ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 38, Number 5, 2008

DYNAMIC BEHAVIOR OF A DELAYED IMPULSIVE SEIRS MODEL IN EPIDEMIOLOGY

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ABSTRACT. A delayed SEIRS epidemic model with pulse vaccination is investigated. Using Krasnoselskii's fixed-point theorem, the infection-free periodic solution is obtained. Some new threshold values $\mathcal{R}_1$, $\mathcal{R}_2$ and $\mathcal{R}_3$ are obtained for dynamic behavior of the solutions. We point out, if $\mathcal{R}_1 < 1$, the infectious population disappear, i.e., the disease dies out, while if $\mathcal{R}_2 > 1$ or $\mathcal{R}_3 > 1$, the infectious permanent, the infectious population will ultimately remain above a positive level. An explicit formula is obtained by which the eventual lower bound of infectious individuals can be computed when $\mathcal{R}_2 > 1$. Our results indicate that a large pulse vaccination rate will have some active effects to prevent or curtail the spread of the disease. Furthermore, we only proved the existence of $\mathcal{R}_3$ based upon some abstract theories.

1. Introduction. The spread of infectious diseases is often described mathematically by compartmental models. A theory of epidemics was derived by Kermack, a chemist, and Mckendrick, a physician, who worked at the Royal College of Surgeons in Edinburgh between 1900 and 1930. They introduced and used many novel mathematical ideas in studies of populations [**6, 13, 17**]. From then on, most of the research literature assumed that the disease latent period is negligible, i.e., once infected, each susceptible individual (in class S) becomes infectious (in class I), instantaneously and later recovers (in class R) with a permanent or temporary acquired immunity. Today, we usually call these compartmental models SIR models or SIRS models with each letter referring to a 'compartment' in which an individual can reside. The SIR and SIRS models have been studied in much liter-

Keywords and phrases. Krasnoselskii's fixed-point theorem, time delay, pulse vaccination, threshold Value.

This work was supported by The National Natural Science Foundation of P.R. China (10361004), The Major Project of The Ministry of Education and The Natural Science Foundation of Xinjiang University.

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Received by the editors on July 14, 2007 and in revised form on January 11, 2008.

DOI:10.1216/RMJ-2008-38-5-1841

ature [**1–3, 7, 10–12, 18, 20, 24, 25**]. In particular, Mena-Lorca and Hethcote [**18**] considered five SIRS epidemiological models for populations of varying size. Thieme [**24**] considered an SIRS epidemiological model with population size dependent upon contact rate and exponential demographics. In [**25**], an SIR epidemic model with a constant removal rate of infective individuals is proposed to understand the affect of limited resources for treatment of the infective on disease spread. However, many diseases (e.g., tuberculosis, measles, AIDS, SARS, etc.) have an incubation period. The disease will incubate inside the host for a period of time before the host becomes infectious. A susceptible individual first goes through a latent period (often called the exposed or in class E) after infection before becoming infectious. The models obtained by the compartmental approach are said to be SEI, SEIS, SEIR and SEIRS, respectively. Earlier global stability results on SEI and SEIR epidemic models without delays are considered in much literature [**15–17, 26**]. For instance, Li [**15**] considered an SEI epidemic model with general contact rate that has an infectious force in both the latent and infected periods. The global dynamics of this model have been completely obtained. Recently, the asymptotic behavior of solutions for an autonomous delayed SEIRS epidemic model with saturation incidence was studied [**26**]. If the basic reproduction number is greater than one, then the disease is permanent. The sufficient conditions for global stability of endemic equilibrium are obtained.

Recently, pulse vaccination epidemic models have been the subject of intense theoretical analysis [**7, 8, 9, 19, 20, 21, 23**]. In [**9**], Gao et al. investigated the following SEIR epidemic model with time delays.

$$\begin{cases} \left.\begin{aligned} S'(t) &= \mu - \mu S(t) - \beta S(t)I(t), \\ E'(t) &= \beta S(t)I(t) - \beta e^{-\mu\omega} S(t-\omega)I(t-\omega) - \mu E(t), \\ I'(t) &= \beta e^{-\mu\omega} S(t-\omega)I(t-\omega) - (\mu + r)I(t), \\ R'(t) &= rI(t) - \mu R(t), \end{aligned}\right\} & t \neq k\tau, \\ \left.\begin{aligned} S(t^+) &= (1-\theta)S(t), \\ E(t^+) &= E(t), \\ I(t^+) &= I(t), \\ R(t^+) &= R(t) + \theta S(t). \end{aligned}\right\} & t = k\tau \end{cases} \quad k \in \mathbf{N}$$

where $\mathbf{N} = \{0, 1, 2, \dots\}$. All parameters in the above model are nonnegative and the detailed meanings can be found in [**9**]. In [**19**], the following SEIRS epidemic model with two profitless delays and nonlinear incidence is proposed.

$$\left\{\begin{array}{l}\left.\begin{array}{l} S'(t) = A - \dfrac{\beta S(t) I(t)}{1+\alpha S(t)} - \mu S(t) - (1-p)\mu I(t) \\ \qquad + rI(t-\omega)e^{-\mu\omega}, \\ E'(t) = \dfrac{\beta S(t) I(t)}{1+\alpha S(t)} - \dfrac{\beta e^{-\mu\tau} S(t-\tau) I(t-\tau)}{1+\alpha S(t-\tau)} \\ \qquad - \mu E(t) + (1-p)\mu I(t), \\ I'(t) = \dfrac{\beta e^{-\mu\tau} S(t-\tau) I(t-\tau)}{1+\alpha S(t-\tau)} - (r+d+\mu) I(t), \\ R'(t) = rI(t) - rI(t-\omega)e^{-\mu\omega} - \mu R(t), \end{array}\right\} \quad \begin{array}{l} t \neq kT, \\ k \in \mathbf{N} \end{array} \\ \left.\begin{array}{l} S(t^+) = (1-\delta) S(t), \\ E(t^+) = E(t), \\ I(t^+) = I(t), \\ R(t^+) = R(t) + \delta S(t), \end{array}\right\} \quad t = kT, \end{array}\right.$$

where all the parameters A, β, μ, α, γ, ω, τ, d are nonnegative and $p \in (0, 1)$. For detailed meanings we refer the reader to [**19**]. For these models in [**8, 9, 19**], there is a common feature: the letter E does not appear in other equations; hence, authors only need to consider the subsystems which do not contain the equation about the exposed E. It is well known that the aim of vaccination only guarantees that individuals who are successfully vaccinated cannot be infected before the bacterin expires. In general, there are different vaccines shown to be reliable for a disease. Each type of vaccine may not show 100 percent efficacy. If the impulsive vaccination is applied every $\tau > 0$ years and $\theta \in (0, 1]$ denotes the proportion of those vaccinated successfully, we can construct the following SEIRS epidemic model with time delays

and bilinear incidence:

$$
\left\{
\begin{array}{l}
\left.
\begin{array}{l}
S'(t) = A - bS(t) - \beta S(t)I(t) + \eta e^{-b\tau} R(t-\tau), \\
E'(t) = \beta S(t)I(t) - (b+\varepsilon)E(t), \\
I'(t) = \varepsilon E(t) - (b+\alpha+\gamma)I(t), \\
R'(t) = \gamma I(t) - bR(t) - \eta e^{-b\tau} R(t-\tau),
\end{array}
\right\} \; t \neq k\tau, \; k \in \mathbf{N} \\
\left.
\begin{array}{l}
S(t^+) = (1-\theta)S(t), \\
E(t^+) = E(t), \\
I(t^+) = I(t), \\
R(t^+) = R(t) + \theta S(t).
\end{array}
\right\} \; t = k\tau,
\end{array}
\right. \tag{1.1}
$$

where $N(t) = S(t) + E(t) + I(t) + R(t)$ denotes the total population at time t; the letters S, E, I and R stand, respectively, for susceptible, exposed, infectious and recovered; A is the constant recruitment rate into the population, b is the natural death rate of the population, β is the average number of adequate contacts of an infectious individual per unit time, ε is the rate constant at which the exposed individuals become infective, so that $1/\varepsilon$ is the mean latent period. Infectious hosts suffer an extra disease-related death with constant rate α, γ is the recovery rate of infectious individuals, η is the rate of losing immunity, $\eta > 0$ implies that the recovered individuals would lose the immunity, $\eta = 0$ implies that the recovered individuals has permanent immunity and τ in the term '$\eta e^{-b\tau} R(t-\tau)$' denotes the immune period of recovered individuals. We see that the immune period was coincident with the interval between two pulses. Here, we give some epidemiological implications: For some successfully vaccinated susceptible individuals, these individuals will gradually lose their immunity after the bacterin is expired, i.e., the period τ. Therefore, τ is also explained as the best times of losing immunity. At time t, the proportion of the susceptible from the recovered individuals should be taken $\eta e^{-b\tau} R(t-\tau)$. All coefficients are positive constants.

The total population size $N(t)$ can be determined by the differential equation

$$
N'(t) = A - bN(t) - \alpha I(t), \tag{1.2}
$$

which is derived by adding the equations in system (1.1). Thus, the total population size may vary in time. From (1.2), we have

$$A-(b+\alpha)N(t)\le N'(t)\le A-bN(t).$$

It follows that

$$\frac{A}{b+\alpha}\le \liminf_{t\to\infty} N(t)\le \limsup_{t\to\infty} N(t)\le \frac{A}{b}.$$

The initial condition of (1.1) is given as

$$\begin{aligned} &S(u)=\phi_1(u), \qquad E(u)=\phi_2(u), \qquad I(u)=\phi_3(u),\\ &R(u)=\phi_4(u), \quad -\tau\le u\le 0 \end{aligned} \tag{1.3}$$

where $\phi=(\phi_1,\phi_2,\phi_3,\phi_4)^T\in C_+$ for all $-\tau\le\theta\le 0$, and C_+ denotes the Banach space $C([-\tau,0],\mathbf{R}^4_+)$ of continuous functions mapping the interval $[-\tau,0]$ into $\mathbf{R}^4_+$ where

$$\mathbf{R}^4_+=\{(x_1,x_2,x_3,x_4)\in\mathbf{R}^4 : x_i\ge 0,\ i=1,2,3,4\}$$

and designates the norm of an element ϕ in C by

$$\|\phi\|=\sup_{-\tau\le u\le 0}\{|\phi_1(u)|,|\phi_2(u)|,|\phi_3(u)|,|\phi_4(u)|\}.$$

By biological meaning, we further assume that $\phi_i(0)>0$ for $i=1,2,3,4$. The domain of system (1.1) is

$$\Omega=\left\{(S,E,I,R)\in\mathbf{R}^4_+ : 0\le S+E+I+R\le\frac{A}{b}\right\},$$

and it is easy to prove that Ω is a positive invariant set. The solution of system (1.1) is a piecewise continuous function $\Phi:\mathbf{R}_+\to\mathbf{R}^4_+$, $\Phi(t)$ is continuous on $(k\tau,(k+1)\tau]$, $k\in\mathbf{N}$ and $\Phi(k\tau^+)=\lim_{t\to k\tau^+}\Phi(t)$ exists. In fact, the righthand side of system (1.1) can ensure the existence and uniqueness of solutions of system (1.1).

The organization of this paper is as follows: In the next section, we will state some lemmas which will be essential to our proofs. Using Krasnoselskii's fixed-point theorem, we establish sufficient conditions

for the existence of infection-free periodic solution. The sufficient conditions for the global attractivity of infection-free periodic solutions is obtained in Section 3. In Section 4, we will discuss the permanence of the disease of model (1.1).

2. Preliminary. The following Krasnoselskii's fixed-point theorem may be a proper vehicle for the proof. It can be found in [**4, 5, 22**].

Lemma 2.1. *Let $\mathcal{B}$ be a closed, convex and nonempty subset of a Banach space $(\mathcal{X}, \|\cdot\|)$. Mappings $T_i : \mathcal{B} \to \mathcal{X}$, $i = 1, 2$, satisfy*:

(1) *$T_1 x + T_2 y \in \mathcal{B}$ for each $x, y \in \mathcal{B}$,*

(2) *T_1 is continuous and compact,*

(3) *T_2 is a contraction.*

Then $T_1 + T_2$ has at least one fixed point on $\mathcal{B}$.

We will use a basic result from Theorem 3.2.1 in [**14**] to obtain the following lemma.

Lemma 2.2. *Consider the following equation*

$$x'(t) = ax(t - \omega) - bx(t),$$

where b, $\omega > 0$; $x(t) > 0$ for $t \in [-\omega, 0]$. We have

(1) *if $|a| < b$, then $\lim_{t \to +\infty} x(t) = 0$,*

(2) *if $a > b$, then $\lim_{t \to +\infty} x(t) = +\infty$.*

Lemma 2.3. *Consider the following impulsive differential equation*

$$\begin{cases} u'(t) = a - bu(t) & t \neq k\tau,\ k \in \mathbf{N} \\ u(t^+) = (1 - \theta)u(t) & t = k\tau \end{cases} \tag{2.1}$$

where $a > 0$, $b > 0$ and $0 < \theta \leq 1$. Then there exists a unique periodic solution of system (2.1) which is globally asymptotically stable.

Lemma 2.4. *Consider the following impulsive delayed differential equation*

$$\begin{cases} u'(t) = a - bu(t) - cu(t-\tau) & t \neq k\tau,\ k \in \mathbf{N} \\ u(t^+) = (1-\theta)u(t) & t = k\tau, \end{cases} \tag{2.2}$$

where $a > 0$, $b > c > 0$ and $0 < \theta \leq 1$. Then system (2.2) exists as a unique nonnegative periodic solution which is globally asymptotically stable.

Proof. Let $\mathcal{X} = C[-\tau, 0]$ be a Banach space of continuous functions mapping the interval $[-\tau, 0]$ into $\mathcal{R}$ with the topology of uniform convergence. For convenience, we designate the norm of an element ϕ in $\mathcal{X}$ by $\|\phi\| = \sup_{-\tau \leq t \leq 0} |\phi(t)|$.

Suppose that system (2.2) has the following initial condition

$$u(t) = \phi(t), \quad \phi \in \mathcal{X}. \tag{2.3}$$

On the interval $0 < t \leq \tau$, the function u is given by

$$u(t) = e^{-bt}\Big((1-\theta)\phi(0) + \int_0^t [a - c\phi(s-\tau)]e^{bs}\,ds\Big). \tag{2.4}$$

If a $\phi \in \mathcal{X}$ exists such that

$$e^{-bt}\Big((1-\theta)\phi(0) + \int_0^t [a - c\phi(s-\tau)]e^{bs}\,ds\Big) = \phi(t-\tau), \quad t \in [0, \tau], \tag{2.5}$$

then system (2.2) has a periodic solution as follows

$$u(t) = \phi(t - (k+1)\tau), \quad k\tau < t \leq (k+1)\tau, \quad k \in \mathbf{N}. \tag{2.6}$$

In fact, we can prove this statement by using induction. We know this relation is true for $k = 0$. Assume that it is true for some $k \geq 0$, i.e.,

$$u(t) = \phi(t - (k+1)\tau), \quad k\tau < t \leq (k+1)\tau. \tag{2.7}$$

Let's derive the solution of (2.2) on $((k+1)\tau, (k+2)\tau]$ from this assumption. From system (2.2) and assumption (2.7), we know that

$$\begin{cases} u'(t) = a - bu(t) - c\phi(t - (k+2)\tau) & (k+1)\tau < t \leq (k+2)\tau \\ u(t^+) = (1-\theta)u((k+1)\tau) = (1-\theta)\phi(0) & t = (k+1)\tau. \end{cases} \tag{2.8}$$

By the variation of constants formula, we obtain

$$\begin{aligned} u(t) &= e^{-bt}\Big((1-\theta)\phi(0)e^{(k+1)\tau} + \int_{(k+1)\tau}^{t} [a - c\phi(s-(k+2)\tau)]e^{bs}\,ds\Big) \\ &= e^{-b(t-(k+1)\tau)}\Big((1-\theta)\phi(0) + \int_{0}^{t-(k+1)\tau} [a - c\phi(s-\tau)]e^{bs}\,ds\Big) \end{aligned}$$

for $(k+1)\tau < t \le (k+2)\tau$. Since $0 < t-(k+1)\tau \le \tau$, by (2.5) we have

$$u(t) = \phi(t-(k+2)\tau), \quad (k+1)\tau < t \le (k+2)\tau.$$

This means exactly that (2.6) holds for $k+1$. The statement about (2.6) is true. Therefore, we only show that integral equation (2.5) has a solution for obtaining a periodic solution of system (2.2).

Take transform $t = \xi - \tau$ and ξ still denoted by t. Then (2.5) becomes

(2.9)
$$e^{-b(t+\tau)}\Big((1-\theta)\phi(0) + \int_{0}^{t+\tau} [a - c\phi(s-\tau)]e^{bs}\,ds\Big) = \phi(t), \quad t \in [-\tau, 0].$$

Obviously, integral equation (2.9) is equivalent to (2.5).

Set $\mathcal{B} = \{\phi \in \mathcal{X} : 0 \le \phi \le a/b\}$ and mappings $T_i : \mathcal{B} \to \mathcal{X}$ where, for any $\phi \in \mathcal{B}$,

$$\begin{aligned} (T_1\phi)(t) &= e^{-b(t+\tau)}(1-\theta)\phi(0), \\ (T_2\phi)(t) &= e^{-b(t+\tau)} \int_{0}^{t+\tau} [a - c\phi(s-\tau)]e^{bs}\,ds. \end{aligned}$$

The existence of the solution of integral equation (2.9) is equivalent whether or not the mapping $T_1 + T_2$ has a fixed point. In the following, we will prove that mapping $T_1 + T_2$ has a fixed point on $\mathcal{B}$ by using Lemma 2.1. It is easy to validate that $\mathcal{B}$ is a closed convex subset of $\mathcal{X}$ and T_1 is continuous and compact by the Ascoli-Areza theorem. Take arbitrarily ϕ_1 and $\phi_2 \in \mathcal{B}$. Since $b > c$, then $0 \le \phi_2(t) \le a/b < a/c$. Thus,

$$\begin{aligned} (T_1\phi_1 + T_2\phi_2)(t) = e^{-b(t+\tau)}\Big((1-\theta)\phi_1(0) \\ + \int_{0}^{t+\tau} [a - c\phi_2(s-\tau)]e^{bs}\,ds\Big) \ge 0 \end{aligned} \tag{2.10}$$

On the other hand, one has

(2.11)
$$\begin{aligned}(T_1\phi_1 + T_2\phi_2)(t) &= e^{-b(t+\tau)}\left((1-\theta)\phi_1(0) + \int_0^{t+\tau} [a - c\phi_2(s-\tau)]e^{bs}\,ds\right)\\ &\le e^{-b(t+\tau)}\left(\frac{a}{b} + \int_0^{t+\tau} ae^{bs}\,ds\right)\\ &= \frac{a}{b}.\end{aligned}$$

Equations (2.10) and (2.11) imply $T_1\phi_1 + T_2\phi_2 \in \mathcal{B}$. To see that T_2 is a contraction, if ϕ_1 and $\phi_2 \in \mathcal{B}$, then

$$\begin{aligned}&\|T_2\phi_1 - T_2\phi_2\|\\ &\quad= \sup_{-\tau\le t\le 0}\left|ce^{-b(t+\tau)}\int_0^{t+\tau}[\phi_2(s-\tau) - \phi_1(s-\tau)]e^{bs}\,ds\right|\\ &\quad\le \|\phi_1 - \phi_2\|\sup_{-\tau\le t\le 0}\left|e^{-b(t+\tau)}\int_0^{t+\tau} ce^{bs}\,ds\right|\\ &\quad= \frac{c}{b}(1 - e^{-b\tau})\|\phi_1 - \phi_2\|.\end{aligned}$$

Therefore, the set $\mathcal{B}$ and mappings T_1 and T_2 satisfy all conditions of Lemma 2.1. Then $T_1 + T_2$ has at least one fixed point on $\mathcal{B}$.

Finally, we will prove that for any two solutions $u_1(t)$ and $u_2(t)$ with initial values ϕ_1 and $\phi_2 \in \mathcal{X}$, respectively, we have

$$\lim_{t\to\infty}(u_1(t) - u_2(t)) = 0. \tag{2.12}$$

Let $x(t) = |u_1(t) - u_2(t)|$. Then $x(t)$ satisfies

$$\begin{cases} D^+x(t) \le -bx(t) + cx(t-\tau) & k\tau < t \le (k+1)\tau,\ k \in \mathbf{N}\\ x(t^+) = (1-\theta)x(t) & t = k\tau.\end{cases} \tag{2.13}$$

Consider the following auxiliary delayed differential system

$$y'(t) = -by(t) + cy(t-\tau). \tag{2.14}$$

Assume $y(t)$ is the solution of system (2.14) with initial function $|\phi_1 - \phi_2|$. Then we claim that

$$0 \le x(t) \le y(t), \quad k\tau < t \le (k+1)\tau, \quad k \in \mathbf{N}. \tag{2.15}$$

We use mathematical induction on the number k to prove statement (2.15). For $k = 0$, i.e., as $0 < t \le \tau$, we obtain by (2.13) and (2.14)

$$x(t) \le (1-\theta)x(0)e^{-bt} + e^{-bt}\int_0^t ce^{bs}|\phi_1(s-\tau) - \phi_2(s-\tau)|\,ds$$
$$y(t) = x(0)e^{-bt} + e^{-bt}\int_0^t ce^{bs}|\phi_1(s-\tau) - \phi_2(s-\tau)|\,ds,$$

which implies that $0 \le x(t) \le y(t)$ as $0 < t \le \tau$

Assume that the assertion has been proven for some $k = n$, i.e., $0 \le x(t) \le y(t)$ for $n\tau < t \le (n+1)\tau$. When $(n+1)\tau < t \le (n+2)\tau$, (2.13) and (2.14) imply that

$$x(t) \le (1-\theta)x((n+1)\tau)e^{-b(t-(n+1)\tau)} + e^{-bt}\int_{(n+1)\tau}^t ce^{bs}x(s-\tau)\,ds,$$
$$y(t) = y((n+1)\tau)e^{-b(t-(n+1)\tau)} + e^{-bt}\int_{(n+1)\tau}^t ce^{bs}y(s-\tau)\,ds.$$

By inductive assumption, $0 \le x(t) \le y(t)$ for $(n+1)\tau < t \le (n+2)\tau$. By Lemma 2.2, the zero solution of system (2.14) is globally uniformly asymptotically stable as $b > c$. That is, $\lim_{t\to\infty} y(t) = 0$ which implies $\lim_{t\to\infty} x(t) = 0$. Thus, (2.12) is valid.

Generally speaking, system (2.2) exists as a unique periodic solution which is globally asymptotically stable, and the lemma is proved.

Remark 2.1. Since the unique nonnegative periodic solution with period τ obtained by Lemma 2.3 depends on a, b, c and θ, we denote this solution by $\Phi(t, a, b, c, \theta)$. We write simply $\Phi(t)$.

Remark 2.2. From Lemma 2.3, we know that $0 \le \Phi \le a/b$. In fact, we can get $\Phi(t) > 0$. Since $b > c$, then $\Phi < a/c$. From (2.9), we have

$$\Phi(t) \ge e^{-b(t+\tau)}\int_0^{t+\tau}[a - c\Phi(s-\tau)]e^{bs}\,ds > 0.$$

Therefore, $\Phi(t)$ is a strictly positive, bounded periodic solution.

3. Global attractivity of IFPS. We firstly demonstrate the existence of an infection-free periodic solution (IFPS for short), in which infectious individuals are entirely absent from the population permanently, i.e., $I(t) \equiv 0$ for all $t \geq 0$. Under this condition, the growth of susceptible, recovered individuals and total population must satisfy the following impulsive system

$$\begin{cases} \left.\begin{aligned} S'(t) &= A - bS(t) + \eta e^{-b\tau} R(t-\tau), \\ E'(t) &= -(b+\varepsilon)E(t), \\ R'(t) &= -bR(t) - \eta e^{-b\tau} R(t-\tau), \end{aligned}\right\} & \begin{aligned} t &\neq k\tau, \\ k &\in \mathbf{N} \end{aligned} \\ \left.\begin{aligned} S(t^+) &= (1-\theta)S(t), \\ E(t^+) &= E(t), \\ R(t^+) &= R(t) + \theta S(t) \end{aligned}\right\} & \begin{aligned} t &= k\tau, \\ k &\in \mathbf{N}. \end{aligned} \end{cases} \tag{3.1}$$

From the second and the fifth equations of (3.1), we easily obtain $\lim_{t\to\infty} E(t) = 0$. Further, if $I(t) \equiv 0$, it follows from (1.2) that $\lim_{t\to\infty} N(t) = A/b$. So, $S(t) + R(t) \to (A/b)$ as $t \to \infty$. Therefore, we have the following limit system of (3.1)

(3.2)

$$\begin{cases} S'(t) = A\left(1 + \dfrac{\eta e^{-b\tau}}{b}\right) - bS(t) - \eta e^{-b\tau} S(t-\tau), & k\tau < t \leq (k+1)\tau, \\ & k \in n \\ S(t^+) = (1-\theta)S(t) & t = k\tau. \end{cases}$$

According to Lemma 2.4, we know that if $b > \eta e^{-b\tau}$, then system (3.2) has a unique positive periodic solution with period τ which is globally asymptotically stable. We denote this periodic solution by $\widetilde{S}_e(t)$.

Regarding the global attractivity of IFPS $(\widetilde{S}_e(t), 0, 0, (A/b) - \widetilde{S}_e(t))$, we have the following result.

Theorem 3.1. *If $b > \eta e^{-b\tau}$ and $\mathcal{R}_1 < 1$, then the infection-free periodic solution $(\widetilde{S}_e(t), 0, 0, (A/b) - \widetilde{S}_e(t))$ of system* (1.1) *is globally attractive on Ω, where*

$$\mathcal{R}_1 = \frac{\beta\varepsilon A(b + \eta e^{-b\tau})(1 - e^{-b\tau})}{b^2(b+\varepsilon)(b+\alpha+\gamma)(1-(1-\theta)e^{-b\tau})}.$$

Proof. Since $\mathcal{R}_1 < 1$, we can choose r_1, $r_2 > 0$ and sufficiently small $\epsilon_0 > 0$ such that

$$\frac{\beta\xi}{b+\alpha+\gamma} < \frac{r_2}{r_1} < \frac{b+\varepsilon}{\varepsilon}, \tag{3.3}$$

where $\xi = (A(b+\eta e^{-b\tau})(1-e^{-b\tau})/b^2(1-(1-\theta)e^{-b\tau})) + \epsilon_0$.

It follows from the first equation of system (1.1) that $S'(t) \le A + \eta e^{-b\tau}(A/b) - bS(t)$. Thus, we consider the following comparison impulsive differential system:

$$\begin{cases} x'(t) = A + \eta e^{-b\tau}(A/b) - bx(t) & t \ne k\tau,\ k \in \mathbf{N} \\ x(t^+) = (1-\theta)x(t) & t = k\tau. \end{cases} \tag{3.4}$$

By Lemma 2.3, we see that the periodic solution of system (3.4) is

$$\tilde{x}_e(t) = \Gamma + (x^* - \Gamma)e^{-b(t-k\tau)}, \quad k\tau < t \le (k+1)\tau,$$

which is globally asymptotically stable, where

$$\Gamma = \frac{A(b+\eta e^{-b\tau})}{b^2}, \quad x^* = \Gamma\frac{(1-\theta)(1-e^{-b\tau})}{1-(1-\theta)e^{-b\tau}}.$$

Let $(S(t), E(t), I(t), R(t))$ be a solution of system (1.1) with initial value (1.3) and $S(0^+) = S_0 > 0$ and let $x(t)$ be the solution of system (3.4) with initial value $x(0^+) = S_0$. By the comparison theorem of impulsive differential equations, an integer $k_1 > 0$ exists such that

$$S(t) \le x(t) < \tilde{x}_e(t) + \epsilon_0, \quad k\tau < t \le (k+1)\tau, \quad k > k_1. \tag{3.5}$$

Thus,

$$\begin{gathered} S(t) < \tilde{x}_e(t) + \epsilon_0 \le \frac{A(b+\eta e^{-b\tau})(1-e^{-b\tau})}{b^2(1-(1-\theta)e^{-b\tau})} + \epsilon_0 = \xi, \\ k\tau < t \le (k+1)\tau, \quad k > k_1. \end{gathered} \tag{3.6}$$

Let us consider the following continuous function

$$V(t) = r_1 E(t) + r_2 I(t). \tag{3.7}$$

We see that, for $t > k_1\tau$, the time derivative of $V(t)$ along the solutions of system (1.1) satisfies

(3.8)
$$\begin{aligned} V'(t) &= r_1\beta S(t)I(t) - r_1(b+\varepsilon)E(t) + r_2\varepsilon E(t) - r_2(b+\alpha+\gamma)I(t) \\ &\le [r_2\varepsilon - r_1(b+\varepsilon)]E(t) + [r_1\beta\xi - r_2(b+\alpha+\gamma)]I(t) \\ &\le -\rho V(t) \end{aligned}$$

where

$$\rho = \min\left\{\frac{r_1(b+\varepsilon) - r_2\varepsilon}{r_1},\ \frac{r_2(b+\alpha+\gamma) - r_1\beta\xi}{r_2}\right\} > 0.$$

Equation (3.8) implies $V(t) \to 0$ as $t \to \infty$. By the nonnegativity of $E(t)$ and $I(t)$, we have

$$\lim_{t\to\infty} E(t) = 0, \qquad \lim_{t\to\infty} I(t) = 0. \tag{3.9}$$

Therefore, for any sufficiently small $\epsilon_1 \in (0,1)$, there exists an integer $k_2 > k_1$ satisfying $E(t) < \epsilon_1$ and $I(t) < \epsilon_1$ for all $t > k_2\tau$.

From system (1.2), we have

$$N'(t) \ge A - bN(t) - \alpha\epsilon_1, \quad t > k_2\tau. \tag{3.10}$$

Consider the following comparison system

$$z'(t) = (A - \alpha\epsilon_1) - bz(t), \quad t > k_2\tau.$$

It is easy to obtain $\lim_{t\to\infty} z(t) = (A - \alpha\epsilon_1)/b$. By the comparison theorem, there is an integer $k_3 > k_2$ such that

$$N(t) \ge \frac{A - \alpha\epsilon_1}{b} - \epsilon_1 \tag{3.11}$$

for all $t > k_3\tau$ because ϵ_1 can be arbitrarily small and $\limsup_{t\to\infty} N(t) \le (A/b)$. Hence,

$$\lim_{t\to\infty} N(t) = \frac{A}{b}. \tag{3.12}$$

From (3.12), an integer $k_4 \ge k_3$ exists such that

$$N(t) > \frac{A}{b} - \epsilon_1 \tag{3.13}$$

for all $t > k_4\tau$.

Let $W(t) = |S(t) - \widetilde{S}_e(t)|$. Therefore, between two pulses

$$\begin{aligned} D^+W(t) &\le \left|\eta e^{-b\tau}\left(N(t-\tau) - \frac{A}{b}\right)\right| \\ &\quad + \beta S(t)I(t) + \eta e^{-b\tau}(E(t-\tau) + I(t-\tau)) \\ &\quad - bW(t) + \eta e^{-b\tau}W(t-\tau) \\ &\le K\epsilon_1 - bW(t) + \eta e^{-b\tau}W(t-\tau) \end{aligned}$$

for $t > (k_4+1)\tau$, where $K = \beta(A/b) + 3\eta e^{-b\tau}$. When $t = k\tau$, $W(t^+) = (1-\theta)W(t)$.

Consider the following delayed differential equations for $t \ge k_4\tau + \tau$,

$$w'(t) = K\epsilon_1 - bw(t) + \eta e^{-b\tau}w(t-\tau). \tag{3.14}$$

By Lemma 2.2, for any fixed solution $w(t)$ of (3.14), one has $\lim_{t\to\infty} w(t) = (K\epsilon_1/b - \eta e^{-b\tau})$. Furthermore, it is similar to (2.15) that $0 \le W(t) \le w(t)$ for all $t \ge k_4\tau + \tau$. Here $w(t)$ is the solution of (3.14) with initial condition $w(t) = |S(t) - \widetilde{S}_e(t)|$ for $t \in ((k_4+1)\tau, (k_4+2)\tau]$. Thus, there exists an integer $k_5 > k_4$ such that

$$0 \le W(t) \le \frac{K\epsilon_1}{b - \eta e^{-b\tau}} + \epsilon_1 \tag{3.15}$$

for all $t \ge k_5\tau$. Because ϵ_1 can be arbitrarily small, it follows from (3.15) that

$$\lim_{t\to\infty} S(t) = \widetilde{S}_e(t). \tag{3.16}$$

Finally, it follows from (3.9), (3.12) and (3.16) that the infection-free periodic solution $(\widetilde{S}_e(t), 0, 0, (A/b) - \widetilde{S}_e(t))$ of system (1.1) is globally attractive. The proof of Theorem 3.1 is completed. □

According to Theorem 3.1, we can easily obtain the following result.

Corollary 3.1. *Assume* $b > \eta e^{-b\tau}$. *The infection-free periodic solution* $(\widetilde{S}_e(t), 0, 0, (A/b) - \widetilde{S}_e(t))$ *of system* (1.1) *is globally attractive provided that* $\theta > \theta^*$, *where*

$$\theta^* = 1 - \frac{b^2(b+\varepsilon)(b+\alpha+\gamma)e^{b\tau} - \beta\varepsilon A(b+\eta e^{-b\tau})(e^{b\tau}-1)}{b^2(b+\varepsilon)(b+\alpha+\gamma)}.$$

Remark 3.1. Theorem 3.1 determines the global attractivity of the IFPS of system (1.1) on Ω for case $\mathcal{R}_1 < 1$. Its epidemiological implication is that the infectious population vanishes, i.e., the disease dies out. Corollary 3.1 implies that the disease will disappear if the pulse vaccination rate is larger than θ^*.

4. Permanence of disease. In this section, we say the disease becomes endemic if the infectious population persists above certain positive level for a long period.

Definition 4.1. In system (1.1), the disease is said to be permanent if there is a positive constant q such that $\liminf_{t\to+\infty} I(t) > q$ holds for any positive solution $(S(t), E(t), I(t), R(t))$ of system (1.1) with initial condition (1.3).

Denote two quantities

$$\mathcal{R}_2 = \frac{\beta\varepsilon A(1-\theta)(1-e^{-b\tau})}{b(b+\varepsilon)(b+\alpha+\gamma)(1-(1-\theta)e^{-b\tau})} \tag{4.1}$$

and

$$I^* = \frac{b}{\beta}(\mathcal{R}_2 - 1). \tag{4.2}$$

Theorem 4.1. *If* $\mathcal{R}_2 > 1$, *then there exists a positive constant* q *such that every positive solution* $(S(t), E(t), I(t), R(t))$ *of system* (1.1) *satisfies* $I(t) \geq q$ *for* t *large enough.*

Proof. Since $\mathcal{R}_2 > 1$, we easily see that $I^* > 0$. Then there exists a sufficiently small $\epsilon > 0$ such that

$$\frac{\beta\varepsilon\delta}{(b+\varepsilon)(b+\alpha+\gamma)} > 1 \tag{4.3}$$

where

$$\delta = \frac{A(1-\theta)(1-e^{-(b+\beta I^*)\tau})}{(b+\beta I^*)(1-(1-\theta)e^{-(b+\beta I^*)\tau})} - \epsilon.$$

By (4.3), there exist r_1 and $r_2 > 0$ such that

$$\frac{\beta\delta}{b+\alpha+\gamma} > \frac{r_2}{r_1} > \frac{b+\varepsilon}{\varepsilon}. \tag{4.4}$$

We claim that it is impossible that $I(t) \le I^*$ for all $t \ge t_0 (t_0$ is any nonnegative number). Suppose the contrary. Then, as $t \ge t_0$

$$\begin{aligned} S'(t) &= A - bS(t) - \beta S(t)I(t) + \eta e^{-b\tau} R(t-\tau) \\ &\ge A - (b+\beta I^*)S(t). \end{aligned}$$

Consider the following comparison system

$$\begin{cases} v'(t) = A - (b+\beta I^*)v(t) & t \ne k\tau,\ k \in \mathbf{N} \\ v(t^+) = (1-\theta)v(t) & t = k\tau. \end{cases} \tag{4.5}$$

By Lemma 2.3, we obtain

$$\tilde{v}_e(t) = \frac{A}{b+\beta I^*} + \left(v^* - \frac{A}{b+\beta I^*}\right) e^{-(b+\beta I^*)(t-k\tau)}, \quad k\tau < t \le (k+1)\tau$$

is the unique globally asymptotically stable periodic solution of system (4.5). Here

$$v^* = \frac{A(1-\theta)(1-e^{-(b+\beta I^*)\tau})}{(b+\beta I^*)(1-(1-\theta)e^{-(b+\beta I^*)\tau})}.$$

Thus, there exists a $T_1 > 0$ satisfying

$$S(t) > \tilde{v}_e(t) - \epsilon \ge v^* - \epsilon = \delta \tag{4.6}$$

for all $t \ge t_0 + T_1$.

Let us define

$$V(t) = r_1 E(t) + r_2 I(t).$$

Then, along the solutions of (1.1), we have

$$\begin{aligned} V'(t) &= r_1\beta S(t)I(t) - r_1(b+\varepsilon)E(t) + r_2\varepsilon E(t) - r_2(b+\alpha+\gamma)I(t) \\ &\ge [r_2\varepsilon - r_1(b+\varepsilon)]E(t) + [r_1\beta\delta - r_2(b+\alpha+\gamma)]I(t) \\ &\ge \rho V(t) \end{aligned} \tag{4.7}$$

for all $t \geq t_0 + T_1$, where

$$\rho = \min\left\{\frac{r_1(b+\varepsilon) - r_2\varepsilon}{r_1},\ \frac{r_1\beta\delta - r_2(b+\alpha+\gamma)}{r_2}\right\} > 0. \tag{4.8}$$

This implies $V(t) \to +\infty$ as $t \to +\infty$. This is contrary to $V(t) \leq (r_1 + r_2)(A/b)$ for large enough t. Hence, the claim is proved. □

From the claim, we will discuss the following two possibilities.

(i) $I(t) \geq I^*$ for all large t;

(ii) $I(t)$ oscillates about I^* for all large t.

Since $\rho > 0$, we can choose a large enough $T_2 > 0$ satisfying $r_2 I^* e^{\rho T_2 - (b+\alpha+\gamma)T_1} > (r_1 + r_2)(A/b)$. Finally, we will show that $I(t) \geq I^* e^{-(b+\alpha+\gamma)(T_1+T_2)} \triangleq q$ as t is sufficiently large. Evidently, we only need consider case (ii). Let t_1 and t_2 be sufficiently large times satisfying

$$I(t_1) = I(t_2) = I^*,$$
$$I(t) < I^* \text{ as } t \in (t_1, t_2).$$

In the following, we will show that $t_2 - t_1 \leq T_1 + T_2$. In fact, assume that $t_2 - t_1 > T_1 + T_2$. Thus, proceeding exactly as the proof for the above claim, we see that $S(t) > \delta$ for all $t \in [t_1 + T_1, t_2]$. Further, $I'(t) \geq -(b+\alpha+\gamma)I(t)$ and $I(t_1) = I^*$ imply $I(t_1 + T_1) \geq I^* e^{-(b+\alpha+\gamma)T_1}$. By (4.7), we know that $V'(t) \geq \rho V(t)$ for all $t \in [t_1 + T_1, t_2]$. Integrating from $t_1 + T_1$ to t_2, we obtain

$$\begin{aligned} V(t_2) &\geq V(t_1 + T) e^{\rho(t_2 - t_1 - T_1)} \geq r_2 I(t_1 + T) e^{\rho T_2} \\ &\geq r_2 I^* e^{\rho T_2 - (b+\alpha+\gamma)T_1} > (r_1 + r_2)\frac{A}{b}. \end{aligned}$$

This is a contradiction with $V(t) \leq (r_1 + r_2)(A/b)$. So, $I'(t) \geq -(b+\alpha+\gamma)I(t)$, $I(t_1) = I^*$ and $t_2 - t_1 \leq T_1 + T_2$ can derive $I(t) \geq q$ for all $t \in [t_1, t_2]$. The proof of Theorem 4.1 is completed. □

Denote

$$\theta_* = 1 - \frac{b(b+\varepsilon)(b+\alpha+\gamma)e^{b\tau}}{b(b+\varepsilon)(b+\alpha+\gamma) + \beta\varepsilon A(e^{b\tau} - 1)}$$

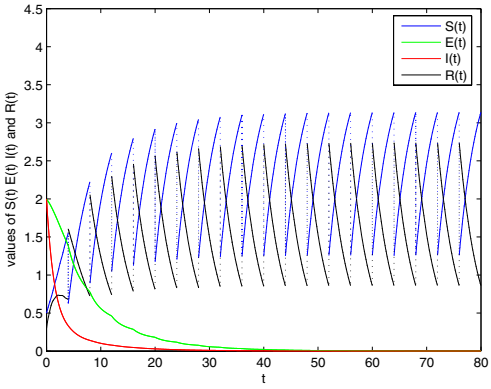

FIGURE 1. Movement paths of S and I as functions of time t. Here, the initial functions are $\phi_1(u) = 0.5 + 0.5\sin(2\pi u)$, $\phi_2(u) = 1 + \cos(2\pi u)$, $\phi_3(u) = 2$ and $\phi_4(u) = 0.3 + 0.3\sin(2\pi u)$ for $u \in [-4, 0]$. The infection-free periodic solution is GA.

and

$$\tau_* = -\frac{1}{b}\ln\left[\frac{\beta\varepsilon A(1-\theta) - b(b+\varepsilon)(b+\alpha+\gamma)}{(1-\theta)(\beta\varepsilon A - b(b+\varepsilon)(b+\alpha+\gamma))}\right].$$

Corollary 4.1. *Assume that $\theta < \theta_*$ or $\tau > \tau_*$. Then the disease is permanent.*

Remark 4.1. If we set $A = 0.8$, $b = 0.2$, $\beta = 0.5$, $\varepsilon = 0.1$, $\alpha = 0.2$, $\gamma = 0.3$, $\eta = 0.2$, $\theta = 0.6$ and $\tau = 4$, then $\mathcal{R}_1 = 0.9266 < 1$. According to Theorem 3.1, we know that the disease will disappear (see Figure 1). If we set $A = 1$, $b = 0.2$, $\beta = 0.6$, $\varepsilon = 0.1$, $\alpha = 0.2$, $\gamma = 0.1$, $\eta = 0.1$, $\theta = 0.2$ and $\tau = 4$, then $\mathcal{R}_2 = 1.1004$. According to Theorem 4.1, the disease will be permanent (see Figure 2). Computer observation validates our theoretical result.

Let $L = \min_{[-\tau,0]}\{\widetilde{S}_e(t)\}$, where $\widetilde{S}_e(t)$ is as in Theorem 3.1. By Remark 2.2, we have $L > 0$.

Theorem 4.2. *If $b > \eta e^{-b\tau}$ and $\mathcal{R}_3 > 1$, then there exists a positive constant q such that every positive solution $(S(t), E(t), I(t), R(t))$ of*

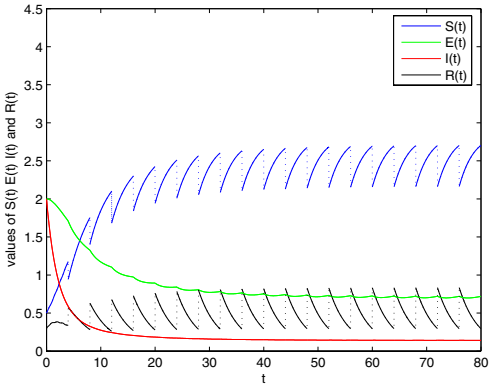

FIGURE 2. Movement paths of S, E, I and R as functions of time t. Here, the initial functions are $\phi_1(u) = 0.5 + 0.5\sin(2\pi u)$, $\phi_2(u) = 1 + \cos(2\pi u)$, $\phi_3(u) = 2$ and $\phi_4(u) = 0.3 + 0.3\sin(2\pi u)$ for $u \in [-4, 0]$. The disease is permanent.

system (1.1) *satisfies* $I(t) \geq q$ *for* t *large enough. Here,*

$$\mathcal{R}_3 = \frac{\beta\varepsilon L}{(b+\varepsilon)(b+\alpha+\gamma)}. \tag{4.9}$$

Proof. Since $\mathcal{R}_3 > 1$, then there exists a small enough $\epsilon > 0$ such that

$$\frac{\beta\varepsilon}{(b+\varepsilon)(b+\alpha+\gamma)}(L - K_0\epsilon) > 1,$$

where $K_0 = 1 + (\beta(A/b) + \eta e^{-b\tau}(3 + (\alpha/b) + (\beta A/b^2))/b - \eta e^{-b\tau})$.

We claim that it is impossible that $I(t) \leq \epsilon$ for all $t \geq t_0$ (t_0 is any nonnegative number). Suppose the contrary. Then, as $t \geq t_0$, the second equations of (1.1) and (1.2) imply that

$$E'(t) \leq \beta\frac{A}{b}\epsilon - bE(t), N'(t) \geq A - bN(t) - \alpha\epsilon.$$

From the above differential inequalities, there exists a $T_0 > 0$ such that

$$E(t) \leq \frac{\beta A}{b^2}\epsilon + \epsilon, \quad N(t) \geq \frac{A}{b} - \left(1 + \frac{\alpha}{b}\right)\epsilon$$

for all $t \geq t_0 + T_0$. Set $M(t) = |S(t) - \widetilde{S}_e(t)|$. Similarly to (3.15), there is a $T_1 \geq T_0$ satisfying

$$M(t) \leq K_0\epsilon \tag{4.10}$$

for all $t \geq t_0 + T_1$. Equation (4.10) implies that

$$S(t) \geq \widetilde{S}_e(t) - K_0\epsilon \geq L - K_0\epsilon$$

as $t \geq t_0 + T_1$. The rest of the proof is completely analogical to Theorem 4.1. We omit it, i.e., we can easily obtain $I(t) \geq \epsilon e^{-(b+\alpha+\gamma)(T_1+T_2)} \triangleq q > 0$. Here, T_2 can be chosen similarly to the proof in Theorem 4.1.

Conclusion. We have studied the dynamical behavior of a delayed SEIRS epidemic model with pulse vaccination and saturation incidence rate. We introduced some thresholds $\mathcal{R}_1$ and $\mathcal{R}_2$ (see Theorems 3.1 and 4.1), and we further obtained that if $\mathcal{R}_1 < 1$ then the disease will be extinct. If $\mathcal{R}_2 > 1$, then the disease will be permanent which means that after some period of time the disease will become endemic. Corollaries 3.1 and 4.1 show that $\theta > \theta^*$ implies the disease will fade out, whereas $\theta < \theta_*$ or $\tau > \tau_*$ implies that the disease will be uniformly persistent. Our results indicate that a large pulse vaccination rate will lead to eradication of the disease. Unfortunately, we cannot give the explicit formula of L. Only from the mathematical point of view, we prove the permanence of the disease as $\mathcal{R}_3 > 1$. Furthermore, we cannot establish the comparison of $\mathcal{R}_3$ and $\mathcal{R}_1$, $\mathcal{R}_2$.

In this paper, we have discussed two cases: (1) $\mathcal{R}_1 < 1$ (or $\theta > \theta^*$), (2) $\mathcal{R}_2 > 1$ (or $\theta < \theta_*$). Obviously, $\mathcal{R}_2 \leq \mathcal{R}_1$. When $\mathcal{R}_2 \leq 1 \leq \mathcal{R}_1$, the dynamical behavior of model (1.1) was not clear. For the pulse vaccination rate between θ_* and θ^*, the extinction and uniform persistence of the disease has not been obtained. This work will be left to our future consideration.

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