

**NECESSARY AND SUFFICIENT CONDITIONS FOR
GLOBAL ATTRACTIVITY OF HOPFIELD-TYPE
NEURAL NETWORKS WITH TIME DELAYS**

SHANGGUO ZHANG, WANBIAO MA AND YANG KUANG

ABSTRACT. In this paper, two classes of two- and three-dimensional Hopfield-type neural networks with time delays are considered by using a completely different method from known results. Some necessary and sharper sufficient criteria for the global attractivity of equilibria of the neural networks are presented.

1. Introduction. It is well known that early study on stability of Hopfield-type neural networks mainly dealt with ordinary differential equation models (see, for example, [11, 12]) in which the updating and propagation are assumed to occur *instantaneously*. However, strictly speaking, the integration and communication delays are ubiquitous both in biological and in artificial neural networks. Hence, investigation on the stability of neural networks with time delays has attracted considerable interest in recent years (see, for example, [1–9, 19, 20] and [14–18, 21–25]).

Hopfield-type neural networks have a broad spectrum of application in various optimization, associative memories, and engineering problems (see, for example, [11, 12]). As is known, engineering applications of neural networks, such as optimization and association, rely crucially on the dynamical behaviors of neural networks. Therefore, qualitative analysis of neurodynamics is indispensable for the practical design of neural network models and tools. When neural networks are applied as associative memories, the equilibria represent the stored patterns, and their stability means that the stored patterns can be retrieved even in the presence of noise. When applied as optimization solvers, the equilibria of the networks characterize all possible optimal solutions of the

2000 AMS *Mathematics subject classification.* Primary 34K20, 92B20.

Keywords and phrases. Hopfield-type neural networks, time delays, global attractivity.

The research is partially supported by NNSF of China (10671011).

Received by the editors on September 17, 2007, and in revised form on January 14, 2008.

DOI:10.1216/RMJ-2008-38-5-1829 Copyright ©2008 Rocky Mountain Mathematics Consortium

optimization problems, and the stability of the networks then ensures the convergence to the optimal solution.

Generally speaking, theoretical analysis on global dynamical properties (such as global asymptotic stability of equilibria or periodic solutions) of neural networks with time delays is still a difficult but a challenging problem in both biology and mathematics. It is known from numerous known results (see, for example, [1–9, 19, 20] and [14–18, 21–25]) that the classic method of Liapunov functions or functionals still plays an important role. On the other hand, the construction of Liapunov functions or functionals for a given system is usually very skillful and complicated. As a result, for a given system, different Liapunov functions or functionals usually result in different stability criteria. In fact, even for some lower-dimensional systems, to give any necessary or sharper sufficient criterion for global asymptotic properties is still a very important problem.

The purpose of the paper is to try to develop a completely different method for analysis of global attractivity of equilibria of two classes of lower dimensional Hopfield-type neural networks with time delays, and to give some necessary and shaper sufficient conditions. Our results show that time delays are actually *harmless* for global asymptotic properties of systems to be considered.

This paper is organized as follows. In the following section, we shall give some preliminaries which include a description of an n -dimensional Hopfield-type neural network with time delays and an important lemma associated with the Hopfield-type neural network with time delays. In Section 3, by using some computational techniques in matrix theory, we shall give some necessary and sufficient conditions for global attractivity of two- and three-dimensional Hopfield-type neural networks with time delays. Our results show that time delays are harmless for the global attractivity of systems to be considered. In the last section, we shall give a numerical simulation example, which shows that our result can be applied to more general systems and that the sufficient condition in Section 3 may also be necessary.

2. Preliminaries. We consider the following Hopfield-type neural network with time delays,

$$(1) \quad \dot{u}_i(t) = -b_i u_i(t) + \sum_{j=1}^n a_{ij} f_j(u_j(t - \tau_{ij})), \quad i = 1, 2, \dots, n$$

for $t \geq 0$, where b_i and a_{ij} are real constants and the time delays τ_{ij} are nonnegative.

Based on some well-known biological meanings, it is assumed that the following conditions are satisfied for (1):

(H₁) $b_i > 0$ and $b_i \geq |a_{ii}|$.

(H₂) $f_i(0) = 0$ and $f_i(u)$ saturates at ± 1 for any $u \in R$, i.e., $\lim_{u \rightarrow \pm\infty} f_i(u) = \pm 1$.

(H₃) $f'_i(u)$ is continuous such that $f'_i(u) > 0$ for any $u \in R$, $f'_i(0) = 1$ and $0 < f_i(u) < u$ for any $u > 0$, where

$$\bar{f}_i(u) = \max \{f_i(u), -f_i(-u)\}, \quad u \geq 0, i = 1, 2, \dots, n.$$

As usual, the initial condition for (1) is given as $u_i(s) = \phi_i(s)$, $-r \leq s \leq 0$, $i = 1, 2, \dots, n$, where $r = \max\{\tau_{ij} \mid i, j = 1, 2, \dots, n\}$, $\phi_i(s)$, $i = 1, 2, \dots, n$, are continuous on $[-r, 0]$. Note that from Lemma 1 below, it is not difficult to show that under (H₁)–(H₃), the solution of (1) satisfying the above initial condition is existent and unique on $R_+ \equiv [0, +\infty)$ (see, for example, [13]).

It is clear that (1) always has an equilibrium $u = (0, 0, \dots, 0)^T$, i.e., (1) has the trivial solution $u_i(t) = 0$, $i = 1, 2, \dots, n$, for $t \geq -\tau$.

With the same arguments as in [16], we have the following lemma which plays a very important role in the paper.

Lemma 1. *If (H₁)–(H₃) are satisfied, then for any solution $(u_1(t), \dots, u_n(t))^T$ of (1), it follows that $\limsup_{t \rightarrow +\infty} |u_i(t)| \leq M_i$, $i = 1, 2, \dots, n$, where the nonnegative constants M_i satisfy*

$$M_i = \frac{1}{b_i} \sum_{j=1}^n |a_{ij}| \bar{f}_j(M_j), \quad i = 1, 2, \dots, n.$$

Proof. First, we have from (H₂)–(H₃) and (1) that, for $t \geq 0$,

$$D^+ |u_i(t)| \leq -b_i |u_i(t)| + \sum_{j=1}^n |a_{ij}|, \quad i = 1, 2, \dots, n,$$

from which it is easily seen that $\limsup_{t \rightarrow +\infty} |u_i(t)| \leq M_{i0} = (1/b_i) \sum_{j=1}^n |a_{ij}|$, $i = 1, 2, \dots, n$. Thus, for any sufficiently small constant $\eta > 0$, there exists sufficiently large time $T_0 = T_0(\eta) > 0$ such that for $t \geq T_0$, it has $|u_i(t-r)| \leq M_{i0} + \eta$, $i = 1, 2, \dots, n$, which, together with (H₃) and (1), yield that for $t \geq T_0$,

$$D^+ |u_i(t)| \leq -b_i |u_i(t)| + \sum_{j=1}^n |a_{ij}| \bar{f}_j(M_{j0} + \eta), \quad i = 1, 2, \dots, n.$$

Note that one can take $\eta \rightarrow 0$ as $t \rightarrow +\infty$. We also have that $\limsup_{t \rightarrow +\infty} |u_i(t)| \leq M_{i1}$, $i = 1, 2, \dots, n$, where

$$M_{i1} = \frac{1}{b_i} \sum_{j=1}^n |a_{ij}| \bar{f}_j(M_{j0}) \leq M_{i0}, \quad i = 1, 2, \dots, n.$$

By repeating the above procedure, we can obtain the positive and decreasing sequences $\{M_{ik}\}$, $i = 1, 2, \dots, n$, such that for $k = 0, 1, 2, \dots$, it has $\limsup_{t \rightarrow +\infty} |u_i(t)| \leq M_{ik}$, $i = 1, 2, \dots, n$, and

$$M_{ik+1} = \frac{1}{b_i} \sum_{j=1}^n |a_{ij}| \bar{f}_j(M_{jk}), \quad i = 1, 2, \dots, n.$$

Let M_i , $i = 1, 2, \dots, n$, denote the limits of $\{M_{ik}\}$, $i = 1, 2, \dots, n$, as $k \rightarrow +\infty$, respectively. Thus, we have that $\limsup_{t \rightarrow +\infty} |u_i(t)| \leq M_i$, $i = 1, 2, \dots, n$, and

$$M_i = \frac{1}{b_i} \sum_{j=1}^n |a_{ij}| \bar{f}_j(M_j), \quad i = 1, 2, \dots, n.$$

This completes the proof of Lemma 1. \square

3. Main results. In this section, we shall give necessary and sufficient conditions for global attractivity of equilibria of two- and three-dimensional Hopfield neural networks with time delays.

First, for global attractivity of the equilibrium $(0, 0)^T$ of (1) for the case of $n = 2$, it has the following

Theorem 1. *The equilibrium $(0, 0)^T$ of (1) for $n = 2$ is globally attractive for any time delays $\tau_{ij} \geq 0$, $i, j = 1, 2$, if and only if the following inequality holds,*

$$(2) \quad (b_1 - |a_{11}|)(b_2 - |a_{22}|) - |a_{12}a_{21}| \geq 0.$$

Proof. (Sufficiency). From Lemma 1, we only need to show that $M_i = 0$, $i = 1, 2$.

In fact, if there are $M_1 = 0$ and $M_2 > 0$, from Lemma 1 and (H_3) it can be seen that $|a_{22}|/b_2 = M_2/f_2(M_2) > 1$, which contradicts $|a_{22}| \leq b_2$. If $M_1 > 0$ and $M_2 = 0$ hold, it is also a contradiction to $|a_{11}| \leq b_1$.

If $M_1 > 0$ and $M_2 > 0$, from Lemma 1 and (H_3) it can be seen that $b_i M_i < \sum_{j=1}^2 |a_{ij}| M_j$, $i = 1, 2$, which is equivalent to the following inequalities

$$(3) \quad \begin{cases} d_1 = (b_1 - |a_{11}|)M_1 - |a_{12}|M_2 < 0, \\ d_2 = -|a_{21}|M_1 + (b_2 - |a_{22}|)M_2 < 0. \end{cases}$$

It is clear from the inequalities (3) that $b_1 - |a_{11}|$ and $|a_{12}|$ cannot be zero simultaneously.

If $b_1 > |a_{11}|$, from (3) it can be seen that

$$((b_2 - |a_{22}|) - |a_{21}||a_{12}|/(b_1 - |a_{11}|))M_2 = d_2 + |a_{21}|d_1/(b_1 - |a_{11}|) < 0,$$

which implies that

$$((b_1 - |a_{11}|)(b_2 - |a_{22}|) - |a_{21}||a_{12}|)M_2 = (b_1 - |a_{11}|)d_2 + |a_{21}|d_1 < 0.$$

If $|a_{12}| > 0$, it can be seen from (3) that

$$((b_1 - |a_{11}|)(b_2 - |a_{22}|)/|a_{12}| - |a_{21}|)M_1 = d_2 + (b_2 - |a_{22}|)d_1/|a_{12}| < 0,$$

which implies that

$$((b_1 - |a_{11}|)(b_2 - |a_{22}|) - |a_{21}||a_{12}|)M_1 = (b_2 - |a_{22}|)d_1 + |a_{12}|d_2 < 0.$$

Hence, it follows from (2) that the above inequalities are not true. This shows that $M_1 = M_2 = 0$. Therefore, the equilibrium $(0, 0)^T$ of (1) for $n = 2$ is globally attractive for any time delays $\tau_{ij} \geq 0$, $i, j = 1, 2$.

(*Necessity*). Note that from assumption (H_3) , the linearized system of (1) at the equilibrium $(0, 0)^T$ is of the following form,

$$(4) \quad \dot{u}_i(t) = -b_i u_i(t) + \sum_{j=1}^2 a_{ij} u_j(t - \tau_{ij}), \quad i = 1, 2.$$

By using standard analysis techniques for the characteristic equation of (4) (see, for example, [4, 13, 22]), it is not difficult to show that, if

$$(b_1 - |a_{11}|)(b_2 - |a_{22}|) - |a_{12}a_{21}| < 0,$$

then the positive constants $\tau_k > 0$, $k = 1, 2, \dots$, exist such that system (4) has Hopf bifurcations for $\tau_{ij} = \tau_k > 0$, $i, j = 1, 2$; $k = 1, 2, \dots$. This completes the proof of Theorem 1. \square

Remark 1. Theorem 1 shows that the time delays τ_{ij} , $i, j = 1, 2$, are factually *harmless* for global attractivity of the equilibrium $(0, 0)^T$ of (1) for the case of $n = 2$.

Next, let us further consider global attractivity of the equilibrium $(0, 0, 0)^T$ of (1) for $n = 3$. This shows the following

Theorem 2. *The equilibrium $(0, 0, 0)^T$ of (1) for $n = 3$ is globally attractive for any time delays $\tau_{ij} \geq 0$, $i, j = 1, 2, 3$, if all the principal minors of C are nonnegative, where the matrix $C = (c_{ij})_{3 \times 3}$ is defined as follows,*

$$(5) \quad c_{ii} = b_i - |a_{ii}| \geq 0, \quad c_{ij} = -|a_{ij}| \quad i \neq j, \quad i, j = 1, 2, 3.$$

Proof. From Lemma 1, one only needs to show that $M_i = 0$, $i = 1, 2, 3$.

If there is some $M_i = 0$, it follows from the proof of Theorem 1 and the assumption of Theorem 2 that $M_i = 0$, $i = 1, 2, 3$. Hence, in the following, we assume that $M_i > 0$, $i = 1, 2, 3$. Furthermore, it is

easily shown from Lemma 1 that $\sum_{j=1}^3 |a_{ij}| > 0$, $i = 1, 2, 3$, if $M_i > 0$, $i = 1, 2, 3$. Hence, from Lemma 1 and (H_3) , it follows that

$$(6) \quad b_i M_i = \sum_{j=1}^3 |a_{ij}| \bar{f}_j(M_j) < \sum_{j=1}^3 |a_{ij}| M_j, \quad i = 1, 2, 3.$$

Let d_1, d_2 and d_3 be negative constants such that

$$(7) \quad \begin{cases} (b_1 - |a_{11}|)M_1 - |a_{12}|M_2 - |a_{13}|M_3 = d_1 < 0, \\ -|a_{21}|M_1 + (b_2 - |a_{22}|)M_2 - |a_{23}|M_3 = d_2 < 0, \\ -|a_{31}|M_1 - |a_{32}|M_2 + (b_3 - |a_{33}|)M_3 = d_3 < 0. \end{cases}$$

We first consider the case of $b_i - |a_{ii}| > 0$ for some i . Without loss of generality, we assume that $b_1 > |a_{11}|$. Hence, from (7) it follows that

$$\begin{cases} \left(b_2 - |a_{22}| - \frac{|a_{12}a_{21}|}{b_1 - |a_{11}|} \right) M_2 - \left(|a_{23}| + \frac{|a_{21}a_{13}|}{b_1 - |a_{11}|} \right) M_3 \\ \qquad \qquad \qquad = d_2 + \frac{|a_{21}|}{b_1 - |a_{11}|} d_1 < 0, \\ - \left(|a_{32}| + \frac{|a_{12}a_{31}|}{b_1 - |a_{11}|} \right) M_2 + \left(b_3 - |a_{33}| - \frac{|a_{13}a_{31}|}{b_1 - |a_{11}|} \right) M_3 \\ \qquad \qquad \qquad = d_3 + \frac{|a_{31}|}{b_1 - |a_{11}|} d_1 < 0, \end{cases}$$

which is equivalent to the following inequalities

$$(8) \quad \begin{cases} ((b_2 - |a_{22}|)(b_1 - |a_{11}|) - |a_{12}a_{21}|) M_2 \\ - (|a_{23}|(b_1 - |a_{11}|) + |a_{21}a_{13}|) M_3 = (b_1 - |a_{11}|)d_2 + |a_{21}|d_1 < 0, \\ - (|a_{32}|(b_1 - |a_{11}|) + |a_{12}a_{31}|) M_2 \\ + ((b_3 - |a_{33}|)(b_1 - |a_{11}|) - |a_{13}a_{31}|) M_3 \\ \qquad \qquad \qquad = (b_1 - |a_{11}|)d_3 + |a_{31}|d_1 < 0. \end{cases}$$

Note that, by the assumption of Theorem 2, it follows that

$$\begin{aligned} A_{11} &\equiv (b_2 - |a_{22}|)(b_1 - |a_{11}|) - |a_{12}a_{21}| \geq 0, \\ A_{22} &\equiv (b_3 - |a_{33}|)(b_1 - |a_{11}|) - |a_{13}a_{31}| \geq 0. \end{aligned}$$

Set

$$\begin{aligned} A_{12} &\equiv |a_{23}|(b_1 - |a_{11}|) + |a_{21}a_{13}|, \quad A_{21} \equiv |a_{32}|(b_1 - |a_{11}|) + |a_{12}a_{31}|, \\ D_2 &\equiv (b_1 - |a_{11}|)d_2 + |a_{21}|d_1, \quad D_3 \equiv (b_1 - |a_{11}|)d_3 + |a_{31}|d_1. \end{aligned}$$

Since $M_2 > 0$, $M_3 > 0$, $D_2 < 0$ and $D_3 < 0$, from (8) we can see that $A_{11} + A_{12} > 0$ and $A_{22} + A_{21} > 0$. If $A_{11} > 0$, we have from (8) that $(A_{22} - A_{12}A_{21}/A_{11})M_3 = D_3 + D_2A_{21}/A_{11}$, from which it follows

$$(9) \quad (A_{11}A_{22} - A_{12}A_{21})M_3 = A_{11}D_3 + A_{21}D_2 < 0.$$

If $A_{12} > 0$, again we have from (8) that $(A_{11}A_{22}/A_{12} - A_{21})M_2 = D_3 + A_{22}D_2/A_{12}$, from which it also follows that

$$(10) \quad (A_{11}A_{22} - A_{12}A_{21})M_3 = A_{12}D_3 + A_{22}D_2 < 0.$$

By some simple computations, it follows that

$$A_{11}A_{22} - A_{12}A_{21} = (b_1 - |a_{11}|) \text{Det}(C) \geq 0.$$

Therefore, the equalities (9) and (10) are contradictive.

Now we consider the case of $c_{ii} = b_i - |a_{ii}| = 0$ for $i = 1, 2, 3$. If at least one column vector of the matrix C is a zero vector, for example, $|a_{12}| = |a_{32}| = 0$, we have from (7) that

$$\begin{cases} (b_1 - |a_{11}|)M_1 - |a_{13}|M_3 = d_1 < 0, \\ -|a_{31}|M_1 + (b_3 - |a_{33}|)M_3 = d_3 < 0. \end{cases}$$

This is reduced to the cases of the two-dimensional system. It follows from the proof of Theorem 1 that $M_1 = M_3 = 0$, and hence, $M_2 = 0$.

We assume that any column vector of the matrix C is not a zero vector. Note that $c_{ii} = 0$, $i = 1, 2, 3$; it follows from the assumption of Theorem 2 that

$$a_{12}a_{21} = 0, \quad a_{13}a_{31} = 0, \quad a_{23}a_{32} = 0, \quad a_{12}a_{23}a_{31} = 0, \quad a_{13}a_{21}a_{32} = 0.$$

If $a_{13} \neq 0$ and $a_{23} = 0$, it follows that $a_{31} = 0$; hence, $a_{21} \neq 0$, $a_{12} = 0$ and $a_{32} \neq 0$. Hence, $a_{13}a_{21}a_{32} \neq 0$, which is a contradiction. If $a_{13} \neq 0$ and $a_{23} \neq 0$, it follows that $a_{31} = 0$ and $a_{32} = 0$; hence, $a_{21} \neq 0$ and $a_{12} \neq 0$. Hence, $a_{12}a_{21} \neq 0$, which is a contradiction.

Therefore, the contradictions show that the assumption $M_i > 0$, $i = 1, 2, 3$, is not true. This completes the proof of Theorem 2. \square

Remark 2. Based on Lemma 2.5 in [2], it can be shown that the condition in Theorem 2 is also necessary for the global attractivity of

the equilibrium $(0, 0, 0)^T$ of (1) for any time delays $\tau_{ij} \geq 0, i, j = 1, 2, 3$, if

$$(H_4) \text{Det} (b_i \delta_{ij} + a_{ij})_{3 \times 3} \neq 0$$

holds. Here $\delta_{ij} = 1$ and 0 for $i = j$ and $i \neq j$, respectively. On the other hand, the numerical simulations in the following section strongly suggest that condition (H_4) may not be necessary for the global attractivity of the equilibrium $(0, 0, 0)^T$ of (1) for any time delays $\tau_{ij} \geq 0, i, j = 1, 2, 3$.

4. Numerical simulation example. In this section, we shall give an illustrative example which shows that Theorem 2 can be used for more general systems. The numerical simulations below show that the conjecture in Section 3 is true for the example.

Let us consider the following three-dimensional Hopfield-type neural network with time delays,

$$(11) \begin{cases} \dot{u}_1(t) = -u_1(t) + a_{11} \tanh(u_1(t - \tau_{11})), \\ \dot{u}_2(t) = -u_2(t) + a_{21} \tanh(u_1(t - \tau_{21})) + a_{22} \tanh(u_2(t - \tau_{22})), \\ \dot{u}_3(t) = -u_3(t) + a_{32} \tanh(u_2(t - \tau_{32})) + a_{33} \tanh(u_3(t - \tau_{33})), \end{cases}$$

where $\tau_{ij} \geq 0$ and $a_{ij}, i, j = 1, 2, 3$, are constants.

It is easy to see that (11) satisfies the assumptions (H_1) – (H_3) . Furthermore, it is easily checked that the matrix C satisfies the conditions of Theorem 2, if and only if the inequalities

$$(12) \quad |a_{ii}| \leq 1, \quad i = 1, 2, 3$$

hold. Hence, it follows from Theorem 2 that, if (12) is satisfied, the equilibrium $(0, 0, 0)^T$ of (11) is globally attractive any time delays $\tau_{ij} \geq 0, i, j = 1, 2, 3$.

We would like to point out here that, if $|a_{ii}| < 1, i = 1, 2, 3$, holds, it follows from some known results (see, for example, [3, 5, 6, 7, 9, 14, 15, 19–21, 23–26] and the references therein) that the equilibrium $(0, 0, 0)^T$ of (11) is factually globally exponentially stable for any time delays $\tau_{ij} \geq 0, i, j = 1, 2, 3$.

On the other hand, it is clear that most known results (also see, for example, [1–9, 14, 15, 17–26]) cannot be applied to (11) for the case of $|a_{ii}| = 1$ for some i .

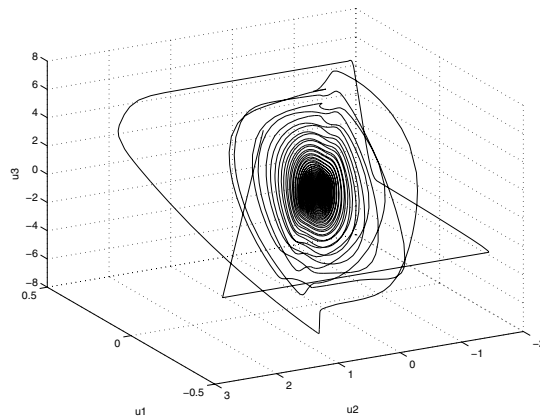


FIGURE 1. $a_{33} = 1, \tau = 30$.

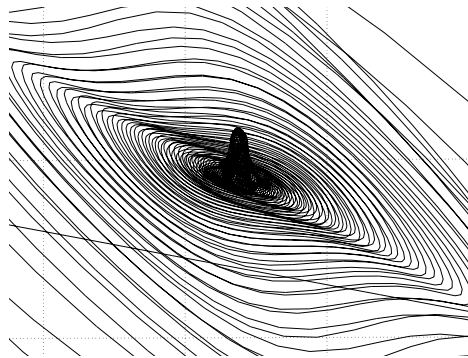


FIGURE 2. $a_{33} = 1.1, \tau = 30$.

To give a numerical simulation for (11), let us choose $a_{11} = a_{22} = -1$, $a_{33} = 1$, $a_{21} = 5$, $a_{32} = 6$ and $\tau = 2\tau_{11} = 3\tau_{21} = \tau_{22} = (5/3)\tau_{32} = (3/2)\tau_{33}$. Figure 1 shows that the equilibrium $(0, 0, 0)^T$ of (11) is globally attractive for any $\tau \geq 0$. If we let $a_{33} = 1.1 > 1$, $\tau = 30$, and the other parameters are the same as the above, Figure 2 shows that the equilibrium $(0, 0, 0)^T$ of (11) becomes unstable and some solution shall tend to some nonzero constant vectors or nonconstant periodic orbits for large $\tau \geq 0$.

REFERENCES

1. J. Bélair, S.A. Campbell and P. van den Driessche, *Frustration, stability, and delay-induced oscillations in a neural network model*, SIAM J. Appl. Math. **56** (1996), 245–255.
2. S.A. Campbell, *Delay independent stability for additive neural networks. New millennium special issue on neural networks and neurocomputing—Theory, models, and applications*, Part I, Differential Equations Dynam. Syst. **9** (2001), 115–138.
3. J. Cao and J. Li, *The stability in neural networks with interneuronal transmission delays*, Appl. Math. Mech. **19** (1998), 457–462.
4. C. Chen and W. Song, *Nonlinear behavior in a delayed neural network model*, Information Control **30** (2001), 240–243.
5. T. Chen, *Global convergence of delayed dynamical systems*, IEEE Trans. Neural Networks **12** (2001), 1532–1536.
6. T. Chu, *An exponential convergence estimate for analog neural networks with delay*, Physics Letters A **283** (2001), 113–118.
7. T. Chu, Z. Zhang and Z. Wang, *A decomposition approach to analysis of competitive-cooperative neural networks with delay*, Physics Letters A **312** (2003), 339–347.
8. T. Chu, C. Zhang and Z. Zhang, *Necessary and sufficient condition for absolute stability of normal neural networks*, Neural Networks **16** (2003), 1223–1227.
9. K. Gopalsamy and X. He, *Stability in asymmetric Hopfield nets with transmission delays*, Physica D **76** (1994), 344–358.
10. J. Hofbauer and W.H.J. So, *Diagonal dominance and harmless off-diagonal delays*, Proc. Amer. Math. Soc. **128** (2000), 2675–2682.
11. J.J. Hopfield, *Neural networks and physical systems with emergent collective computational abilities*, Proc. Natl. Acad. Sci. **79** (1982), 2554–2558.
12. ———, *Neurons with graded response have collective computational properties like those of two-state neurons*, Proc. Natl. Acad. Sci. **81** (1984), 3088–3092.
13. Y. Kuang, *Delay differential equations with applications in population dynamics*, Academic Press, New York, 1993.
14. X. Liao, *Mathematical intension of the physical variables in Hopfield neural networks with time delays*, Science in China E **33** (2003), 127–136.
15. X. Liao, K. Wong and J. Yu, *Novel stability conditions for cellular neural networks with time delays*, Internat. J. Bifurcation Chaos **11** (2001), 1853–1864.
16. W. Ma, T. Hara and Y. Takeuchi, *Stability of a 2-dimensional neural network with time delays*, J. Biol. Systems **8** (2000), 177–193.
17. C.M. Marcus and R.M. Westervelt, *Stability of analog networks with delay*, Physics Rev. A **39** (1989), 347–359.
18. L. Olien and J. Bélair, *Bifurcations, stability, and monotonicity properties of a delayed neural network model*, Physica D **102** (1997), 349–363.
19. P. van den Driessche, J. Wu and X. Zou, *Stabilization role of inhibitory self-connections in a delayed neural network*, Physica **150** (2001), 84–90.

20. P. van den Driessche and X. Zou, *Global attractivity in delayed Hopfield neural network models*, SIAM J. Appl. Math. **58** (1998), 1878–1890.
21. L. Wang and D. Xu, *Stability analysis of Hopfield neural networks with time delay*, Appl. Math. Mech. **23** (2002), 65–70.
22. J. Wei and S. Ruan, *Stability and bifurcation in a neural network model with two delays*, Physica D **130** (1999), 255–272.
23. D. Xu, H. Zhao and H. Zhu, *Global dynamics of Hopfield neural networks involving variable delays*, Comp. Math. Appl. **42** (2001), 39–45.
24. J. Zhang, *Global exponential stability of neural networks with variable delays*, IEEE Trans. Circuits Syst. I **50** (2003), 288–291.
25. Y. Zhang, *Global exponential convergence of recurrent neural networks with variable delays*, Theoret. Comput. Sci. **312** (2004), 281–293.
26. Y. Zhang, P.A. Heng and K.S. Leung, *Convergence analysis of cellular neural networks with unbounded delay*, IEEE Trans. Circuits Syst. I **48** (2001), 680–687.

DEPARTMENT OF MATHEMATICS AND MECHANICS, SCHOOL OF APPLIED SCIENCE,
UNIVERSITY OF SCIENCE AND TECHNOLOGY BEIJING, BEIJING 100083, CHINA AND
NORTHEASTERN UNIVERSITY AT QINHUANGDAO, QINHUANGDAO, HEBEI 066004,
CHINA

Email address: shgzhang78@163.com

DEPARTMENT OF MATHEMATICS AND MECHANICS, SCHOOL OF APPLIED SCIENCE,
UNIVERSITY OF SCIENCE AND TECHNOLOGY BEIJING, BEIJING 100083, CHINA

Email address: wanbiao_ma@yahoo.co.jp

DEPARTMENT OF MATHEMATICS AND STATISTICS, COLLEGE OF LIBERAL ARTS AND
SCIENCES, ARIZONA STATE UNIVERSITY, TEMPE, AZ 85287-1804

Email address: kuang@asu.edu