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ON THE HEIGHTS OF HAPPY NUMBERS

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ABSTRACT. A happy number is a positive integer a such that $S_2^m(a) = 1$ for some $m \geq 0$; here $S_2(a)$ is the sum of the squares of its decimal digits and $S_2^m(a) = S_2(S_2^{m-1}(a))$. The height of a happy number is the least $m \geq 0$ such that $S_2^m(a) = 1$. In this paper, we give a general method to find theoretically the least happy number of any given height. For instance, we determine the least happy numbers of heights 11 and 12.

1. Introduction. The function S_2 is defined so that for any $a \in Z^+$, $S_2(a)$ is the sum of the squares of its decimal digits. For $a \in Z^+$, let $S_2^0(a) = a$, and for $m \geq 1$, let $S_2^m(a) = S_2(S_2^{m-1}(a))$. A happy number is a positive integer a such that $S_2^m(a) = 1$ for some $m \geq 0$. It is well known [**5**] that 4 is not a happy number and that, in fact, for all $a \in Z^+$, a is not a happy number if and only if $S_2^m(a) = 4$ for some $m \geq 0$. The height of a happy number is the least $m \geq 0$ such that $S_2^m(a) = 1$. Hence, 1 is a happy number of height 0, 10 is a happy number of height 1, 13 is a happy number of height 2, 23 is a happy number of height 3, 19 is a happy number of height 4, and 7 is a happy number of height 5.

In [**4**] (problem E34), Guy asks several questions about happy numbers such as: do there exist sequences of consecutive happy numbers of arbitrary length? What is the bound of the least happy number with height h? In 2000, El-sedy and Siksek [**1**] gave an affirmative answer to the former question. For the latter question, by computing the heights of each happy number less than 400, it is easy to find the least happy numbers of heights up to 6. (These, as well as the least happy number of height 7, can be found in [**4, 6**].) In 1994, Guy [**4**] reports that McCrainie verified the value of the least happy number of height 7 and determined the value of the least happy number of height 8.

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Recently, Grundman and Teeple [**2**] introduced a method to find the least happy numbers of given heights. Using this method they found the least happy numbers of heights 7 to 10. In their paper, they noted some obstacles that need to be dealt with in order to get additional least happy number of given heights. In this paper, we first solve the two problems left in their paper, our main tool being Largrange's four squares theorem. Then we can find the least happy number of any given height theoretically; for instance, we determine the least happy numbers of heights 11 and 12.

Denote by σ_k the least happy number of height k. Then $\sigma_0 = 1$, $\sigma_1 = 10$, $\sigma_2 = 13$, $\sigma_3 = 23$, $\sigma_4 = 19$, $\sigma_5 = 7$, etc. The main results we obtain in the paper are:

Theorem 1. *For any $k \geq 6$, the total number of digits in σ_k is no more than $[\sigma_{k-1}/81] + 4$ and no less than $[\sigma_{k-1}/81]$. The number of digits equal to 9 in σ_k is no less than $[\sigma_{k-1}/81] - 15$.*

Theorem 2. *$S_2(\sigma_{k+1}) = \sigma_k$ for any $k \geq 6$.*

Theorem 3. $\sigma_{11} = 179 \times 10^{\sigma_{10}-114/81} - 1$, $\sigma_{12} = 47 \times 10^{\sigma_{11}-52/81} - 1$.

2. Proofs of Theorem 1 and Theorem 2. First we let $\sigma_k = 81a_k + b_k$, where $0 \leq b_k \leq 80$.

Proof of Theorem 1. Largrange's theorem states that every positive integer is the sum of four squares. So for any $0 \leq b_{k-1} \leq 80$, the equation $b_{k-1} = c_1^2 + c_2^2 + c_3^2 + c_4^2$ has at least one solution. Therefore, $m = c_1c_2c_3c_4\overbrace{99\cdots 9}^{a_{k-1}}$ is a happy number of height k since $S_2(m) = \sigma_{k-1}$. Since $\sigma_k \leq m$, the number of the digits of σ_k is no more than $a_{k-1}+4$. On the other hand, it's obvious that this number of digits is no less than a_{k-1}.

Now let l_k denote the number of 9s in the decimal expansion of σ_k. From the fact $S_2(\sigma_k) \geq \sigma_{k-1}$, we obtain $81a_{k-1} \leq \sigma_{k-1} \leq S_2(\sigma_k) \leq 8^2 \times (a_{k-1} + 4 - l_k) + 9^2 \times l_k$, so $l_k \geq a_{k-1} - 15$. This completes the proof of Theorem 1. $\square$

Corollary. *For any σ_k, the number of its digits which are not equal to* 9 *is no more than* 19.

Proof of Theorem 2. For $6 \le k \le 9$, Theorem 2 is true by $\sigma_6 = 356$, $\sigma_7 = 78999$, $\sigma_8 = 3789 \times 10^{973} - 1$, $\sigma_9 = 78889 \times 10^{(\sigma_8 - 305)/81} - 1$, $\sigma_{10} = 259 \times 10^{(\sigma_9 - 93)/81} - 1$ [**2**]. We assume that it is also true for $8 \le k \le m-1$. Our goal is to prove that $S_2(\sigma_m) = \sigma_{m-1}$.

First of all, it is easy to see that any happy number less than 100 has a height no more than 5. Note that if $a \ge 100$ then $S_2(a) < a$ [**3**]. By the definition of σ_m, we have $\sigma_m > S_2(\sigma_m) \ge \sigma_{m-1}$, $m \ge 6$. Hence, for $m \ge 8$, the number of digits of σ_m equal to 9 is no less than $[\sigma_{m-1}/81] - 15 \ge [\sigma_7/81] - 15 = 960$. We write $\sigma_{m-1} = AB$, where $B = \overbrace{99\cdots 9}^{L}$, $L \ge 960$, and each digit of A is less than 9, i.e., $\sigma_{m-1} = 10^L A + B$.

Now assume that $S_2(\sigma_m) = \sigma_{m-1} + d$, $d > 0$. Since $S_2(\sigma_m) \le 81 \times (a_{m-1} + 4)$, then $81a_{m-1} + b_{m-1} + d = \sigma_{m-1} + d \le 81(a_{m-1} + 4)$, so $d \le d + b_{m-1} \le 324$. Note that the last digit of A is not 9, so $S_2(\sigma_m) = \sigma_{m-1} + d = AB + d = (A+1)0\ldots0(d-1)$. Hence $S_2^2(\sigma_m) = S_2(\sigma_{m-1} + d) = S_2(A+1) + S_2(d-1) < S_2(A) + 9^2 + 3 \times 9^2 < S_2(A) + L \times 9^2 = S_2(\sigma_{m-1}) = \sigma_{m-2}$. But as $S_2^2(\sigma_m)$ is a happy number of height $m-2$, it must satisfy $S_2^2(\sigma_m) \ge \sigma_{m-2}$; a contradiction. Therefore, we have $d = 0$. By induction, this means that $S_2(\sigma_m) = \sigma_{m-1}$. This completes the proof of Theorem 2.

3. Calculating the least happy number of any height. Theorem 1 and Theorem 2 provide a way to find the least happy number of any height, one by one. Let $\sigma_{k-1} = 81a_{k-1} + b_{k-1}$, $0 \le b_k \le 80$. We define the minimum solution $U(k) = u_1 u_2 \cdots u_{19}$ if $u_1, u_2, \ldots, u_{19}$ satisfies the equation

$$(1) \qquad y_1^2 + y_2^2 + \cdots + y_{19}^2 = b_{k-1} + 15 \times 9^2, \quad 0 \le y_i \le 9$$

and for any solution $X = (x_1, x_2, \ldots, x_{19})$ of (1), $U(k) \le x_1 x_2 \cdots x_{19}$. With the notation above, $\sigma_k = U(k)\overbrace{99\cdots 9}^{a-15}$ (here some digits in $U(k)$ may be 9).

From the above discussion, the only thing we need to do is to find the minimum solution for any $0 \le b_{k-1} \le 80$. After the computer search, we find the number of digits which are not equal to 9 in the minimum solution of no more than 7, that is to say the number of digits which is equal to 9 in σ_k is no less than $[\sigma_{k-1}/81] + 1 - 7 = [\sigma_{k-1}/81] - 6$, (if $b \neq 0$, the number of digits of σ_k is no less than $[\sigma_{k-1}/81] + 1$, otherwise it is obvious). So we list the numbers $0 \le b_{k-1} \le 80$ and corresponding σ_k. For brevity, we denote σ_k as (X_k, Y_k); X_k is the number which is not equal to 9 in the digits of σ_k, and Y_k means the number of 9 in σ_k subtract $[\sigma_{k-1}/81]$. For example, $\sigma_7 = 975 \times 81 + 24$, then $b_7 = 24$. From the following table we get $X_8 = 3788, Y_8 = -2$; then $\sigma_8 = 3788\overbrace{99\cdots9}^{[\sigma_{k-1}/81]-2} = 3788\overbrace{99\cdots9}^{973}$.

b_{k-1}	X_k	Y_k	b_{k-1}	X_k	Y_k	b_{k-1}	X_k	Y_k
0	0	0	27	5688	-2	54	127	0
1	1	0	28	368	-1	55	5888	-2
2	688	-2	29	25	0	56	388	-1
3	1688	-2	30	888	-2	57	578	-1
4	2	0	31	1888	-2	58	37	0
5	12	0	32	78	-1	59	137	0
6	2688	-2	33	178	-1	60	888888	-4
7	37888	-3	34	2888	-2	61	56	0
8	58	-1	35	468	-1	62	78888	-3
9	3	0	36	6	0	63	488	-1
10	13	0	37	16	0	64	8	0
11	3688	-2	38	58888	-3	65	18	0
12	258	-1	39	3888	-2	66	6888	-2
13	8888	-3	40	26	0	67	16888	-2
14	18888	-3	41	378	-1	68	28	0

b_{k-1}	X_k	Y_k	b_{k-1}	X_k	Y_k	b_{k-1}	X_k	Y_k
15	788	-2	42	577	-1	69	128	0
16	4	0	43	8888888	-5	70	26888	-2
17	77	-1	44	568	-1	71	378888	-3
18	177	-1	45	36	0	72	588	-1
19	68	-1	46	4888	-2	73	38	0
20	168	-1	47	88	-1	74	57	0
21	277	-1	48	188	-1	75	157	0
22	38888	-3	49	7	0	76	2588	-1
23	268	-1	50	17	0	77	88888	-3
24	3788	-2	51	288	-1	78	188888	-3
25	5	0	52	46	0	79	7888	-2
26	15	0	53	27	0	80	48	0

4. Proof of Theorem 3. To calculate σ_k, since $\varphi(2\times 3^t) = 2\times 3^{t-1}$, the only thing we need to know is a remainder of σ_{k-i}, $i \le k$, divided by 2×3^{3i}. Next, we calculate σ_{11} and σ_{12}. We begin with σ_8, from Lemma 1, with the help of a computer, $\sigma_8 = 3789 \times 10^{973} - 1 \equiv 589013 \pmod{2\times 3^{12}}$, $\sigma_8 \equiv 62 \pmod{3^4}$; thus, $\sigma_9 = 78889 \times 10^{\sigma_8 - 62/81 - 3} - 1$, and $(\sigma_8 - 62/81) - 3 \equiv 7268 \pmod{2 \times 3^8}$. By using Fermat's little theorem, we have $\sigma_9 \equiv 78889 \times 10^{7268} - 1 \equiv 13863 \pmod{2 \times 3^9}$; then $\sigma_9 \equiv 12 \pmod{81}$. Therefore, $\sigma_{10} = 259 \times 10^{\sigma_9 - 12/81 - 1} - 1$. From $(\sigma_9 - 12/81) - 1 \equiv 170 \pmod{2 \times 3^5}$, $\sigma_{10} \equiv 259 \times 10^{170} - 1 \equiv 519 \pmod{2 \times 3^6}$, thus $\sigma_{10} \equiv 33 \pmod{81}$. Finally, we have $\sigma_{11} = 179 \times 10^{(\sigma_{10} - 33/81) - 1} - 1$, and $\sigma_{11} \equiv 179 \times 10^5 - 1 \equiv 52 \pmod{81}$, $\sigma_{12} = 47 \times 10^{\sigma_{11} - 52/81} - 1$. This completes the proof of Theorem 3.

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