\mathcal{M}$ and the set of completely multiplicative functions by $\mathcal{CM}$.

We now prove the following result.

Theorem 1. *Let* $f, g, h \in \mathcal{M}$, $n = \pi_{i=1}^{k} p_i^{\alpha_i}$ *and* $m = \pi_{i=1}^{k} p_i^{\beta_i}$, $\alpha_i \geq 0$ *and* $\beta_i \geq 0$. *Then*

$$G_{(f\circ g)}(n,m) = \prod_{i=1}^{k} \sum_{j=0}^{\min\{\alpha_i,\beta_i\}} f(p_i^j)g(p_i^{\beta_i - j})$$

and

$$G_{(h\circ f\circ g)}(n,m) = \prod_{i=1}^{k} \sum_{j=0}^{\min\{\alpha_i,\beta_i\}} h(p_i^j)f(p_i^{\alpha_i - j})g(p_i^{\beta_i - j}).$$

Proof. Assuming $(nm, n'm') = 1$, it follows that $(n,n') = (n,m') = (m,n') = (m,m') = 1 \Rightarrow ((n,m),(n',m')) = 1$ and that $(n,m)(n',m') = (nn',mm')$ and, hence,

$$\begin{aligned} G_{(f\circ g)}(nn',mm') &= \sum_{d|(nn',mm')} f(d)g\left(\frac{mm'}{d}\right) \\ &= \sum_{d|(n,m)(n',m')} f(d)g\left(\frac{mm'}{d}\right) \\ &= \sum_{\substack{d_1|(n,m) \\ d_2|(n',m') \\ (d_1,d_2)=1}} f(d_1d_2)g\left(\frac{mm'}{d_1d_2}\right) \end{aligned}$$

$$= \sum_{d_1|(n,m)} f(d_1) g\left(\frac{m}{d_1}\right) \sum_{d_2|(n',m')} f(d_2) g\left(\frac{m'}{d_2}\right)$$
$$= G_{(f\circ g)}(n,m) G_{(f\circ g)}(n',m'),$$

that is, $G_{(f\circ g)}(n,m)$ is multiplicative. Consequently, $G_{(f\circ g)}(n,m)$ is known when n and m are powers of the same prime p from which it follows that:

$$G_{(f\circ g)}\left(p^\alpha, p^\beta\right) = \sum_{j=0}^{\min\{\alpha,\beta\}} f(p^j) g(p^{\beta-j}), \tag{4}$$

and hence the result. A similar argument applies for the other function. □

Remarks. It follows that:

$$G_{(f\circ e\circ g)} = G_{(f\circ g)}$$

and when $n = m$ that

$$G_{(f\circ e\circ g)}(m,m) = G_{(f\circ g)}(m,m) = (f * g)(m),$$

where $e(n) = 1$ for all n.

Further, a routine calculation shows that if $f = g = h$ is completely multiplicative, then

$$G_{(f\circ f\circ f)}(n,m) = \sum_{d|(n,m)} f(d) f\left(\frac{n}{d}\right) f\left(\frac{m}{d}\right)$$
$$= \frac{f(n)f(m)}{f((n,m))} \frac{\Pi_{i=1}^k \left(1 - f(p_i)^{\min\{\alpha_i,\beta_i\}+1}\right)}{\sum_{d|(n,m)} \mu(d) f(d)}$$
$$= \frac{f(n)f(m)}{f((n,m))} \Pi_{i=1}^k \sum_{j=0}^{\min\{\alpha_i,\beta_i\}} f(p_i^j).$$

Theorem 2. *Let* $f \in \mathcal{M}$, $n = \pi_{i=1}^k p_i^{\alpha_i}$, $m = \pi_{i=1}^k p_i^{\beta_i}$, $\alpha_i \geq 0$ *and* $\beta_i \geq 0$. *Then* $f \in \mathcal{CM}$ *if and only if*

$$G_{(f\circ f)}(n,m) = f(m)\tau((n,m)),$$

where $\tau((n,m)) = \sum_{d|(n,m)} 1$.

Proof. Suppose $f \in \mathcal{CM}$. Then

$$\begin{aligned} G_{(f\circ f)}(n,m) &= \sum_{d|(n,m)} f(d) f\left(\frac{m}{d}\right) \\ &= \prod_{i=1}^{k} \sum_{j=0}^{\min\{\alpha_i,\beta_i\}} f(p_i^j) f(p_i^{\beta_i - j}) \\ &= \prod_{i=1}^{k} \sum_{j=0}^{\min\{\alpha_i,\beta_i\}} f(p)^{\beta_i} \\ &= \prod_{i=1}^{k} f(p^{\beta_i}) \Pi_{i=1}^{k} \left(1 + \min\{\alpha_i, \beta_i\}\right) \\ &= f(m)\tau((n,m)). \end{aligned}$$

Conversely,

$$\begin{aligned} G_{(f\circ f)}(n,m) = f(m)\tau((n,m)) &\Longrightarrow G_{(f\circ f)}(p^\alpha, p^\beta) \\ &= \sum_{j=0}^{\min\{\alpha,\beta\}} f(p^j) f(p^{\beta - j}) = f(p^\beta)\tau(p^{\min\{\alpha,\beta\}}) \end{aligned}$$

if and only if

$$f(p^a) f(p^{\beta - a}) = f(p^\beta)$$

for all primes p and $\beta \geq a \geq 1$, and hence the result. □

By setting $n = m$ in Theorem 2, we obtain the following known result, see [**2**].

Corollary 1. *Let* $f \in \mathcal{M}$. *Then,* $f \in \mathcal{CM}$ *if and only if* $f * f = f\tau$.

Apostol [**1**] proves that if $f \in \mathcal{CM}$, then $f \cdot (g * h) = (f \cdot g) * (f \cdot h)$ for all $g, h \in \mathcal{A}$, where $\cdot$ denotes ordinary multiplication of functions. The analogue of this result is as follows.

Theorem 3. *Let $f \in \mathcal{M}$. Then $f \in \mathcal{CM}$ if and only if $G_{(e_0 \circ f)} \cdot G_{(g \circ h)} = G_{(f \cdot g) \circ (f \cdot h)}$ for all $g, h \in \mathcal{A}$, where $e_0(n) = [1/n]$.*

Proof. Suppose $f \in \mathcal{CM}$. Then

$$\begin{aligned} G_{(f\cdot g)\circ(f\cdot h)}(n,m) &= \sum_{d|(n,m)} (f\cdot g)(d)(f\cdot h)\left(\frac{m}{d}\right) \\ &= \sum_{d|(n,m)} f(d) f\left(\frac{m}{d}\right) g(d) h\left(\frac{m}{d}\right) \\ &= f(m) \sum_{d|(n,m)} g(d) h\left(\frac{m}{d}\right) = f(m) G_{(g\circ h)}(n,m) \\ &= (G_{(e_0\circ f)} \cdot G_{(g\circ h)})(n,m). \end{aligned}$$

Conversely, if $G_{(e_0 \circ f)} \cdot G_{(g \circ h)} = G_{(f \cdot g)\circ(f \cdot h)}$ for all $g, h \in \mathcal{A}$, then setting $g = \mu$ and $h = \mu^{-1}$ gives $G_{(e_0 \circ f)} \cdot G_{(\mu \circ \mu^{-1})} = G_{f\mu \circ f\mu^{-1}}$ for all natural numbers n, m. And, in particular, when $n = m$ we obtain the desired result. □

We now apply Theorem 1 to reprove the following result, proved in [**6**, pages 57–58].

Theorem 4. *Let $f \in \mathcal{M}$. If n and r contain the same ν prime divisors, then*

$$\sum_{d|r} f\left(\frac{nr}{d}\right) f^{-1}(d) = (-1)^{\nu} \sum_{\substack{t|n \\ \gamma(t)=\gamma(n)}} f\left(\frac{n}{t}\right) f^{-1}(rt),$$

where $\gamma(n)$ denotes the core of n.

Proof. Let $n = \pi_{i=1}^{k} p_i^{\alpha_i}$, $r = \pi_{i=1}^{k} p_i^{\beta_i}$. Then,

$$G_{(f^{-1}\circ f)}(r, rn) = \sum_{d|(r,rn)} f^{-1}(d) f\left(\frac{nr}{d}\right) = \sum_{d|r} f^{-1}(d) f\left(\frac{nr}{d}\right).$$

Consequently,

$$G_{(f^{-1}\circ f)}(p^{\beta}, p^{\alpha+\beta}) = \sum_{j=0}^{\beta} f^{-1}(p^j) f(p^{\alpha+\beta-j}),$$

and since

$$\begin{aligned}\sum_{j=0}^{\beta} f^{-1}(p^j) f(p^{\alpha+\beta-j}) &+ \sum_{j=1}^{\alpha} f^{-1}(p^{\beta+j}) f(p^{\alpha-j}) \\ &= \sum_{d|p^{\alpha+\beta}} f^{-1}(d) f\left(\frac{p^{\alpha+\beta}}{d}\right) = 0 \Longrightarrow \sum_{j=0}^{\beta} f^{-1}(p^j) f(p^{\alpha+\beta-j}) \\ &= -\sum_{j=1}^{\alpha} f^{-1}(p^{\beta+j}) f(p^{\alpha-j}) = -\sum_{\substack{t|p^{\alpha} \\ \gamma(t)=\gamma(p^{\alpha})}} f\left(\frac{p^{\alpha}}{t}\right) f^{-1}(p^{\beta} t) \\ &= \sum_{d|p^{\beta}} f^{-1}(d) f\left(\frac{p^{\alpha+\beta}}{d}\right),\end{aligned}$$

and hence, the result. □

Definition. Let $f \in \mathcal{M}$. Then f is said to be a quadratic function [**8, 9**], if it is the Dirichlet product of two completely multiplicative functions.

The following result is also known.

Theorem 5 [**9**]. *A quadratic function f satisfies the identity*

$$f(nm) = \sum_{d|(n,m)} f\left(\frac{n}{d}\right) f\left(\frac{m}{d}\right) f_1(d) f_2(d) \mu(d),$$

where $f = f_1 * f_2$ and $f_1, f_2 \in \mathcal{CM}$, where μ is the Möbius function.

It follows from this theorem and Theorem 1 with $h = f_1 f_2 \mu$ that:

$$\begin{aligned}\prod_{i=1}^{k} f(p_i^{\alpha_i+\beta_i}) &= \prod_{i=1}^{k} \sum_{j=0}^{\min\{\alpha_i,\beta_i\}} f_1(p_i^j) f_2(p_i^j) \mu(p_i^j) f(p_i^{\alpha_i-1}) f(p_i^{\beta_i-1}) \\ &= \prod_{i=1}^{k} \Big[f(p_i^{\alpha_i}) f(p_i^{\beta_i}) - f_1(p_i) f_2(p_i) f(p_i^{\alpha_i-1}) f(p_i^{\beta_i-1}) \Big] \\ &= \prod_{i=1}^{k} \Big[f(p_i^{\alpha_i}) f(p_i^{\beta_i}) - f^{-1}(p_i^2) f(p_i^{\alpha_i-1}) f(p_i^{\beta_i-1}) \Big],\end{aligned}$$

since $f = f_1 * f_2$, from which it follows that

$$f(p^{\alpha+\beta}) = f(p^\alpha) f(p^\beta) - f^{-1}(p^2) f(p^{\alpha-1}) f(p^{\beta-1}),$$

and, in particular with $\beta = 1$, that

$$f(p^{\alpha+1}) = f(p^\alpha) f(p) - f^{-1}(p^2) f(p^{\alpha-1}).$$

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