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SUFFICIENT CONDITIONS FOR FACTORABLE MATRICES TO BE BOUNDED OPERATORS ON $\mathcal{A}_k$

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ABSTRACT. A factorable matrix A is a lower triangular matrix with entries $a_{nk} = a_n b_k$. The sequence space $\mathcal{A}_k$ is defined in (2). In this paper we determine sufficient conditions for a nonnegative factorable matrix A to be a bounded operator on $\mathcal{A}_k$, i.e., $A \in B(\mathcal{A}_k)$. As corollaries we obtain sufficient conditions for the discrete Cesáro, terraced, and p-Cesáro matrices defined by Rhaly, to be in $B(\mathcal{A}_k)$.

1. Introduction. Let T be an infinite matrix, $\{s_n\}$ a sequence. Then the nth term, t_n, of the T transform of $\{s_n\}$ is given by

$$t_n = \sum_{v=0}^{\infty} t_{nv} s_v.$$

A series $\sum x_n$, with partial sums s_n, is said to be k-absolutely summable by T for $k \geq 1$, written $\sum x_n$ is $|T|_k$, if

$$\sum_{n=1}^{\infty} n^{k-1} |\Delta\, t_{n-1}|^k < \infty, \tag{1}$$

where Δ is the forward difference operator defined by $\Delta\, t_{n-1} = t_{n-1} - t_n$.

Let (C, α) denote the Cesáro matrix of order $\alpha > -1$, σ_n^α its nth transform of a sequence $\{s_n\}$. Using (1) with $t_n = \sigma_n^\alpha$, Flett [**3**] proved that, if a series $\sum x_n$ is summable $|C, \alpha|_k$ then it is summable $|C, \beta|_r$ for $\alpha > -1$, $r \geq k \geq 1$, $\beta > \alpha + 1/k - 1/r$. Setting $r = k$ gives an inclusion type theorem for Cesáro matrices.

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Let $\sum x_n$ be a given infinite series with partial sums s_n. Then, for $k \geq 1$,

$$(2) \qquad \mathcal{A}_k := \left\{ \{s_n\} : \sum_{n=1}^{\infty} n^{k-1} \,|\, x_n|^k < \infty, \; x_n = s_n - s_{n-1} \right\}.$$

A matrix T is said to be a bounded linear operator on $\mathcal{A}_k$, written $T \in B(\mathcal{A}_k)$, if $T : \mathcal{A}_k \to \mathcal{A}_k$.

In 1970, Das [**2**], using definitions (1) and (2), proved that every conservative Hausdorff matrix maps $\mathcal{A}_k$ to $\mathcal{A}_k$, i.e., $H \in B(\mathcal{A}_k)$.

In [**9**], it is shown that $(C, \alpha) \in B(\mathcal{A}_k)$ for each $\alpha > -1$, demonstrating that being a conservative matrix is not a necessary condition for a matrix to belong to $B(\mathcal{A}_k)$.

A factorable matrix A is a lower triangular matrix with entries $a_{nk} = a_n b_k$, $0 \leq k \leq n$. In this paper we determine sufficient conditions for a nonnegative factorable matrix $A \in B(\mathcal{A}_k)$. As corollaries we obtain sufficient conditions for the three classes of Rhaly matrices to be in $B(\mathcal{A}_k)$.

We may associate with A two matrices $\overline{A}$ and $\hat{A}$ defined by

$$\bar{a}_{nv} = \sum_{i=v}^{n} a_{ni} \quad \text{and} \quad \hat{a}_{nv} = \bar{a}_{nv} - \bar{a}_{n-1,v},$$

respectively.

Let $\sum x_n$ be a given infinite series with partial sums $\{s_n\}$. We shall denote the nth term of the A-transform of $\{s_n\}$ by t_n. Then

$$\begin{aligned} t_n &= \sum_{i=0}^{n} a_{ni} s_i = \sum_{i=0}^{n} a_{ni} \sum_{v=0}^{i} x_v \\ &= \sum_{v=0}^{n} x_v \sum_{i=v}^{n} a_{ni} = \sum_{v=0}^{n} \bar{a}_{nv} x_v, \end{aligned}$$

and

$$\begin{aligned} y_n := \Delta\, t_{n-1} &= \sum_{v=0}^{n-1} \bar{a}_{n-1,v} x_v - \sum_{v=0}^{n} \bar{a}_{nv} x_v \\ &= -\sum_{v=0}^{n} \hat{a}_{nv} x_v, \end{aligned}$$

since $\bar{a}_{n-1,n} = 0$.

Thus, $A \in B(\mathcal{A}_k)$ means that

$$\sum_{n=1}^{\infty} n^{k-1} |x_n|^k < \infty \Longrightarrow \sum_{n=1}^{\infty} n^{k-1} |y_n|^k < \infty. \tag{3}$$

Define sequences $\{x_n^*\}$ and $\{y_n^*\}$ by $x_n^* = n^{1/k^*} x_n$, $y_n^* = n^{1/k^*} y_n$, $n > 0$, and $x_0^* = x_0$, $y_0^* = y_0$, where k^* denotes the conjugate index of k. Then (3) is equivalent to

$$\sum_{n=1}^{\infty} |x_n^*|^k < \infty \Longrightarrow \sum_{n=1}^{\infty} |y_n^*|^k < \infty.$$

Note that

$$\begin{aligned} y_n^* &= -n^{1/k^*} \left(\hat{a}_{n0} x_0 + \sum_{v=1}^{n} \hat{a}_{nv} x_v \right) \\ &= -n^{1/k^*} \hat{a}_{n0} x_0^* - \sum_{v=1}^{n} \left(\frac{n}{v} \right)^{1/k^*} \hat{a}_{nv} x_v^* \\ &= \sum_{v=0}^{n} b_{nv} x_v^*, \end{aligned}$$

where the matrix $B = (b_{nv})$ has entries

$$b_{nv} = \begin{cases} -n^{1/k^*} \hat{a}_{n0} & v = 0, \\ -(n/v)^{1/k^*} \hat{a}_{nv} & 1 \le v \le n, \\ 0 & v > n. \end{cases} \tag{4}$$

Consequently, the statement $A : \mathcal{A}_k \to \mathcal{A}_k$ is equivalent to $B : l^k \to l^k$.

Lemma 1. *If $A = (a_{nv})$ is a factorable matrix, then the associated matrices $\bar{A} = (\bar{a}_{nv})$ and $\hat{A} = (\hat{a}_{nv})$ can be written as a sum of two factorable matrices.*

Proof. Let A be a factorable matrix.

$$\bar{a}_{nv} = \sum_{r=v}^{n} a_{nr} = \sum_{r=v}^{n} a_n b_r = a_n \sum_{r=v}^{n} b_r.$$

If we write

$$\mathbf{b}_n^{(1)} = b_0 + b_1 + \cdots + b_n,$$

then

$$\sum_{r=v}^{n} b_r = \mathbf{b}_n^{(1)} - \mathbf{b}_{v-1}^{(1)},$$

and we have that

$$\bar{a}_{nv} = \begin{cases} a_n \left(\mathbf{b}_n^{(1)} - \mathbf{b}_{v-1}^{(1)}\right) & 0 \le v \le n, \\ 0 & v > n. \end{cases}$$

Thus, we can write $\overline{A} = B + C$, where

$$b_{nv} = \begin{cases} a_n \mathbf{b}_n^{(1)} & 0 \le v \le n, \\ 0 & v > n, \end{cases}$$
$$c_{nv} = \begin{cases} -a_n \mathbf{b}_{v-1}^{(1)} & 1 \le v \le n, \\ 0 & v > n, \text{ or } v = 0. \end{cases}$$

Now, since

$$\begin{aligned} \hat{a}_{nv} &= \bar{a}_{nv} - \bar{a}_{n-1,v} \\ &= a_n \left(\mathbf{b}_n^{(1)} - \mathbf{b}_{v-1}^{(1)}\right) - a_{n-1} \left(\mathbf{b}_{n-1}^{(1)} - \mathbf{b}_{v-1}^{(1)}\right) \\ &= \left(a_n \mathbf{b}_n^{(1)} - a_{n-1} \mathbf{b}_{n-1}^{(1)}\right) + (a_{n-1} - a_n)\, \mathbf{b}_{v-1}^{(1)} \\ &= -\Delta \left(a_{n-1} \mathbf{b}_{n-1}^{(1)}\right) + \Delta\,(a_{n-1})\, \mathbf{b}_{v-1}^{(1)}, \end{aligned}$$

then

$$\hat{a}_{nv} = \begin{cases} -\Delta\left(a_{n-1}\mathbf{b}_{n-1}^{(1)}\right) + \Delta\left(a_{n-1}\right)\mathbf{b}_{v-1}^{(1)} & 0 \le v \le n, \\ 0 & v > n. \end{cases}$$

Hence, $\widehat{A} = D + E$, where

$$d_{nv} = \begin{cases} -\Delta\left(a_{n-1}\right)\mathbf{b}_{n-1}^{(1)} & 0 \le v \le n, \\ 0 & v > n, \end{cases}$$

and

$$e_{nv} = \begin{cases} \Delta\left(a_{n-1}\right)\mathbf{b}_{v-1}^{(1)} & 1 \le v \le n, \\ 0 & v > n \text{ or } v = 0. \end{cases} \qquad \square$$

Theorem 2. *Let A be a factorable matrix, and let $k > 1$. Then $A \in B(\mathcal{A}_k)$ if*

(i)

$$\left[\sum_{v=0}^{n-1}\left(a_{n-1,v} - a_{nv}\right) - a_{nn}\right] = O\left(\frac{1}{n\log n}\right),$$

(ii)

$$\sum_{r=0}^{v} b_r = O\left((v+1)\,b_v\right),$$

and

(iii)

$$\sum_{v=0}^{n-1}\left(a_{n-1,v} - a_{nv}\right) = O\left(\frac{1}{n+1}\right)$$

are satisfied.

Proof. From (4) we may write $B = D + E$, where $D = (d_{nv})$ is the matrix whose first column entries are defined by

$$d_{n0} = -n^{1/k^*}\hat{a}_{n0}, \quad \left(d_{00} = -\hat{a}_{00} = -a_{00}\right),$$

all other entries are zero, and E is the matrix whose entries are defined by

$$e_{nv} = \begin{cases} -(n/v)^{1/k^*}\hat{a}_{nv} & 1 \le v \le n, \\ 0 & v > n \text{ or } v = 0. \end{cases}$$

From Lemma 1 the matrix E may be written as the sum of matrices $F = (f_{nv})$ and $G = (g_{nv})$, where

$$f_{nv} = \begin{cases} n^{1/k^*}\Delta\left(a_{n-1}\mathbf{b}_{n-1}^{(1)}\right) 1/v^{1/k^*} & 1 \le v \le n, \\ 0 & v > n \text{ or } v = 0, \end{cases}$$
$$g_{nv} = \begin{cases} -n^{1/k^*}\Delta\left(a_{n-1}\right)(\mathbf{b}_{v-1}^{(1)}/v^{1/k^*}) & 1 \le v \le n, \\ 0 & v > n \text{ or } v = 0, \end{cases}$$

so we may write $B = D+F+G$. Omitting the first row and first column of F and G, which contains all zeros, what remains are factorable matrices.

For any $\{s_n\} \in l^k$, let μ_n denote the nth term of the $D = (d_{nv})$ matrix transformation of $\{s_n\}$. Then

$$\mu_n = \sum_{v=0}^{\infty} d_{nv}s_v = -n^{1/k^*}\hat{a}_{n0}s_0,$$

and, using (i) we have

$$\begin{aligned}\left(\sum_{n=1}^{\infty}\left|\sum_{v=0}^{n} d_{nv}s_v\right|^k\right)^{1/k} & \\ &= \left(\sum_{n=1}^{\infty}\left|-n^{1/k^*}\hat{a}_{n0}s_0\right|^k\right)^{1/k} \\ &= O(1)\left(\sum_{n=1}^{\infty} n^{k/k^*}\left|\hat{a}_{n0}\right|^k\right)^{1/k} \\ &= O(1)\left(\sum_{n=1}^{\infty} n^{k-1}\left|a_{nn} + \sum_{v=0}^{n-1}(a_{nv} - a_{n-1,v})\right|^k\right)^{1/k} \\ &= O(1)\left(\sum_{n=1}^{\infty} n^{k-1}\left|\frac{1}{n\log n}\right|^k\right)^{1/k}\end{aligned}$$

$$= O(1)\left(\sum_{n=1}^{\infty}\frac{1}{n\log^k n}\right)^{1/k}$$
$$= O(1).$$

Hence $D : l^k \to l^k$.

Since F is a factorable matrix, from Corollary 1 of [**1**], $F : l^k \to l^k$, $k > 1$, if for $n = 1, 2, \ldots,$

$$x_n \sum_{v=1}^{n} y_v^{k^*} \le K y_n^{1/(k-1)},$$

i.e.,

$$(5) \qquad n^{1/k^*}\left|\Delta\left(a_{n-1}\mathbf{b}_{n-1}^{(1)}\right)\right| \sum_{v=1}^{n}\left(\frac{1}{v^{1/k^*}}\right)^{k^*} \le K\left(\frac{1}{n^{1/k^*}}\right)^{1/(k-1)}$$

is satisfied. Since

$$\begin{aligned}\Delta\left(a_{n-1}\mathbf{b}_{n-1}^{(1)}\right) &= a_{n-1}\mathbf{b}_{n-1}^{(1)} - a_n\mathbf{b}_n^{(1)}\\ &= a_{n-1}\sum_{v=0}^{n-1} b_v - a_n\sum_{v=0}^{n-1} b_v - a_n b_n\\ &= \sum_{v=0}^{n-1}(a_{n-1,v} - a_{nv}) - a_{nn},\end{aligned}$$

and

$$\sum_{v=1}^{n}\frac{1}{v} = 1 + \frac{1}{2} + \cdots + \frac{1}{n} = \log n + c + \alpha_n,$$

then (5) is equivalent to

$$(6) \qquad \left|\sum_{v=0}^{n-1}(a_{n-1,v} - a_{nv}) - a_{nn}\right| n(\log n + c + \alpha_n) \le K,$$

where c is the Euler constant and α_n is a null sequence. Inequality (6) is satisfied if condition (i) is satisfied. So $F : l^k \to l^k$ if condition (i) is satisfied.

Recall that G is a factorable matrix. From [**1**, Corollary 1] $G : l^k \to l^k$, $k > 1$, if, for $n = 1, 2, \dots$,

$$x_n \sum_{v=1}^{n} y_v^{k^*} \le K y_n^{1/(k-1)},$$

or

$$(7) \qquad n^{1/k^*} \,|\, a_n - a_{n-1}| \sum_{v=1}^{n} \left(\frac{\mathbf{b}_{v-1}^{(1)}}{v^{1/k^*}} \right)^{k^*} \le K \left(\frac{\mathbf{b}_{n-1}^{(1)}}{n^{1/k^*}} \right)^{1/(k-1)}$$

is satisfied. From (ii) and (iii),

$$n^{(1/k)+(1/k^*)} |a_n - a_{n-1}| \left(\mathbf{b}_{n-1}^{(1)}\right)^{-1/(k-1)} \sum_{v=1}^{n} \frac{\left(\mathbf{b}_{v-1}^{(1)}\right)^{k^*}}{v}$$
$$= O(1) n \,|a_n - a_{n-1}| \sum_{v=0}^{n-1} b_v = O(1) n \sum_{v=0}^{n-1} (a_{n-1,v} - a_{n,v}) = O(1).$$

So inequality (7) is satisfied if conditions (ii) and (iii) are satisfied.

Combining these facts, $B : l^k \to l^k$, i.e., $A : \mathcal{A}_k \to \mathcal{A}_k$. □

A discrete generalized Cesáro matrix, see [**4**], is a factorable matrix A with nonzero entries $a_{nk} = \lambda^{n-k}/(n+1)$, where $\lambda \in [0, 1]$. The choice $\lambda = 1$, of course, gives the Cesáro matrix of order one.

Corollary 3. *C_λ, the Rhaly discrete Cesáro matrix, is a mapping from $\mathcal{A}_k$ to $\mathcal{A}_k$ for $0 < \lambda < 1$, $k > 1$.*

Proof. C_λ is a factorable matrix. In Theorem 2 we take $A = C_\lambda$. Since

$$\begin{aligned}\sum_{v=0}^{n-1} (a_{n-1,v} - a_{nv}) - a_{nn} &= \sum_{v=0}^{n-1} a_{n-1,v} - \sum_{v=0}^{n} a_{nv} \\ &= \sum_{v=0}^{n-1} \frac{\lambda^{n-1-v}}{n} - \sum_{v=0}^{n} \frac{\lambda^{n-v}}{n+1} \\ &= \frac{1}{n} \left(\frac{1-\lambda^n}{1-\lambda} \right) - \frac{1}{n+1} \frac{\left(1-\lambda^{n+1}\right)}{(1-\lambda)},\end{aligned}$$

we have

$$\begin{aligned} n\log n\left|\sum_{v=0}^{n-1}(a_{n-1,v}-a_{nv})-a_{nn}\right| \\ &= n\log n\left|\frac{1}{n}\left(\frac{1-\lambda^n}{1-\lambda}\right)-\frac{1}{n+1}\frac{\left(1-\lambda^{n+1}\right)}{(1-\lambda)}\right| \\ &= O\left(1\right)\frac{\log n}{(1-\lambda)}\left|\lambda^n-\lambda^{n+1}\right| \\ &= O\left(1\right)\lambda^n\log n \\ &= O\left(1\right). \end{aligned}$$

Therefore, condition (i) of Theorem 2 is satisfied. Since

$$a_{nv}=\frac{\lambda^{n-v}}{(n+1)}=\frac{\lambda^n}{n+1}\frac{1}{\lambda^v},$$

if we take $b_v = 1/\lambda^v$, then condition (ii) of Theorem 2 is automatically satisfied.

$$\begin{aligned} (n+1)\left[\sum_{v=0}^{n-1}(a_{n-1,v}-a_{nv})\right] &= (n+1)\sum_{v=0}^{n-1}\left(\frac{\lambda^{n-v-1}}{n}-\frac{\lambda^{n-v}}{n+1}\right) \\ &= \frac{(n+1)}{n}\sum_{v=0}^{n-1}\lambda^{n-v-1}-\sum_{v=0}^{n-1}\lambda^{n-v} \\ &= O\left(1-\lambda^n\right)=O\left(1\right), \end{aligned}$$

and condition (iii) of Theorem 2 is satisfied. □

A terraced matrix R_a, see [**6**], is a factorable matrix A with nonzero entries $a_{nk} = a_n$, where $\{a_n\}$ is a sequence of real (or complex) numbers. The case $a_n = 1/(n+1)$ yields the Cesáro matrix of order one, and more generally, if we take $a_n = (n+1)^{-p}$ for some $p \geq 1$, we get the p-Cesáro matrix defined in [**5**].

Corollary 4. *A Rhaly matrix R_a maps $\mathcal{A}_k$ into $\mathcal{A}_k$, $k > 1$, if*

(i) $na_{n-1} - (n+1)a_n = O(1/(n\log n))$,

and

(ii) $a_{n-1} - a_n = O(1)/n(n+1)$.

Proof. In Theorem 2 set $A = R_a$. Since

$$\left[\sum_{v=0}^{n-1} (a_{n-1,v} - a_{nv}) - a_{nn}\right] = na_{n-1} - (n+1)\,a_n,$$

condition (i) of Theorem 2 reduces to condition (i) of Corollary 4. Condition (ii) of Theorem 2 is automatically satisfied. Since

$$\left[\sum_{v=0}^{n-1} (a_{n-1,v} - a_{nv})\right] = n\,(a_{n-1} - a_n)\,,$$

condition (iii) of Theorem 2 reduces to condition (ii) of Corollary 4. □

Lemma 5 [**7**]. *For* $p > 1$,

$$n^{p-1}\left(\frac{1}{n^p} - \frac{1}{(n+1)^p}\right) \le p\left(\frac{1}{n} - \frac{1}{n+1}\right).$$

Corollary 6. *C_p, the Rhaly p-Cesáro matrix, maps $\mathcal{A}_k$ into $\mathcal{A}_k$ for $p > 1$, $k > 1$.*

Proof. In Corollary 4 we take $a_n = (n+1)^{-p}$. Using Lemma 5, we get

$$\begin{aligned} n\log n\,[na_{n-1} - (n+1)\,a_n] &= n\log n\left[\frac{n}{n^p} - \frac{n+1}{(n+1)^p}\right] \\ &\le n\,(n+1)\log n\left(\frac{1}{n^p} - \frac{1}{(n+1)^p}\right) \\ &\le \frac{pn\,(n+1)}{n^{p-1}}\left(\frac{1}{n} - \frac{1}{n+1}\right)\log n \\ &\le \frac{\log n}{n^{p-1}} \\ &= O\,(1) \end{aligned}$$

and

$$\begin{aligned} n\,(n+1)\,(a_{n-1} - a_n) &= n\,(n+1)\left[\frac{1}{n^p} - \frac{1}{(n+1)^p}\right] \\ &\le \frac{n\,(n+1)\,p}{n^{p-1}}\left(\frac{1}{n} - \frac{1}{n+1}\right) \\ &= O\left(1/n^{p-1}\right) = O\,(1)\,. \end{aligned}$$

Hence, conditions (i) and (ii) of Corollary 4 are satisfied. □

Using a different technique, Corollaries 3 and 6 of this paper appear as Corollaries 16 and 17 in [**7**].

Corollary 7. *Let* (p_n) *be a positive sequence, and let* $k > 1$*. Then* $(\overline{N}, p_n) \in B(\mathcal{A}_k)$ *if*

$$np_n \asymp (P_n)\,. \tag{8}$$

Proof. In Theorem 2, set $A = (\overline{N}, p_n)$. Since $(\overline{N}, p_n)$ is a factorable matrix,

$$\begin{aligned}\sum_{v=0}^{n-1}(a_{n-1,v} - a_{nv}) - a_{nn} &= \sum_{v=0}^{n-1}\left(\frac{p_v}{P_{n-1}} - \frac{p_v}{P_n}\right) - \frac{p_n}{P_n}\\ &= \left(\frac{1}{P_{n-1}} - \frac{1}{P_n}\right)\sum_{v=0}^{n-1} p_v - \frac{p_n}{P_n}\\ &= \frac{p_n}{P_nP_{n-1}}P_{n-1} - \frac{p_n}{P_n}\\ &= \frac{p_n}{P_n} - \frac{p_n}{P_n} = 0,\end{aligned}$$

and condition (i) of Theorem 2 is automatically satisfied. Condition (ii) of Theorem 2 reduces to $P_v = O(vp_v)$. Since

$$\begin{aligned}\sum_{v=0}^{n-1}(a_{n-1,v} - a_{nv}) &= \sum_{v=0}^{n-1}\left(\frac{p_v}{P_{n-1}} - \frac{p_v}{P_n}\right)\\ &= \left(\frac{1}{P_{n-1}} - \frac{1}{P_n}\right)\sum_{v=0}^{n-1} p_v = \frac{p_n}{P_n},\end{aligned}$$

condition (iii) of Theorem 2 reduces to $np_n = O(P_n)$. □

We know that $(C, 1) \in B(\mathcal{A}_k)$ for $k > 1$. From [**8**, Theorem 5.1], in order for $|C, 1|_k$ to be equivalent to $|\overline{N}, p_n|_k$, it is necessary and sufficient that (8) be satisfied, so condition (8) is also necessary.

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