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GENERATING SINGULARITIES OF WEAK SOLUTIONS OF p-LAPLACE EQUATIONS ON FRACTAL SETS

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ABSTRACT. We study p-Laplace equations $-\Delta_p u = F(x)$ possessing weak solutions in the Sobolev space $W_0^{1,p}(\Omega)$, $\Omega \subset \mathbf{R}^N$, that are singular on prescribed fractal sets having large Hausdorff dimension. With an appropriate choice of $F \in L^{p'}(\Omega)$, the Hausdorff dimension of a singular set of the weak solution can be made arbitrarily close to $N - pp'$ if $pp' < N$. For $p = 2$, that is, for the classical Laplace equation, the bound $N-4$ is optimal, provided $N \geq 4$. Moreover, there exist maximally singular solutions, that is, such that the bound is achieved. The proof is obtained via an explicit lower a priori bound of supersolutions corresponding to special choice of righthand sides that are singular near a fractal set.

1. Introduction. Let Ω be an open set in $\mathbf{R}^N$ and $1 < p < \infty$. Throughout this paper we assume that $p < N$, so that functions from the Sobolev space $W^{1,p}(\Omega)$ may have discontinuities. It is well known that, for any function $F \in L^{p'}(\Omega)$, where $p' = p/(p-1)$ is the conjugate exponent, there exists a unique weak solution u of the boundary value problem involving the p-Laplace equation:

$$(1) \qquad -\Delta_p u = F(x) \quad \text{in } \mathcal{D}'(\Omega), \quad u \in W_0^{1,p}(\Omega).$$

We are interested in how large the Hausdorff dimension of the singular set of solutions of this equation can be, generated by righthand sides from $L^{p'}(\Omega)$. Let us recall the definition of the singular set $\operatorname{Sing} u$.

We say that $a \in \Omega$ is a singular point of a measurable function $u\colon \Omega \to \mathbf{R}$ if there exist positive constants γ, ε, C such that

$$u(x) \geq C \cdot |x-a|^{-\gamma} \quad \text{for almost every } x \in B_\varepsilon(a),$$

where $B_\varepsilon(a)$ is the open ball of radius ε around a.

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The set of all singular points of a measurable function $u\colon \Omega \to \mathbf{R}$ is denoted by $\operatorname{Sing} u$, and we call it the singular set of u. If there exists a set A in Ω and $C > 0$ such that $u(x) \geq C \cdot d(x,A)^{-\gamma}$ almost everywhere in a neighborhood of A, where $d(x,A)$ is the Euclidean distance from the point x to A, we say that u has an order of singularity γ on A. We shall also need the notion of an extended singular set of u, denoted by e-$\operatorname{Sing} u$, containing all $a \in \Omega$, such that

$$\limsup_{r \to 0} \frac{1}{r^N} \int_{B_r(a)} u(y)\, dy = +\infty.$$

It is easy to see that $\operatorname{Sing} u \subseteq$ e-$\operatorname{Sing} u$. The extended singular set also contains weaker types of singularities, like logarithmic singularities a, that is, $u(x) \geq C \log 1/|x-a|$ in a neighborhood of a, and iterated logarithmic singularities.

Let X be a given space (or just a nonempty set) of measurable functions from Ω to $\mathbf{R}$. We define the *lower and upper singular dimension* of X by

$$\begin{aligned} \text{s-}\underline{\dim}\, X &:= \sup\{\dim_H(\operatorname{Sing} u) \colon u \in X\}, \\ \text{s-}\overline{\dim}\, X &:= \sup\{\dim_H(\text{e-}\operatorname{Sing} u) \colon u \in X\}. \end{aligned}$$

Clearly, s-$\underline{\dim}\, X \leq$ s-$\overline{\dim}\, X$. If both values coincide, the common value is denoted by s-dim X which we call the *singular dimension* of X. These dimensions have been introduced and studied in [**6**]. We say that a function $u \in X$ is *maximally singular in* X if $\dim_H(\operatorname{Sing} u) =$ s-dim X. Such functions have been studied in [**3, 7, 8**].

We are interested in finding fractal sets A in Ω possessing Hausdorff's dimension as large as possible, such that there exists a righthand side $F(x) \in L^{p'}(\Omega)$ for which the corresponding weak solution u of p-Laplace equation (1) is singular on A.

Let $X(N,p)$ be the set of weak solutions of (1) generated by all $F \in L^{p'}(\Omega)$. First we would like to estimate s-$\underline{\dim}\, X(N,p)$ from below. We show that s-$\underline{\dim}\, X(N,p) \geq N - pp'$. It is known that s-$\overline{\dim}\, X(N,p) \leq N - p$ since $X(N,p) \subset W^{1,p}(\Omega)$ and s-dim $W^{1,p}(\Omega) = N - p$, see [**6**]. Namely, it can be shown that, for the Sobolev space $W^{k,p}(\Omega)$, $kp < N$, we have

$$\text{s-dim}\, W^{k,p}(\Omega) = N - kp, \tag{2}$$

and moreover, there exist maximally singular Sobolev functions u, that is, such that $\dim_H(\operatorname{Sing} u) = N - kp$, see [**3**].

When $p = 2$, that is, in the case of the usual Laplace equation, we have the optimal result s-dim $X(N, 2) = N - 4$ when $N \geq 4$, see Theorem 2. On the other hand, s-dim $X(N, 2) = 0$ for $N \leq 3$ since all solutions are continuous in this case.

The question of generating singularities when A is a single point set has been solved in [**5**, Theorem 3] using Tolksdorf's comparison principle [**4**]. There it was shown that if $F(x)$ has singularity of order $\gamma > p$ in a point a in Ω, then the weak solution u of (1) has singularity at a of order $(\gamma - p)/(p - 1)$. More precisely, if $F(x) \geq C|x - a|^{-\gamma}$ for almost every $x \in B_\varepsilon(a)$ and $\gamma \in (p, 1 + N/p')$, then we have the following lower estimate for any positive weak solution of (1):

(3)
$$u(x) \geq \left(\frac{C}{N-\gamma}\right)^{p'-1} \frac{p-1}{\gamma - p} \left(|x-a|^{-(\gamma-p)/(p-1)} - \varepsilon^{-(\gamma-p)/(p-1)}\right),$$

for almost every $x \in B_\varepsilon(a)$. A similar estimate can be obtained in the case of $\gamma = p$, see [**5**, Theorem 3]:

$$u(x) \geq \left(\frac{C}{N-p}\right)^{p'-1} \log \frac{\varepsilon}{|x-a|}, \quad \text{for almost every } x \in B_\varepsilon(a). \tag{4}$$

It can formally be obtained from (3) by passing to the limit as $\gamma \to p$.

For a subset $A \subset \mathbf{R}^N$ and $\varepsilon > 0$, by A_ε we denote the ε-neighborhood of A, that is, the set of all points having Euclidean distance from A less than ε. This set is often called the Minkowski saussage of radius ε around A.

2. Main results.

Theorem 1. *Let $1 < p < \infty$, $\Omega \subseteq \mathbf{R}^N$, be an open subset, A a bounded set in $\mathbf{R}^N$ be such that $\overline{A_\varepsilon} \subset \Omega$, and assume that γ satisfies*

$$p \leq \gamma < \min\left\{1 + \frac{N}{p'}, N - \overline{\dim}_B A\right\}. \tag{5}$$

Assume that $F \in L^1_{\mathrm{loc}}(\Omega)$ *and* F *has singularity at least of order* γ *on* A, *that is,*

$$F(x) \geq C \cdot d(x,A)^{-\gamma} \quad \text{for almost every } x \in A_\varepsilon, \tag{6}$$

where ε *and* C *are positive constants, and* $\overline{A_\varepsilon} \subset \Omega$. *Then for any supersolution* $u \in W^{1,p}(\Omega)$ *of*

$$-\Delta_p u = F(x) \quad \text{in } \mathcal{D}'(\Omega) \tag{7}$$

such that $u \geq 0$ *on* Ω, *we have for almost every* $x \in A_\varepsilon$,

$$u(x) \geq \begin{cases} (C/(N-\gamma))^{p'-1}(p-1)/(\gamma-p)\big(d(x,A)^{-(\gamma-p)/(p-1)} - \varepsilon^{-(\gamma-p)/(p-1)}\big) \\ \qquad\qquad \text{for } \gamma > p, \\ (C/(N-p))^{p'-1}\log(\varepsilon/d(x,A)) \\ \qquad\qquad \text{for } \gamma = p. \end{cases} \tag{8}$$

In particular, if $\gamma > p$, *then* u *has singularity at least of order* $(\gamma-p)/(p-1)$ *on* A. *The lower bound of* $u(x)$ *on* A_ε *is sharp.*

Proof. Assume that $\gamma > p$. Let us fix $a \in A$. Using the Harvey-Polking lemma, see [**7**, Lemma 1], from $\gamma < N - \overline{\dim}_B A$ we conclude that the function $F(x) = Cd(x,A)^{-\gamma}$ is in $L^1(A_\varepsilon)$. Hence, since $F(x) \geq C|x-a|^{-\gamma}$ for any $a \in A$, we can apply [**5**, Theorem 3] to obtain (3). From this the desired estimate (8) follows by taking the infimum in (3) over all $a \in A$. The remaining case of $\gamma = p$ is treated similarly. □

Theorem 2. *Let* $1 < p < \infty$ *and* $pp' < N$. Ω *is a bounded subset of* $\mathbf{R}^N$. *Denote by* $X(N,p)$ *the set of all functions* $u \in W_0^{1,p}(\Omega)$ *such that there exists* $F \in L^{p'}(\Omega)$ *satisfying the distribution equation* (1).

(a) *Then*

$$\text{s-}\underline{\dim}\, X(N,p) \geq N - pp'. \tag{9}$$

(b) *For the ordinary Laplace operator, that is, when $p = 2$, we have the precise result*:

$$\text{s-dim } X(N,2) = \begin{cases} N-4 & \text{for } N \geq 5 \\ 0 & \text{for } N \leq 4. \end{cases} \tag{10}$$

Moreover, there exist explicit functions $F \in L^2(\Omega)$ such that the corresponding weak solution $u \in H_0^1(\Omega)$ of problem $-\Delta u = F(x)$ in $\mathcal{D}'(\Omega)$ is maximally singular, that is, $\dim_H(\operatorname{Sing} u) = N - 4$.

Proof. (a) Let A be a compact set in Ω. Let us define $F(x) := d(x,A)^{-\gamma}$ on A_ε and $F(x) := 0$ on $\Omega \setminus A_\varepsilon$, with A and γ to be specified below. We have that $F \in L^{p'}(\Omega)$ provided $p'\mathbf{C} < N - d$, where $d := \overline{\dim}_B A$, see [**7**, Lemma 1]. The condition $p < \gamma < 1 + (N/p')$ in Theorem 1 is meaningful since $p < 1 + (N/p')$ is equivalent with $p < N$, and this follows from $pp' < N$. Hence, in order to be able to apply Theorem 1, we need to see that the inequality $p < \gamma < (N-d)/p'$ is possible for some γ. Such a γ exists provided $p < (N-d)/p'$, that is, when $d < N - pp'$. Let us fix any number $\delta < N - pp'$, which can be arbitrarily close to $N - pp'$.

We may assume without loss of generality that Ω contains the unit cube $[0,1]^N$, since otherwise the set A introduced below can be scaled and translated into Ω. We construct a compact set of the form of the Cantor grill $A := C^{(\alpha)} \times [0,1]^k$, with k defined as follows. If $N - pp'$ is a noninteger we take $k := \lfloor N - pp' \rfloor$, that is, the largest integer part of $N - pp'$ (if $k = 0$ we let $A := C^{(\alpha)}$). If $N - pp'$ is a positive integer we take $k = N - pp' - 1$. Here $C^{(\alpha)}$ is the generalized, uniform Cantor set with parameter $\alpha \in (0, (1/2))$, see Falconer [**1**]. Since

$$\dim_B A = \dim_H A = \frac{2}{\log(1/\alpha)} + k,$$

see [**1**, Corollary 7.4 and product formula 7.5], we can choose α so that

$$d = \dim_B A \in (\delta, N - pp').$$

The function $F(x)$ generated by A then satisfies conditions of Theorem 1, case $\gamma > p$. Hence, for the weak solution u of the corresponding p-Laplace equation we have that $A \subseteq \operatorname{Sing} u$. Since $\delta \leq \dim_H A \leq$

$\dim_H(\operatorname{Sing} u) \le \text{s-dim } X(N,p)$, we can let $\delta \to N - pp'$ to conclude that $\text{s-dim } X(N,p) \ge N - pp'$.

(b) Assume that $N > 4$. Let A_k, $k \ge 1$, be a sequence of subsets of Ω such that

$$\overline{\dim}_B A_k < N - 4, \quad \lim_{k\to\infty} (\dim_H A_k) = N - 4, \tag{11}$$

and there exists an $\varepsilon_k > 0$ such that $\overline{(A_k)_{\varepsilon_k}} \subset \Omega$. As we saw in step (a), such sets can be constructed using generalized Cantor sets. Let us choose numbers γ_k such that

$$2 < \gamma_k < \frac{N - \overline{\dim}_B A_k}{2}. \tag{12}$$

Now define the sequence of functions

$$F_k(x) := \begin{cases} d(x, A_k)^{-\gamma_k} & \text{for } x \in (A_k)_{\varepsilon_k} \\ 0 & \text{for } x \in \Omega \setminus (A_k)_{\varepsilon_k}. \end{cases} \tag{13}$$

As in step (a) we see that all of them are in $L^2(\Omega)$. For any k the corresponding weak solution $u_k \in H_0^1(\Omega)$ of $-\Delta u_k = F_k(x)$ is positive and such that $A_k \subseteq \operatorname{Sing} u_k$, see (a). The function

$$F(x) := \sum_{k=1}^{\infty} c_k \frac{F_k(x)}{\|F_k\|_{L^2}} \tag{14}$$

is in $L^2(\Omega)$ provided c_k are positive and $\sum_k c_k < \infty$. For the corresponding weak solution $u \in H_0^1(\Omega)$ of $-\Delta u = F(x)$, we have

$$u(x) = \sum_{k=1}^{\infty} c_k \frac{u_k(x)}{\|F_k\|_{L^2}}.$$

From Theorem 1 we conclude that $\cup_k A_k \subseteq \operatorname{Sing} u$. Using countable stability of the Hausdorff dimension (see Falconer [**1**, page 29]) we obtain

$$\dim_H(\operatorname{Sing} u) \ge \dim_H \left(\bigcup_k A_k \right) = \sup_k (\dim_H A_k) = \lim_k d_k = N - 4.$$

On the other hand, from regularity theory of elliptic equations we know that $u \in H^2(\Omega) := W^{2,2}(\Omega)$ (see, e.g., Gilbarg and Trudinger [**2**]), therefore, using (2) we obtain the converse inequality:

$$\dim_H(\operatorname{Sing} u) \leq \text{s-dim } H^2(\Omega) = N - 4.$$

This proves that $\dim_H(\operatorname{Sing} u) = N - 4$, that is, u is maximally singular.

For $N = 4$ all solutions are in $H^2(\Omega)$, and since s-dim $H^2(\Omega) = 0$, we have

$$\text{s-dim } X(4,2) \leq \text{s-dim } H^2(\Omega) = 0,$$

that is, s-dim $X(4,2) = 0$.

For $N \leq 3$ the Sobolev space $H^2(\Omega)$ is imbedded into a space of continuous functions (see, e.g., [**2**]), so that the extended singular set of u is empty in these cases. In particular, s-dim $X(N,2) = 0$ for $N \leq 3$. □

Remark 1. We do not know if the bound $N - pp'$ in Theorem 2 (a) is optimal for $p \neq 2$. For $p = 2$ and $N = 4$ we may have e-Sing $u \neq \varnothing$ for some $u \in X(4,2)$, although $\dim_H(\text{e-Sing } u) = 0$ in this case.

Remark 2. In the proof of Theorem 2 (b) we have constructed a class of maximally singular weak solutions of (1) possessing singular sets of the form $\cup_k A_k$, satisfying conditions (11) and (12). It would be interesting to know if every maximally singular weak solution of (1) has a singular set representable in this form.

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