A GAUGE INVARIANT UNIQUENESS THEOREM FOR CORNERS OF HIGHER RANK GRAPH ALGEBRAS

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ABSTRACT. For a finitely aligned k-graph Λ with X a set of vertices in Λ, we define a universal C^*-algebra called $C^*(\Lambda, X)$ generated by partial isometries. We show that $C^*(\Lambda, X)$ is isomorphic to the corner $P_X C^*(\Lambda) P_X$, where P_X is the sum of vertex projections in X. We then prove a version of the Gauge Invariant Uniqueness theorem for $C^*(\Lambda, X)$ and then use the theorem to prove various results involving fullness, simplicity and Morita equivalence as well as results relating to application in symbolic dynamics.

1. Introduction. Much study has been done lately in regards to higher rank graphs (also known as k-graphs) and their associated graph algebras since their first appearance in [**12**]. As k-graphs are a higher-dimensional generalization of directed graphs (which can be regarded as one-dimensional), it is important to be able to adapt the known results for directed graphs to the field of k-graphs. So far this has been done with a reasonable amount of success, for example, see [**1, 13, 19, 23**] to name a few; however, the complex nature of k-graphs often makes the proofs of these adapted results much more complicated than the previous ones.

Corners of graph algebras naturally arise in many places when studying graph algebras, see [**10, 14, 26, 27**] for example, and have been shown to be a necessary tool in the understanding of arbitrary graph algebras. In particular, there is an important link between graph algebras and symbolic dynamics, see [**2, 4, 9**], since directed graphs represent subshifts of finite type, see [**15**]. Transferring results from symbolic dynamics to graph algebras frequently involves using corners.

It is the goal of this paper to provide tools for dealing with corners of k-graph algebras generated by vertex projections. As such, we describe a universal C^*-algebra generated by partial isometries which

is isomorphic to a corner of a graph algebra. We then show that this algebra has a version of the Gauge Invariant Uniqueness theorem (Theorem 3.5) which tells us when mappings that respect the gauge action are injective. We then show the facility our definition provides by proving various applications. As such, we gain some new results as well as some generalizations of existing results for directed graphs. In particular, we obtain conditions for checking Morita equivalence of graph algebras using corners and also realize the AF core of a k-graph as a corner.

We begin in Section 2 with the preliminaries involved in finitely aligned k-graphs and their associated graph algebra since for the most part of this paper we restrict ourselves to this class of k-graph. In Section 3, given a set X of vertices in a k-graph Λ, we define a (Λ, X)-family of partial isometries subject to a set of relations similar to the Cuntz-Krieger relations of [20] that generates a universal C^*-algebra $C^*(\Lambda, X)$. We then prove a Gauge Invariant Uniqueness theorem (Theorem 3.5) for $C^*(\Lambda, X)$ which generalizes [20, Theorem 4.2] and then use this theorem to show that $C^*(\Lambda, X)$ is isomorphic to a corner of $C^*(\Lambda)$.

In Section 4 we describe saturated and hereditary sets of vertices and use them to find conditions for fullness of our corners (Corollary 4.4). We also describe the Morita equivalence class (Proposition 4.2) of a corner based on saturated and hereditary sets. In Section 5 we examine a class of k-graph morphisms which induce maps between corners and in particular how these can be used to show Morita equivalence of k-graph algebras (Corollary 5.4).

In Section 6 we establish necessary conditions for a corner to be simple (Proposition 6.2) and, in particular, if our k-graph is row finite the condition is also sufficient (Proposition 6.4). In Section 7 we briefly look at some corners that are generated by more general projections using the dual graph defined in [1].

Finally in Section 8 we look at skew product graphs and establish a connection between certain fixed-point algebras of k-graphs and corners of skew product graphs. In particular, we give a condition for the AF core of a k-graph algebra to be Morita equivalent to a skew product graph naturally associated to it.

2. Preliminaries. Throughout this paper we let $\mathbf{N} := \{0, 1, 2, \dots\}$ be the set of counting numbers and regard $\mathbf{N}^k$ as an abelian monoid with identity $0 = (0, 0, \dots, 0)$ and canonical generators $e_i = (0, \dots, 1, \dots, 0)$, (1 is the ith coordinate). For $n \in \mathbf{N}^k$ we write n_i as the ith coordinate of n. There is a partial order l on $\mathbf{N}^k$ given by $m \leq n$ if $m_i \leq n_i$ for all $1 \leq i \leq k$, with $m < n$ if $m \leq n$ and $m \neq n$. For $m, n \in \mathbf{N}^k$ we write $m \vee n$ and $m \wedge n$ for their coordinate-wise maximum and minimum, respectively.

Definition 2.1. A k-graph is a pair (Λ, d) consisting of a countable category Λ and a *degree* functor $d : \Lambda \to \mathbf{N}^k$ which satisfies the *factorization property*: for every $\lambda \in \Lambda$ and $m, n \in \mathbf{N}^k$ with $d(\lambda) = m + n$ there exist unique $\mu, \nu \in \Lambda$ such that $d(\mu) = m$, $d(\nu) = n$ and $\lambda = \mu\nu$, see [**12**] for more details. A k-graph morphism is a functor between two k-graphs which respects the degree map.

Throughout this paper we will simply write Λ instead of (Λ, d) whenever it is clear what we mean. Since we regard k-graphs as analogues of directed graphs, we will sometimes refer to morphisms as paths (denoted with Greek letters $\lambda, \mu, \nu, \dots$) and objects as vertices (denote $u, v, w, \dots$), and we will write s and r for the domain and codomain maps, respectively.

Definition 2.2. For all $n \in \mathbf{N}^k$ we define $\Lambda^n := \{\lambda \in \Lambda : d(\lambda) = n\}$. The factorization property ensures that $\mathrm{Obj}\,(\Lambda)$ can be identified with Λ^0, and we will regard them as the same thing. Given any $v \in \Lambda^0$ and $n \in^k$ we define $v\Lambda^n := \{\lambda \in \Lambda^n : r(\lambda) = v\}$ and $\Lambda^n v := \{\lambda \in \Lambda^n : s(\lambda) = v\}$. Similarly, for any $X \subseteq \Lambda^0$, we define $X\Lambda^n := \cup_{v \in X} v\Lambda^n$ and $\Lambda^n X := \cup_{v \in X} \Lambda^n v$ and $X\Lambda := \{\lambda \in \Lambda : r(\lambda) \in X\}$.

Definition 2.3. A k-graph is *row finite* if the set $v\Lambda^n$ is finite for all $v \in \Lambda^0$ and $n \in \mathbf{N}^k$. We call a vertex $v \in \Lambda^0$ a *source* if $v\Lambda^{e_i} = \varnothing$ for some $1 \leq i \leq k$ and a *sink* if $\Lambda^{e_i} v = \varnothing$ for some $1 \leq i \leq k$.

Definition 2.4. A k-graph is *locally convex* if, for all $v \in \Lambda^0$ and $i, j \in \{1, \dots, k\}$ such that $i \neq j$ and $v\Lambda^{e_i}$ and $v\Lambda^{e_j}$ are nonempty, then for all $\lambda \in v\Lambda^{e_i}$ the set $s(\lambda)\Lambda^{e_j}$ is nonempty.

Definition 2.5. For $\lambda \in \Lambda$ and $m \leq n \leq d(\lambda)$, the factorization property gives unique paths $\lambda' \in \Lambda^m$, $\lambda'' \in \Lambda^{n-m}$ and $\lambda''' \in \Lambda^{d(\lambda)-n}$ such that $\lambda = \lambda'\lambda''\lambda'''$. We denote λ'' by $\lambda(m, n)$, so $\lambda' = \lambda(0, m)$ and $\lambda''' = \Lambda(n, d(\lambda))$.

Definition 2.6. For $\lambda, \mu \in \Lambda$, we write

$$\Lambda^{\min}(\lambda, \mu) := \{(\alpha, \beta) : \lambda\alpha = \mu\beta, d(\lambda\alpha) = d(\lambda) \vee d(\mu)\}$$

for the collection of pairs which give *minimal common extensions* of λ and μ. For any set $F \subset \Lambda$:

$$\mathrm{MCE}\,(F) = \left\{\lambda \in \Lambda : d(\lambda) = \bigvee_{\alpha \in F} d(\alpha), \lambda(0, d(\alpha)) = \alpha \text{ for all } \alpha \in F\right\},$$

and $\vee F = \cup_{G \subset F} \mathrm{MCE}\,(G)$. We say the Λ is *finitely aligned* if $\Lambda^{\min}(\lambda, \mu)$ is finite (possibly empty) for all $\lambda, \mu \in \Lambda$.

Definition 2.7. A set $E \subset v\Lambda$ is *exhaustive* if for every $\mu \in v\Lambda$ there exists $\lambda \in E$ such that $\Lambda^{\min}(\lambda, \mu) \neq \varnothing$.

For this paper we are only concerned with finite exhaustive sets. This is reflected in Definition 2.8 (iv) and Definition 3.1 (iii). We note that if Λ is row finite with no sources, then for any $v \in \Lambda^0$ and $n \in \mathbf{N}^k$ the set $v\Lambda^n$ is a finite exhaustive set.

Definition 2.8. Let (Λ, d) be a finitely aligned k-graph. A *Cuntz-Krieger Λ-family* is a collection $\{t_\lambda : \lambda \in \Lambda\}$ of partial isometries in a C^*-algebra satisfying:

(i) $\{t_v : v \in \Lambda^0\}$ is a collection of mutually orthogonal projections;

(ii) $t_\lambda t_\mu = t_{\lambda\mu}$ whenever $s(\lambda) = r(\mu)$;

(iii) $t_\lambda^* t_\mu = \sum_{(\alpha, \beta) \in \Lambda^{\min}(\lambda, \mu)} t_\alpha t_\beta^*$ for all $\lambda, \mu \in \Lambda$; and

(iv) $\prod_{\lambda \in E}(t_v - t_\lambda t_\lambda^*) = 0$ for all $v \in \Lambda^0$ and finite exhaustive $E \subset v\Lambda$.

Remarks 2.9. (1) Relation (iii) implies that $t_\lambda^* t_\lambda = t_{s(\lambda)}$ and that $t_\lambda^* t_\mu = 0$ if $\Lambda^{\min}(\lambda, \mu) = \varnothing$. Also, the finitely aligned condition is necessary for relation (iii) to make sense. See [**20**, Definition 2.5] for more details.

(2) Relation Definition 2.8 (iv) allows vertex projections to be written as a linear combination of path projections. That is, for any $v \in \Lambda$ with a finite exhaustive set $E \subset v\Lambda$:

$$t_v = \sum_{\lambda \in \vee E} c_\lambda t_\lambda t_\lambda^*.$$

Given a finitely aligned k-graph (Λ, d), there exists a C^*-algebra $C^*(\Lambda)$ generated by a Cuntz-Krieger Λ-family $\{s_\lambda : \lambda \in \Lambda\}$ which is universal in the following sense: given a Cuntz-Krieger Λ-family $\{t_\lambda : \lambda \in \Lambda\}$ of bounded operators on a Hilbert space $\mathcal{H}$, there exists a unique homomorphism $\pi : C^*(\Lambda) \to \mathcal{B}(\mathcal{H})$ such that $\pi(s_\lambda) = t_\lambda$ for all $\lambda \in \Lambda$. Given a Cuntz-Krieger Λ-family $\{t_\lambda : \lambda \in \Lambda\}$, then as a consequence of Definition 2.8 (i)–(iii) and [20, Lemma 2.7], we have

$$C^*(\{t_\lambda : \lambda \in \Lambda\}) = \overline{\text{span}}\, \{t_\lambda t_\mu^* : \lambda, \mu \in \Lambda, s(\lambda) = s(\mu)\}.$$

Given any finitely aligned k-graph (Λ, d), then it has a strongly continuous *gauge action* $\gamma^\Lambda : \mathbf{T}^k \to \text{Aut}(C^*(\Lambda))$ determined by $\gamma_z^\Lambda(s_\lambda) = z^{d(\lambda)} s_\lambda$ where $z \in \mathbf{T}^k$ and for any $m \in \mathbf{N}^k$ we have $z^m = z_1^{m_1} \dots z_k^{m_k}$. The fixed-point algebra $C^*(\Lambda)^{\gamma^\Lambda}$ is AF and is equal to $\overline{\text{span}}\, \{s_\lambda s_\mu^* : d(\lambda) = d(\mu)\}$ and is called the AF *core* of $C^*(\Lambda)$, see [20, Theorem 3.1].

3. Cuntz-Krieger (Λ, X)-families. We now wish to describe corners of finitely aligned k-graph algebras determined by vertex projections as universal C^*-algebras generated by partial isometries. A similar method was used in [25, Section 2] to describe corners of certain directed graph algebras. The notation used in this paper is also comparable to that of [22, Section 3] where rank 2 Cuntz-Krieger algebras are defined in a similar way.

Definition 3.1. Let (Λ, d) be a finitely aligned k-graph, and let $X \subseteq \Lambda^0$ be nonempty. A *Cuntz-Krieger (Λ, X)-family* is a collection of partial isometries

$$\{T_{\alpha,\beta} : \alpha, b \in X\Lambda \text{ and } s(\alpha) = s(\beta)\}$$

(with notation $T_\lambda := T_{\lambda,\lambda}$ for each $\lambda \in \Lambda$) subject to the following relations:

For any $\alpha, \beta, \lambda, \mu \in X\Lambda$ with $s(\alpha) = s(\beta)$ and $s(\lambda) = s(\mu)$,

(i) $T_{\alpha,\beta}^* = T_{\beta,\alpha}$;

(ii) $T_{\alpha,\beta} T_{\lambda,\mu} = \sum_{(\beta',\lambda') \in \Lambda^{\min}(\beta,\lambda)} T_{\alpha\beta', \mu\lambda'}$; and

(iii) $\prod_{\lambda\nu \in E}(T_\lambda - T_{\lambda\nu}) = 0$ for all $\lambda \in X\Lambda$ and finite exhaustive $E \subset \lambda\Lambda$.

Remarks 3.2. (1) As a result of relations (i) and (ii) $\{T_\lambda : \lambda \in X\Lambda\}$ is a set of commuting projections, and in particular $\{T_v : v \in X\}$ is a set of mutually orthogonal projections.

(2) When $X = \Lambda^0$ these relations reduce to a Cuntz-Krieger Λ-family. That is, the set $\{T_{\lambda, s(\lambda)} : \lambda \in \Lambda\}$ satisfies Definition 2.8.

Given a finitely aligned k-graph Λ and $X \subseteq \Lambda^0$ there exists a C^*-algebra generated by a Cuntz-Krieger (Λ, X)-family $\{T_{\alpha,\beta} : \alpha, \beta \in X\Lambda : s(\alpha) = s(\beta)\}$, denoted $C^*(\Lambda, X)$, which is universal in the following sense: given a Cuntz-Krieger (Λ, X)-family $\{t_{\alpha,\beta} : \alpha, \beta \in X\Lambda : s(\alpha) = s(\beta)\}$ there exists a unique homomorphism π_t of

$C^*(\Lambda, X)$ such that $\pi_t(T_{\alpha,\beta}) = t_{\alpha,\beta}$ for all $\alpha, \beta \in X\Lambda$ with $s(\alpha) = s(\beta)$.

It was shown in [20, Proposition 2.12] that for any finitely aligned k-graph Λ there exists a nondegenerate Cuntz-Krieger Λ-family $\{s_\lambda : \lambda \in \Lambda\}$. By the universal property of $C^*(\Lambda, X)$ there is a homomorphism $\phi : B \to C^*(\Lambda)$ given by $\phi(T_{\alpha,\beta}) = s_\alpha s_\beta^*$. Hence, for every finitely aligned k-graph Λ with $X \subseteq \Lambda^0$ there exists a nondegenerate Cuntz-Krieger (Λ, X)-family.

By the same argument as [20, Lemma 2.7(iv)], it can be shown that

$$C^*(\Lambda, X) = \overline{\operatorname{span}}\{T_{\alpha,\beta} : \alpha, \beta \in X\Lambda, \ s(\alpha) = s(\beta)\}.$$

By the universality of $C^*(\Lambda, X)$ and a standard $\varepsilon/3$ argument, $C^*(\Lambda, X)$ has a strongly continuous *gauge action* X of $\mathbf{T}^k$ given by

$$\gamma_z^X(T_{\alpha,\beta}) = z^{d(\alpha)-d(\beta)} T_{\alpha,\beta}.$$

We call the fixed-point algebra $C^*(\Lambda, X)^{\gamma^X}$ the *core* of $C^*(\Lambda, X)$.

Using a standard argument, it can be shown that

$$C^*(\Lambda, X)^{\gamma^X} = \overline{\text{span}} \{T_{\alpha,\beta} \in C^*(\Lambda, X) : d(\alpha) = d(\beta)\},$$

see [12, Lemma 3.1] and [5, Lemma 2.2] for example.

Our next goal is to prove that $C^*(\Lambda, X)^{\gamma^X}$ is AF. We proceed by showing that every finite set of generators $\{T_{\alpha,\beta}\}$ is contained in a finite-dimensional subalgebra of $C^*(\Lambda, X)^{\gamma^X}$, which suffices. Following from [20, Definition 3.3], for any finite set $E \subset X\Lambda$ then $\prod E$ is the smallest set containing E such that span $\{T_{\alpha,\beta} \in C^*(\Lambda, X)^{\gamma^X} : \alpha, \beta \in \prod E\}$ is closed under multiplication.

Lemma 3.3. *Let Λ be a finitely aligned k-graph with $X \subseteq \Lambda^0$. Then the fixed-point algebra $C^*(\Lambda, X)^{\gamma^X}$ is AF.*

Proof. Let $F = \{T_{\alpha_i,\beta_i} \in C^*(\Lambda, X)^{\gamma^X}\}_{i=1}^n$, and let $E = \{\alpha_i, \beta_i\}_{i=1}^n$. Then the set $\prod E$ is finite by [20, Lemma 3.2], and span $\{T_{\alpha,\beta} \in C^*(\Lambda, X)^{\gamma^X} : \alpha, \beta \in \prod E\}$ is a finite-dimensional subalgebra of $C^*(\Lambda, X)^{\gamma^X}$ containing F. Hence, $C^*(F)$ is finite-dimensional. Therefore, $C^*(\Lambda, X)^{\gamma^X}$ is AF by [6, Theorem 2.2]. $\square$

We now wish to prove an analogue of the Gauge Invariant Uniqueness theorem for C^*-algebras generated by Cuntz-Krieger (Λ, X)-families. Recall the following definition from [20, Proposition 3.5]: for any $\lambda \in \prod E$,

$$Q(T)_\lambda^{\prod E} = T_\lambda \prod_{\substack{\lambda\nu \in \prod E \\ d(\nu) > 0}} (T_\lambda - T_{\lambda\nu}).$$

By [20, Proposition 3.5], the set $\{Q(T)_\lambda^{\prod E} : \lambda \in \prod E\}$ is a set of mutually orthogonal projections. Our claim is that every ideal in $C^*(\Lambda, X)^{\gamma^X}$ contains a $Q(T)_\lambda^{\prod E}$ for some finite $E \subset X\Lambda$ and $\lambda \in \prod E$.

By claim (ii) of [20, Corollary 3.7], $T_\mu Q(T)_\sigma^{\prod E} = 0$ for any $\mu, \sigma \in \prod E$ with $\sigma(0, d(\mu)) \neq \mu$. If $\{\lambda\nu \in \prod E\}$ is exhaustive then clearly $Q(T)_\lambda^{\prod E} = 0$ by Definition 3.1 (iii). Also [20, Proposition 3.13] shows

that $Q(T)_\lambda^{\prod E} = 0$ only if $\{\lambda\nu \in \prod E\}$ is exhaustive. The following lemma will be useful for proving Theorem 3.5.

Lemma 3.4. *Let $E \subset X\Lambda$ be a finite set, and let $\lambda, \mu \in \prod E$ be such that $T_{\lambda,\mu} \in C^*(\Lambda, X)^{\gamma^X}$. Then*

(1) $T_{\lambda,\mu} = \sum_{i=1}^n c_i T_{\alpha_i,\beta_j}$ *where* $\alpha_i, \beta_i \in \prod E$ *and* $Q(T)_{\alpha_i}^{\prod E} \neq 0$, $Q(T)_{\beta_i}^{\prod E} \neq 0$ *for all* $1 \leq i \leq n$.

(2) $Q(T)_\lambda^{\prod E} T_{\lambda,\mu} = T_{\lambda,\mu} Q(T)_\mu^{\prod E}$, *cf.* [**20**, *Lemma 3.10*].

Note. We cannot use [**20**, Lemma 3.10] in this paper because the proof relies on [**20**, Remark 3.6] which is not applicable to this paper.

Proof. For (1), let $n_E = \vee E$, and let $\alpha \in E^{\leq n_E}$. Since α has no extensions in $\prod E$, then $Q(T)_\alpha^{\prod E} = T_\lambda \neq 0$ for all $\alpha \in E^{\leq n_E}$. Suppose $Q(T)_\lambda^{\prod E} = 0$, and hence $\{\lambda\nu \in \prod E\}$ is a finite exhaustive set. Using Definition 3.1 (iii) we can write

$$T_{\lambda,\mu} = \sum_{\substack{\lambda\nu,\mu\nu \in \prod E \\ d(\nu) \neq 0}} c_\nu T_{\lambda\nu,\mu\nu}.$$

Because $\prod E$ is a finite set and because $d(\alpha) \leq n_E$ for all $\alpha \in \prod E$, we can repeat these steps on each term until the hypothesis is met.

For (2), let $\lambda, \mu, \lambda\nu, \nu \in \prod E$; then

$$T_{\lambda,\mu}(T_\mu - T_{\mu\nu}) = T_{\lambda,\mu} - T_{\lambda\nu,\mu\nu} = (T_\lambda - T_{\lambda\nu})T_{\lambda,\mu}.$$

Since $Q(T)_\lambda^{\prod E}$ is a finite product we can perform the above calculation term by term. $\square$

Theorem 3.5 (Gauge Invariant Uniqueness theorem). *Let Λ be a finitely aligned k-graph with $X \subseteq \Lambda^0$, $\{t_{\alpha,\beta}\}$ a Cuntz-Krieger (Λ, X)-family and π a representation of $C^*(\Lambda, X)$ such that $\pi(T_{\alpha,\beta}) = t_{\alpha,\beta}$. Suppose that $\pi(T_\lambda) \neq 0$ for each $\lambda \in X\Lambda$, and suppose that there is a*

strongly continuous action δ of $\mathbf{T}^k$ on $C^*(t_{\alpha,\beta})$ such that $\delta_z \circ \pi = \pi \circ \gamma_z$ for all $z \in \mathbf{T}^k$. Then π is faithful.

Proof. First suppose $T_{\alpha,\beta} \in C^*(\Lambda, X)^{\gamma^X}$ and $\pi(T_{\alpha,\beta}) = 0$; then since

$$(*) \qquad\qquad T_\alpha = T_{\alpha,\beta} T_{\beta,\alpha},$$

we must have that $\pi(T_\alpha) = 0$ which contradicts our hypothesis.

Next suppose $x = \sum_{i=1}^n c_i T_{\alpha_i, \beta_i} \in C^*(\Lambda, X)^{\gamma^X}$ such that $x \neq 0$ and $\pi(x) = 0$. Let $E = \{\alpha_i, \beta_i\}_{i=1}^n$. By Lemma 3.4 (1) we may assume that $Q(T)_\lambda^{\prod E} \neq 0$ for all $\lambda \in E$. Since E is a finite set, then there exists $1 \leq j \leq n$ such that $\alpha_j(0, d(\alpha_i)) \neq \alpha_i$ and $\beta_j(0, d(\beta_i)) \neq \beta_i$ for all $\alpha_i, \beta_i \in E$ and $i \neq j$. Hence,

$$\begin{aligned}
Q(T)_{\alpha_j}^{\prod E} x Q(T)_{\beta_j}^{\prod E} &= c_j Q(T)_{\alpha_j}^{\prod E} T_{\alpha_j,\beta_j} Q(T)_{\beta_j}^{\prod E} \\
&= c_j Q(T)_{\alpha_j}^{\prod E} Q(T)_{\alpha_j}^{\prod E} T_{\alpha_j,\beta_j} \\
&\qquad \text{(by Lemma 3.4 (2))} \\
&= c_j Q(T)_{\alpha_j}^{\prod E} T_{\alpha_j,\beta_j}.
\end{aligned}$$

However, $\pi(T_{\alpha_j,\beta_j}) \neq 0$ by the above argument, and clearly $\pi(Q(T)_{\alpha_l}^{\prod E}) \neq 0$. Hence, $\pi(x) = 0$ contradicts our hypothesis.

Hence, π is faithful on $\mathcal{F} = \text{span}\{T_{\alpha,\beta} \in C^*(\Lambda, X) : d(\alpha) = d(\beta)\}$ and since $C^*(\Lambda, X)^{\gamma^X}$ is AF by Lemma 3.3 then every nontrivial ideal in $C^*(\Lambda, X)^{\gamma^X}$ must intersect F by [6, Lemma 3.1] and since the kernel of π is an ideal then π must also be faithful on $C^*(\Lambda, X)^{\gamma^X}$. Finally, since π is faithful on $C^*(\Lambda, X)^{\gamma^X}$ which is AF, the remainder of the proof is now standard, see [12, Theorem 3.4] or [20, Theorem 4.2]. $\square$

Remark 3.6. If $X = \Lambda^0$, then Theorem 3.5 is equivalent to the usual Gauge Invariant Uniqueness theorem for finitely aligned k-graphs as seen in [20, Theorem 4.2].

Given a k-graph Λ with $X \subseteq \Lambda^0$ and Cuntz-Krieger Λ-family $\{s_\lambda\}$, then by the same argument as [17, Lemma 3.3.1] the sum $\sum_{v \in X} s_v$ converges to a projection $P_X \in \mathcal{M}(C^*(\Lambda))$ in the strict topology.

Corollary 3.7. *Let Λ be a finitely aligned k-graph and $\{s_\lambda\}$ a Cuntz-Krieger Λ-family. Let $X \subseteq \Lambda^0$ and $\{T_{\alpha,\beta}\}$ be a Cuntz-Krieger (Λ, X)-family. Then*

$$P_X C^*(\Lambda) P_X \cong C^*(\Lambda, X).$$

Proof. Since $\{s_\alpha s_\beta^* : \alpha\beta \in X\Lambda, s(\alpha) = s(\beta)\}$ satisfies Definition 3.1 (i)–(iii), the universal property of $C^*(\Lambda, X)$ implies that there is a homomorphism $\phi : C^*(\Lambda, X) \to P_X C^*(\Lambda) P_X$ given by $\phi(T_{\alpha,\beta}) = s_\alpha s_\beta^*$. Then ϕ is surjective and $\gamma_s^\Lambda(\phi(T_{\alpha,\beta})) = \gamma_z^\Lambda s_\alpha s_\beta^* = \phi(\gamma_z^X(T_{\alpha,\beta}))$ and $\phi(T_\lambda) = s_\lambda s_\lambda^* \neq 0$ for all $\lambda \in X\Lambda$. Therefore by Theorem 3.5, ϕ is also injective. $\square$

Remark 3.8. Using the map ϕ in the proof of Corollary 3.7, we have that

$$C^*(\Lambda, \Lambda^0) \cong C^*(\Lambda).$$

In this case the relations in Definition 3.1 are equivalent to the relations of a Cuntz-Krieger Λ-family as given in Definition 2.8. Hence, when it is convenient, we will identify $C^*(\Lambda, \Lambda^0)$ with $C^*(\Lambda)$ via the mapping $T_{\alpha,\beta} \mapsto s_\alpha s_\beta^*$ in order to avoid conflicts of notation.

4. Fullness of $C^*(\Lambda, X)$. When considering corners it is natural to ask when the corner is full. The answer has a lot to do with saturated hereditary subsets of Λ^0 and their association with gauge invariant ideals in $C^*(\Lambda, \Lambda^0)$, see [**23**] for details.

Definition 4.1. Given a k-graph Λ with $H, S \subset \Lambda^0$, then:

(1) we say H is *hereditary* if for all $v \in H$, then $v\Lambda = v\Lambda H$,

(2) we say S is *saturated* if for any $v \in \Lambda^0$ such that there exists a finite exhaustive set $E \subset v\Lambda S$ then $v \in S$.

If H is the smallest hereditary set containing V and S is the smallest saturated set containing H, then S is the smallest saturated and hereditary set containing V, see [**23**, Lemma 3.2]. We denote the smallest saturated hereditary set containing V by $\Sigma(V)$.

Proposition 4.2. *Given a finitely aligned k-graph Λ with $X \subseteq \Lambda^0$, then $C^*(\Lambda, X)$ is Morita equivalent to $C^*(\Lambda, \Sigma(X))$.*

Proof. By [**21**, Example 3.6], $C^*(\Lambda, X)$ is Morita equivalent to the ideal generated by P_X and by [**23**, Theorem 5.5], see also [**20**, Remark 5.6 (1)], this ideal is equal to the ideal generated by $P_{\Sigma}(X)$.   □

Remark 4.3. If Λ is finitely aligned and $X \subseteq \Lambda^0$, then $C^*(\Lambda, X)$ is not usually a k-graph algebra but is always Morita equivalent to a k-graph algebra because of Proposition 4.2. If Λ is row finite and locally convex and $X \subseteq \Lambda^0$ is a hereditary set, then by [**19**, Theorem 5.2 (c)] $C^*(\Lambda, X) \cong C^*(X\Lambda, X)$ where $X\Lambda$ is a k-graph because X is hereditary. Since $X\Lambda^0 = X$, then it follows that $C^*(X\Lambda, X)$ is a k-graph algebra. If Λ is finitely aligned but not row finite and $X \subseteq \Lambda^0$ is a saturated and hereditary set, then by [**23**, Lemma 3.6] we again have $C^*(\Lambda, X) \cong C^*(X\Lambda, X)$. Hence, for any finitely aligned k-graph with $X \subseteq \Lambda^0$, then by Proposition 4.2 we have $C^*(\Lambda, X)$ is Morita equivalent to $C^*(\Lambda, \Sigma(X))$ with the latter being a k-graph algebra.

In particular, Proposition 4.2 says that $C^*(\Lambda, X)$ is a full corner of $C^*(\Lambda, \Sigma(X))$.

Corollary 4.4. *Given a finitely aligned k-graph Λ with $X \subseteq \Lambda^0$, then $C^*(\Lambda, X)$ is a full corner of $C^*(\Lambda, \Lambda^0)$ if and only if $\Sigma(X) = \Lambda^0$.*

Proof. By Proposition 4.2 if $\Sigma(X) = \Lambda^0$, then $C^*(\Lambda, X)$ is full in $C^*(\Lambda, \Lambda^0)$. Conversely, if $C^*(\Lambda, X)$ is full, then the ideal generated by P_X is equal to the ideal generated by P_{Λ^0} and hence $\Sigma(X) = \Lambda^0$ by [**20**, Theorem 5.5].   □

Corollary 4.5. *Let Λ be a finitely aligned k-graph, and let $X, Y \subseteq \Lambda^0$ be such that $\Sigma(X) = \Sigma(Y)$. Then $C^*(\Lambda, X)$ is Morita equivalent to $C^*(\Lambda, Y)$.*

Proof. $C^*(\Lambda, X)$ is Morita equivalent to $C^*(\Lambda, \Sigma(X)) = C^*(\Lambda, \Sigma(Y))$ which is Morita equivalent to $C^*(\Lambda, Y)$ by Proposition 4.2.   □

Example 4.6. Let $X \subseteq \Lambda^0$ be any subset, and let $X^c = \Lambda^0 \setminus X$ and suppose $\Sigma(X) = \Sigma(X^c)$. This implies $\Sigma(X) = \Lambda^0$ and hence $C^*(\Lambda, X)$ and $C^*(\Lambda, X^c)$ are complimentary corners, see [**7**, Theorem 1.1].

5. k-graph morphisms.

Definition 5.1. Given k-graphs Λ_1, Λ_2 with $X \subseteq \Lambda_1^0$ and a k-graph morphism $\phi : \Lambda_1 \to \Lambda_2$, then we say Φ is *saturated* with respect to X if $\phi : X\Lambda_1 \to \phi(X)\Lambda_2$ is a bijection, cf. [**18**, Definition 3.2, Proposition 3.3]. If $X = \Lambda_1^0$, then we call ϕ a saturated k-graph morphism.

Theorem 5.2. *Given finitely aligned k-graphs Λ_1, Λ_2 with $X \subseteq \Lambda_1^0$ and a k-graph morphism $\Phi : \Lambda_1 \to \Lambda_2$ that is relatively saturated with respect to X, then $C^*(\Lambda_1, X) \cong C^*(\Lambda_2, \phi(X))$.*

Proof. This proof follows the same argument as [**18**, Proposition 3.3] and is repeated here for convenience. Let $\{T_{\alpha,\beta}\}$ be a Cuntz-Krieger (Λ_1, X)-family, and let $\{S_{\gamma,\delta}\}$ be a Cuntz-Krieger $(\Lambda_2, \phi(X))$-family. The relative saturation property ensures that $\{S_{\phi(\alpha),\phi(\beta)}\}$ is a Cuntz-Krieger $(\Lambda_2, \phi(X))$-family and the universal property of $C^*(\Lambda_1, X)$ induces a homomorphism $\phi_* : C^*(\Lambda_1, X) \to C^*(\Lambda_2, \phi(X))$ given by $\phi_*(T_{\alpha,\beta}) = S_{\phi(\alpha),\phi(\beta)}$. Then ϕ_* is surjective since ϕ is saturated and also ϕ is degree preserving since it is a k-graph morphism and hence ϕ_* respects the gauge action. Finally, $\phi_*(T_\lambda) = S_{\phi(\lambda)} \neq 0$, so by Theorem 3.5, ϕ_* is also injective. $\square$

Example 5.3. Let Λ be a finitely aligned k-graph with $H \subset \Lambda$ a hereditary subset. Then $H\Lambda$ is a sub k-graph of Λ and the identity map $i : H\Lambda \to \Lambda$ is a relatively saturated k-graph morphism with respect to H. Hence, we have $C^*(H\Lambda) \cong C^*(\Lambda, H)$ by Theorem 5.2 which is an improvement of [**19**, Theorem 5.2 (c)] which requires Λ to be row finite locally convex and [**23**, Lemma 3.6] which requires H to be saturated and hereditary.

Corollary 5.4. *Given finitely aligned k-graphs Λ_1, Λ_2 and a saturated k-graph morphism $\phi : \Lambda_1 \to \Lambda_2$ such that $\Sigma(\phi(\Lambda_1^0)) = \Lambda_2^0$, then $C^*(\Lambda_1)$ is Morita equivalent to $C^*(\Lambda_2)$.*

Proof. Follows from Theorem 5.2 and Corollary 4.4.   □

Example 5.5. Recall from [**13**] that Ω_k is the k-graph $\{(m, n) \in \mathbf{N}^k \times \mathbf{N}^k : m \le n\}$ with $r(m, n) = m$, $s(m, n) = n$ and $d(m, n) = m - n$ and object space identified with $\mathbf{N}^k$, while Δ^k is the k-graph $\{(m, n) \in \mathbf{Z}_k \times \mathbf{Z}_k : m < n\}$ with the same range, source and degree maps as above and object space identified with $\mathbf{Z}^k$. So there is a natural embedding using the identity map $i : \Omega_k \to \Delta_k$ which is a saturated k-graph morphism. Noting that $\Omega_k^0 = \mathbf{N}^k$ is a hereditary subset of $\Delta_k^0 = \mathbf{Z}^k$, then, as in Example 5.3, $C^*(\Omega_k) \cong C^*(\Delta_k, \mathbf{N}^k)$. However, $\mathcal{S}(\mathbf{N}^k) = \mathbf{Z}^k$ so by Corollary 5.4 $C^*(\Delta_k, \mathbf{N}^k)$ is a full corner of $C^*(\Delta_k)$ and thus we have $C^*(\Omega_k)$ is Morita equivalent to $C^*(\Delta_k)$.

We now look at an application of saturated k-graph morphisms to symbolic dynamics in which we are concerned with the bi-infinite path space of a k-graph.

Definition 5.6. Given a k-graph Λ, the *bi-infinite path space* of Λ is the set of k-graph morphisms

$$\Lambda^\Delta = \{x : \Delta_k \longrightarrow \Lambda\}$$

where Δ_k is as defined in Example 5.5.

A typical construction is the *essential subgraph* which is the largest subgraph with no sinks or sources. The bi-infinite path space of the essential subgraph can be identified with the bi-infinite path space of the original graph, see [**15**, Section 2], and is also identified with the edge shift associated to the graph. This was done for 1-graphs in [**15**, Proposition 2.2.10].

We will now adapt this construction to k-graphs and show conditions for a k-graph and its essential subgraph to have Morita equivalent algebras. Constructing the essential subgraph involves removing 'stranded' vertices that do not lie on any bi-infinite path.

Remark 5.7. If $\Lambda^\Delta = \varnothing$, then the essential subgraph will be empty.

Definition 5.8. Given a finitely aligned k-graph Λ then for any $v \in \Lambda^0$ we say v is *stranded* if there exists $n \in \mathbf{N}^k$ such that $v\Lambda^n = \varnothing$ or $\Lambda^n v = \varnothing$. We denote $\mathcal{S}(\Lambda)$ as the set of all stranded vertices in Λ^0.

For any k-graph Λ we construct the essential subgraph $E(\Lambda)$ by first identifying all the stranded vertices in Λ^0 and then constructing the subcategory with objects $\Lambda^0 \setminus \mathcal{S}(\Lambda)$ and morphisms $\{\lambda \in \Lambda :$ $s(\lambda),\ r(\lambda) \notin \mathcal{S}(\Lambda)\}$, cf. [**15**]. We can constructively define the set of stranded vertices in a recursive manner, cf. [**3**, Remark 3.1], as follows. First, let S_0 be the set of all sinks and sources. Next, let

$$S_{n+1} = S_n \cup \bigcup_{i=1}^{k} \left\{ v \in \Lambda^0 : v\Lambda^{e_i} \subset v\Lambda S_n \right\} \cup \left\{ v \in \Lambda^0 : \Lambda^{e_i} v \subset S_n \Lambda_v \right\},$$

and finally let $\mathcal{S}(\Lambda) = \cup_{n \in \mathbf{N}} S_n$.

It is worth taking a moment to check that $E(\Lambda)$ forms a k-graph.

Lemma 5.9. *Let Λ be a finitely aligned k-graph with $\Lambda^\Delta \neq \varnothing$; then there exists a unique k-graph $E(\Lambda) = (\Lambda^0 \setminus \mathcal{S}(\Lambda))\Lambda(\Lambda^0 \setminus \mathcal{S}(\Lambda))$ such that $E(\Lambda)$ is the largest subgraph of Λ with no sinks or sources and $E(\Lambda)^\Delta = \Lambda^\Delta$.*

Proof. To see that $E(\Lambda)$ is a k-graph, we need to check the factorization property. Suppose $\lambda \in E(\Lambda)$. Then this implies that $\lambda \in \Lambda$ and $r(\lambda)\Lambda^n$ and $\Lambda^n s(\lambda)$ are nonempty for all $n \in \mathbf{N}^k$. Let $p, q \in \mathbf{N}^k$ be such that $p + q = d(\lambda)$. Then there exist $\mu \in \Lambda^p$ and $\nu \in \Lambda^q$ such that $\lambda = \mu\nu$. By the factorization property of Λ, we must have $\Lambda^n s(\mu)$ and $s(\mu)\Lambda^m$ nonempty for all $m \leq p$ and $n \leq q$ and further, since $s(\lambda)$ and $r(\lambda)$ are not stranded, we then have $s(\mu)\Lambda^n$ and $\Lambda^n s(\mu)$ are nonempty for all $n \in \mathbf{N}^k$ and hence $\mu, \nu \in E(\Lambda)$. Therefore, $E(\Lambda)$ is a k-graph.

Clearly if $x \in \Lambda^\Delta$ then $x \in E(\Lambda)^\Delta$ and vice-versa so $\Lambda^\Delta = E(\Lambda)^\Delta$ and clearly $E(\Lambda)$ is unique. To show $E(\Lambda)$ is the largest subgraph with no sinks or sources, suppose S is also a subgraph of Λ with no sinks or sources. Then every $v \in S^0$ is not stranded and hence $S \subset E(\Lambda)$. $\square$

Example 5.10. Let Ω_k and Δ_k be defined as in Example 5.5. Then $E(\Omega_k) = \varnothing$ since every vertex is stranded; however, $E(\Delta_k) = \Delta_k$ since every vertex is not stranded.

6. Simplicity.

Definition 6.1. A k-graph is *cofinal* if $\Sigma(\{v\}) = \Lambda^0$ for all $v \in \Lambda^0$. Also, given $X \subseteq \Lambda^0$, then X is *relatively cofinal* if $\Sigma(\{v\}) = \Sigma(X)$ for all $v \in X$.

Cofinal is usually defined using the infinite path space of Λ, see [**12**, Definition 4.7] or [**23**, Definition 8.4]. A k-graph is cofinal if for every vertex in $v \in \Lambda$ and every infinite path $x \in \Lambda^\infty$, then there exists an $n \in \mathbf{N}^k$ such that $v\Lambda x(n) \neq \varnothing$, while relatively cofinal means every vertex in X has the same property. The problem with this definition is that it is undefined for k-graphs with the property $\Lambda^\infty = \varnothing$. It should also be noted that Λ is cofinal if and only if Λ^0 is relatively cofinal.

Proposition 6.2. *Let Λ be a finitely aligned k-graph with $X \subseteq \Lambda^0$ such that all ideals in $C^*(\Lambda, X)$ are gauge invariant. Then $C^*(\Lambda, X)$ is simple if and only if X is relatively cofinal, cf. [**24**, Theorem 12].*

Proof. By Proposition 4.2 we may assume X is saturated and hereditary and so $C^*(\Lambda, X) \cong C^*(X\Lambda)$ by Theorem 5.2. In particular, if X is relatively cofinal in Λ, then $X\Lambda$ is cofinal. Finally, by [**23**, Proposition 8.5], $C^*(X\Lambda)$ is simple if and only if $X\Lambda$ is cofinal. □

For most purposes Proposition 6.2 is unsatisfactory for determining simplicity since there is not yet a necessary and sufficient condition for the ideals of k-graph to all be gauge invariant. In [**23**, Theorem 7.2] condition (D) is stated for when all the ideals of a finitely aligned k-graph are gauge invariant; however, it is not easily checkable. However, if Λ is row finite with no sources k-graphs we can define simplicity in terms of aperiodicity and cofinality as in [**12**, Section 4].

Recall from [**12**, Definition 4.1] that $x \in \Lambda^\infty$ is periodic if there exists $p \in \mathbf{Z}^k$ such that $x(m + p, n + p) = x(m, n)$ for all $m, n \in \mathbf{N}^k$ (with $m + p \geq 0$) and is eventually periodic if there exists $n \in \mathbf{N}^k$ such that $\sigma^n x$ is periodic (where σ is the shift map). A path in Λ^∞ is *aperiodic* if it is not periodic or eventually periodic.

Definition 6.3. Let Λ be a k-graph with $X \subseteq \Lambda^0$. Then we say X is *relatively aperiodic* if for all $v \in X$ there exists $x \in v\Lambda^\infty$ such that x is aperiodic. We say Λ is *aperiodic* if Λ^0 is relatively aperiodic.

In [**12**, Proposition 4.8] it was shown that if Λ is a row finite k-graph then $C^*(\Lambda)$ is simple if Λ is aperiodic and cofinal. Here we extend this notion to corners of locally convex and row finite k-graphs using the definition of relative aperiodicity.

Proposition 6.4. *Let Λ be a row finite k-graph with no sources, and let $X \subseteq \Lambda^0$ be X relatively aperiodic. Then $C^*(\Lambda, X)$ is simple if and only if X is relatively cofinal.*

Proof. By Proposition 4.2 we may assume X is saturated and hereditary and $C^*(\Lambda, X) \cong C^*(X\Lambda)$ by Theorem 5.2. We note that if X is relatively aperiodic in Λ then $X\Lambda$ is aperiodic. Hence, [**12**, Proposition 4.8] applies and $C^*(\Lambda, X)$ is simple if and only if $X\Lambda$ is cofinal. Since X is relatively cofinal if and only if $X\Lambda$ is cofinal, then this completes the proof.   □

7. Corners generated by subsets of Λ^p. If Λ is a nonaperiodic k-graph, then not all ideals are generated by saturated hereditary subsets of Λ^0, see [**23**, Section 5], and hence not all corners can be generated by subsets of Λ^0. In this section we show that if Λ is any row finite k-graph with no sources then we can easily extend our ideas in this paper to corners generated by certain subsets of Λ^p for some $p \in \mathbf{N}^k$.

Definition 7.1. Let Λ be a k-graph, and let $p \in \mathbf{N}^k$. Then the *dual graph* is $p\Lambda := \{\lambda \in \Lambda : d(\lambda) \geq p\}$ with range, source and degree maps defined as follows: For any $\lambda \in p\Lambda$ with $\lambda = \sigma\lambda' = \lambda''\rho$ and $d(\sigma) = d(\rho) = p$,

$$r_p(\lambda) = \rho, \quad s_p(\lambda) = \sigma, \quad d_p(\lambda) = d(\lambda) - p,$$

and composition defined as follows: For any $\lambda = \lambda'\rho$, $\mu = \rho\mu' \in p\Lambda$ with $r_p(\lambda) = s_p(\mu) = \rho$, then $\lambda \circ_p \mu = \lambda'\rho\mu'$.

For more details of dual higher rank graphs, see [**1**, Section 3]. In particular, $p\Lambda$ is a k-graph and if Λ is row finite with no sources, then $C^*(\Lambda) \cong C^*(p\Lambda)$, see [**1**, Proposition 3.2, Theorem 3.5].

Lemma 7.2. *Let Λ be a row finite k-graph with no sources, and let $X \subseteq (p\Lambda)^0 = \Lambda^p$ for some $p \in \mathbf{N}^k$ be such that $X \cap v\Lambda^p$ is finite for all $v \in \Lambda^0$. Then*

$$P_X C^*(\Lambda) P_X \cong C^*(p\Lambda, X),$$

where $P_X = \sum_{\alpha \in X} s_\alpha s_\alpha^*$.

Proof. Since $\Lambda^{\min}(\alpha, \beta) = \varnothing$ for all $\alpha, \beta \in X$ it follows that $\{s_\alpha s_\alpha^* : \alpha \in X\}$ is a set of mutually orthogonal projections in $C^*(\Lambda)$. Then P_X converges to a projection in $\mathcal{M}(C^*(\Lambda))$ (by the same argument as [**17**, Lemma 3.3.1]), and the rest follows from Corollary 3.7 and [**1**, Theorem 3.5]. $\quad\square$

We can extend Definition 3.1 to include $X \subseteq \Lambda^p$ subject to the hypothesis in Lemma 7.2. Note that if $p = 0$ then Lemma 7.2 reduces to Corollary 3.7.

When we talk about corners generated by arbitrary subsets of Λ, we have to be careful to watch that P_X converges to a projection in $\mathcal{M}(C^*(\Lambda))$. However, in some cases we can still talk about Cuntz-Krieger (Λ, X)-families generated by such arbitrary subsets. For example, let us suppose Λ is row finite with no sinks or sources with $X \subset \Lambda$, and suppose $\mu, \nu \in X$ with $\mu = \nu\nu'$. By definition

$$C^*(p\Lambda, X) = \overline{\operatorname{span}} \{T_{\alpha,\beta} : \alpha, \beta \in X\Lambda, \ \operatorname{sp}(\alpha) = s_p(\beta)\},$$

and hence for any $\lambda \in \mu\Lambda \cup \nu\Lambda$ there is a $T_\lambda \in C^*(p\Lambda, X)$. However, $\mu\Lambda \subset \nu\Lambda$ so $C^*(p\Lambda, X \setminus \{\mu\})$ is identical to $C^*(p\Lambda, X)$ in this case.

8. Skew product graphs. In this section we look at a k-graph construction called a skew product graph $G \times_c \Lambda$ and its relation to fixed-point algebras.

Definition 8.1. Given a k-graph Λ and a functor $c : \Lambda \to G$ where G is a discrete group, then the *skew product* graph $G \times_c \Lambda$ is the k-graph with objects $G \times \Lambda^0$ and morphisms $G \times \Lambda$ with $s(g, \lambda) = (gc(\lambda), s(\lambda))$ and $r(g, \lambda) = (g, r(\lambda))$ and degree map $d(g, \lambda) = d(\lambda)$, see [**12**, Definition 5.1] for details.

In particular, a functor $c : \Lambda \to G$ gives rise to a normal coaction γ_c of G on $C^*(\Lambda, \Lambda^0)$ given by:

$$\gamma_c(T_{\alpha,\beta}) = T_{\alpha,\beta} \otimes 1_{c(\alpha)c(\beta)^{-1}},$$

where for any $g \in G$, then 1_g is the point mass function in $C^*(G)$. Then the fixed-point algebra is $C^*(\Lambda, \Lambda^0) = \overline{\text{span}}\{T_{\alpha,\beta} \in C^*(\Lambda, \Lambda^0) : c(\alpha) = c(\beta)\}$, see [**16**, Section 7] for more details.

Proposition 8.2. *Let Λ be a finitely aligned k-graph, let G be a discrete group with a functor $c : \Lambda \to G$, and let c be the corresponding coaction of G on $C^*(\Lambda, \Lambda^0)$. Then*

$$C^*(\Lambda, \Lambda^0)^{\gamma_c^\Lambda} \cong C^*(G \times_c \Lambda, V),$$

where $V = \{1\} \times \Lambda^0$ and 1 is the identity element of G.

Proof. For any $(g, \lambda) \in G \times_c \Lambda$ with range in V we must have $g = 1$. Also for any $(1, \mu), (1, \nu) \in G \times_c \Lambda$ with the same source we must have $c(\mu) = c(\nu)$ and $s(\mu) = s(\nu)$. Hence $C^*(G \times_c \Lambda, V) = \overline{\text{span}}\{T_{(1,\mu),(1,\nu)} : c(\mu) = c(\nu), s(\mu) = s(\nu)\}$. Thus, by the universal property of $C^*(G \times_c \Lambda, V)$ we define the homomorphism $\phi : C^*(G \times_c \Lambda, V) \to C^*(\Lambda, \Lambda^0)^{\gamma^\Lambda}$ by $\phi(T_{(1,\mu),(1,\nu)}) = T_{\mu,\nu}$. Clearly, ϕ is surjective and respects the gauge action, and since $\phi(T_{(1,\lambda)}) = T_\lambda \neq 0$ for all $\lambda \in \Lambda$, then by Theorem 3.5 ϕ is also injective. $\square$

Proposition 8.2 is a well-known property of directed graphs, see [**8**, Theorem 4.6] and [**11**, Proposition 2.8] to name a few, and as such, it is no surprise that it is also true for finitely aligned k-graphs. However, by using the universal property of corner algebras and Theorem 3.5, we obtain a shorter proof.

Example 8.3. For any finitely aligned k-graph Λ, let $G = \mathbf{Z}^k$ and take the degree map as our functor. Then $C^*(\mathbf{Z}^k \times_d \Lambda, V) \cong C^*(\Lambda, \Lambda^0)^{\gamma_d}$ by Proposition 8.2 where $\gamma_d = \gamma^\Lambda$ is the gauge action.

Now we will find a condition for $C^*(\mathbf{Z}^k \times_d \Lambda, V)$ to be full and hence $C^*(\Lambda, \Lambda^0)^{\gamma^\Lambda}$ to be Morita equivalent to the $C^*(\mathbf{Z}^k \times_d \Lambda)$. Recall from

Section 5 that the essential subgraph $E(\Lambda)$ is the largest subgraph of Λ with no sinks or sources. The bi-infinite path space of $E(\Lambda)$ can be identified with the bi-infinite path space Λ^Δ of Λ.

Definition 8.4. A k-graph Λ is *essentially saturated* if the smallest saturated set containing $E(\Lambda)^0$ is Λ^0.

Remark 8.5. Given a finitely aligned k-graph Λ that is essentially saturated, then $\Lambda^\Delta \neq \varnothing$. Also, $E(\Lambda)^0$ must be a hereditary set and hence Λ must have no sources. Finally, by Corollary 4.4, $C^*(E(\Lambda))$ is Morita equivalent to $C^*(\Lambda)$.

Proposition 8.6. *Let Λ be a finitely aligned k-graph and $V = \{0\} \times \Lambda^0$. Then $C^*(\mathbf{Z}^k \times_d \Lambda, V)$ is a full corner of $C^*(\mathbf{Z}^k \times_d \Lambda)$ if and only if Λ is row finite and essentially saturated.*

Proof. Firstly, for any k-graph Λ such that $E(\Lambda) \neq \varnothing$ and any $v \in E(\Lambda)^0$, the set $\Lambda^n v \neq \varnothing$ for all $n \in \mathbf{N}^k$. Hence, if $H(V)$ is the smallest hereditary set containing V, then $\mathbf{N}^k \times E(\Lambda)^0 \subset H(V)$.

Suppose Λ is row finite and essentially saturated. Given any $(n, v) \in \mathbf{N}^k \times \Lambda^0$, then there is a finite exhaustive set $E_v \subset v(\Lambda)E(\Lambda)^0$. Thus, $E_{(n,v)} = \{(n, \lambda) : \lambda \in E_v\} \subset (n, v)(\mathbf{Z}^k \times_d \Lambda)\mathbf{N}^k \times E(\Lambda)^0$ is a finite exhaustive set, and hence $(n, v) \in \Sigma(V)$.

Now let $(z, v) \in \mathbf{Z}^k \times \Lambda^0$ with $z \notin \mathbf{N}^k$, and let $m = z \vee 0$. Since Λ has no sources, the set $v\Lambda^n \neq \varnothing$ is finite exhaustive for all $v \in \Lambda$ and $n \in \mathbf{N}^k$. Thus, the set $E_{(z,v)}\{(z, \lambda) : \lambda \in v\Lambda^{m-z}\}$ is a finite exhaustive set such that $s(E_{(z,v)}) \subset \mathbf{N}^k \times \Lambda^0$, and hence $(z, v) \in \Sigma(V)$. Therefore, $\Sigma(V) = \mathbf{Z}^k \times \Lambda^0$ is full by Corollary 4.4 and hence $C^*(\mathbf{Z}^k \times_d \Lambda, V)$.

Now suppose that $\Sigma(V) = \mathbf{Z}^k \times \Lambda^0$. Then for any $(z, v) \in \mathbf{Z}^k \times \Lambda^0$ with $z < 0$ there exists a finite exhaustive set $E_{\mathbf{Z}^k} \subset (z, v)(\mathbf{Z}^k \times_d \Lambda)V$. This means that for all $n \in \mathbf{N}^k$ there is a finite exhaustive set $E_N \subset v\Lambda$ such that $d(\lambda) \geq N$ for all $\lambda \in E_N$. Hence, $v\Lambda^N$ must be a finite nonempty set for all $v \in \Lambda^0$ and $N \in \mathbf{N}^k$. Hence, Λ must be row finite and have no sources.

Suppose $(n, v) \in \mathbf{N}^k \times \Lambda^0 \setminus H(V)$. Then there is a finite exhaustive set $E_{(n,v)} \subset (n, v)(\mathbf{Z}^k \times_d \Lambda)H(V)$. For all $(m, w) \in s(E_{(n,v)})$ we must

have $\Lambda^m w \neq \varnothing$. Since n can be any value in $\mathbf{N}^k$ and since $m > n$ then $\Lambda^n w \neq \varnothing$ for all $n \in \mathbf{N}^k$. Thus, $w \in E(\Lambda)^0$ since Λ has no sources and therefore Λ is essentially saturated.   □

Corollary 8.7.   *Given a row finite k-graph Λ that is essentially saturated, then the AF core $C^*(\Lambda)^\gamma$ is Morita equivalent to $C^*(\mathbf{Z}^k \times_d \Lambda)$.*

Proof. Follows from Proposition 8.2 and Proposition 8.6.   □

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