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STABILITY AND BIFURCATION IN A BEDDINGTON-DEANGELIS TYPE PREDATOR-PREY MODEL WITH PREY DISPERSAL

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ABSTRACT. A time delayed predator-prey model with prey dispersal and Beddington-DeAngelis type functional response is investigated. By analyzing the corresponding characteristic equations, the local stability of a positive equilibrium and each of the boundary equilibria is discussed. The existence of Hopf bifurcations at the positive equilibrium is established. By using an iteration technique, sufficient conditions are derived for the global attractiveness of the positive equilibrium of the proposed model. By comparison arguments, sufficient conditions are obtained for the global stability of each of the boundary equilibria of the model. Numerical simulations are carried out to illustrate some main results.

1. Introduction. The traditional mathematical model describing predator-prey interactions consists of the following system of differential equations

$$\begin{aligned} \dot{x}(t) &= a(x) - F(x,y), \\ \dot{y}(t) &= eF(x,y) - c(y), \end{aligned} \tag{1.1}$$

where $x(t)$ and $y(t)$ represent densities of the prey and the predator at time t, respectively. The functions $a(x)$ and $c(y)$ are the intrinsic growth rate of the prey and the mortality rate of the predator, respectively. The function $F(x,y)$ is called the "response function" representing the prey consumption per unit of time. The most popular response functions used in the modeling of predator-prey systems

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are of Michaelis-Menten type (also called Holling type-II) $F(x,y) = fxy/(c+x)$ and the ratio-dependent type $F(x,y) = fxy/(my+x)$. However, it is believed that the Michaelis-Menten type response function does not account for mutual competition among predators (see, for example, [**13**]), while the ratio-dependent type response function allows unrealistic positive growth rate of the predator at low densities [**2, 10, 12**]. The Beddington-DeAngelis response function $F(x,y) = fxy/(a+by+cx)$ was introduced independently by Beddington [**1**] and DeAngelis [**7**] as a solution of the observed problems in classical predator-prey theory. It has a term by in the denominator modeling mutual interference between predators and avoids the "low densities problem" of the ratio-dependent type functional response. In [**3**], Cantrell and Cosner discussed system (1.1) when $a(x) = rx(1-x/K)$, $c(y) = -\mu y$, $F(x,y) = fxy/(1+by+cx)$, where b, c, f, r, K, μ are positive constants. They presented some qualitative analysis of solutions of system (1.1) from the viewpoint of permanence (uniform persistence).

Dispersal is a ubiquitous phenomenon in the natural world. Its importance in understanding the ecological and evolutionary dynamics of populations was mirrored by a large number of mathematical models devoted to it in the scientific literature. Some of the mathematical models dealt with a single population dispersing among patches. Some of them dealt with competition and predator-prey interactions in patchy environments (see, for example, [**6, 8, 17, 18**] and the references cited therein). We note that many authors always assumed that intrinsic growth rates are all continuous and bounded above and below by positive constants. This means that every species lives in a suitable environment. However, the actual living environments of endangered species are not always like this. Because of the ecological effects of human activities and industry, e.g., the location of manufacturing industries and pollution of the atmosphere, rivers and soil, more and more habitats have been broken into patches and some of the patches have been polluted. In some of these patches, and sometimes even in every patch, species will become extinct without contributions from other patches, and hence the species live in a weak patchy environment. In [**4, 5**], Cui and Chen proposed and studied population models with weak patchy environment. In a different way from the former studies (see, for example, [**8, 17, 18**]), they considered the important situation in conservation biology in which species live in a weak patchy environment, in the sense that species will become extinct in some of the isolated patches without contribution from other patches. In [**6**], Cui et al. proposed and studied diffusive stage structured single-species population models. A conservation strategy was put forward by analyzing the asymptotic behavior of solutions of the proposed models.

In this paper, motivated by the work of Cantrell and Cosner [**3**] on a Beddington-DeAngelis type predator-prey model and Cui et al. [**6**] on diffusive stage-structured single-species population models, we are concerned with the effect of prey dispersal between two patches, the Beddington-DeAngelis type functional response and time delay due to the gestation of the predator on the dynamics of a predator-prey system. To this end, we study the following delayed differential system

$$\begin{aligned}\dot{x}_1(t) &= x_1(t)\left(r_1 - a_{11}x_1(t) - \frac{a_{12}y(t)}{1+Bx_1(t)+Cy(t)}\right)\\ &\qquad + D_2x_2(t) - D_1x_1(t),\\ \dot{x}_2(t) &= -r_2x_2(t) + D_1x_1(t) - D_2x_2(t),\\ \dot{y}(t) &= \frac{a_{21}x_1(t-\tau)y(t-\tau)}{1+Bx_1(t-\tau)+Cy(t-\tau)} - ry(t).\end{aligned} \tag{1.2}$$

In system (1.2), it is assumed that the ecosystem is composed of two isolated patches and the breeding area in patch 2 is damaged. $x_1(t)$ and $x_2(t)$ represent densities of the prey at time t in patches 1 and 2, respectively; $y(t)$ represents the density of predator population in patch 1 at time t. The parameters a_{11}, a_{12}, a_{21}, r_1, r_2, r, B, C, D_1 and D_2 are positive constants, where r_1 is the intrinsic growth rate of prey in patch 1, a_{11} is the intra-specific competition rate of prey in patch 1, a_{12} is the capturing rate of the predator in patch 1, r_2 is the death rate of prey in patch 2, D_1 and D_2 are dispersal rates of the prey between the two patches, a_{21}/a_{12} is the conversion rate of nutrients into the reproduction of the predator, r is the death rate of the predator and $\tau \geq 0$ is a constant delay due to the gestation of the predator, that is, mature adult predators can only contribute to the reproduction of predator biomass.

The initial conditions for system (1.2) take the form

$$\begin{aligned}&x_1(\theta) = \phi_1(\theta), \qquad x_2(\theta) = \phi_2(\theta), \qquad y(\theta) = \psi(\theta),\\ &\phi_1(\theta) \geq 0, \quad \phi_2(\theta) \geq 0, \quad \psi(\theta) \geq 0, \quad \theta \in [-\tau, 0],\\ &\phi_1(0) > 0, \quad \phi_2(0) > 0, \quad \psi(0) > 0,\end{aligned} \tag{1.3}$$

where $\Phi = (\phi_1(\theta), \phi_2(\theta), \psi(\theta)) \in C([-\tau, 0], \mathbf{R}^3_{+0})$, is the Banach space of continuous functions mapping the interval $[-\tau, 0]$ into $\mathbf{R}^3_{+0}$, where $\mathbf{R}^3_{+0} = \{(x_1, x_2, x_3) : x_i \geq 0, i = 1, 2, 3\}$.

It is well known by the fundamental theory of functional differential equations [**9**], system (1.2) has a unique solution $(x_1(t), x_2(t), y(t))$ satisfying initial conditions (1.3). It is easy to show that all solutions of system (1.2) with initial conditions (1.3) are defined on $[0, +\infty)$ and remain positive for all $t \geq 0$.

The organization of this paper is as follows. In the next section, by analyzing the corresponding characteristic equations, we discuss the local stability of a positive equilibrium and each of the boundary equilibria of system (1.2). The existence of Hopf bifurcations at the positive equilibrium is proved. Numerical simulations are carried out to illustrate the results above. In Section 3, using an iteration technique, we establish the global attractiveness of a positive equilibrium of system (1.2). By comparison arguments, we discuss the global stability of each of the boundary equilibria of system (1.2). A brief discussion is given in Section 4 to conclude this work.

2. Local stability. In this section, we discuss the local stability of a positive equilibrium and each of boundary equilibria of system (1.2) by analyzing the corresponding characteristic equations. We also study the existence of Hopf bifurcation at the positive equilibrium.

System (1.2) always has a trivial equilibrium $E_0(0, 0, 0)$. If the following holds:

$$\text{(H1)} \qquad r_1(D_2 + r_2) - D_1 r_2 > 0,$$

then system (1.2) has a semi-trivial (boundary) equilibrium $E_1(x_1^0, x_2^0, 0)$, where

$$x_1^0 = \frac{r_1(D_2 + r_2) - D_1 r_2}{a_{11}(D_2 + r_2)}, \quad x_2^0 = \frac{D_1[r_1(D_2 + r_2) - D_1 r_2]}{a_{11}(D_2 + r_2)^2}.$$

Further, if (H1) and the following hold:

$$\text{(H2)} \qquad (a_{21} - Br)\left(r_1 - \frac{D_1 r_2}{D_2 + r_2}\right) - a_{11} r > 0,$$

then system (1.2) has a unique positive equilibrium $E^*(x_1^*, x_2^*, y^*)$, where

$$x_1^* = \frac{-A + \sqrt{\Delta}}{2a_{11}a_{21}C}, \quad x_2^* = \frac{D_1}{D_2 + r_2} x_1^*, \quad y^* = \frac{(a_{21} - Br)x_1^* - r}{Cr};$$

here

$$A = a_{12}(a_{21} - Br) - a_{21}r_1C + \frac{a_{21}r_2CD_1}{D_2 + r_2},$$
$$\Delta = A^2 + 4a_{11}a_{12}a_{21}rC.$$

The characteristic equation of system (1.2) at the equilibrium $E_0(0, 0, 0)$ is of the form

$$(\lambda + r)[\lambda^2 + (D_1 + D_2 + r_2 - r_1)\lambda + D_1r_2 - r_1(D_2 + r_2)] = 0. \tag{2.1}$$

If $r_1(D_2 + r_2) - D_1r_2 > 0$, then $(0, 0, 0)$ is unstable; if $r_1(D_2 + r_2) - D_1r_2 < 0$, then we have $r_1 < D_1$. In this case, $D_1 + D_2 + r_2 - r_1 > 0$, hence, $(0, 0, 0)$ is stable.

The characteristic equation of system (1.2) at the equilibrium $E_1(x_1^0, x_2^0, 0)$ takes the form

$$\left(\lambda + r - \frac{a_{21}x_1^0}{1 + Bx_1^0} e^{-\lambda\tau}\right)[\lambda^2 + (D_2 + r_2 - r_1 + 2a_{11}x_1^0 + D_1) \cdot \lambda + r_1(D_2 + r_2) - D_1r_2] = 0. \tag{2.2}$$

Noting that

$$-r_1 + 2a_{11}x_1^0 + D_1 = \frac{D_1D_2 + r_1(D_2 + r_2) - D_1r_2}{D_2 + r_2} > 0,$$

it is easy to show that the equation

$$\lambda^2 + (D_2 + r_2 - r_1 + 2a_{11}x_1^0 + D_1)\lambda + r_1(D_2 + r_2) - D_1r_2 = 0$$

always has two negative real roots. All other roots are given by roots of equation

$$\lambda + r - \frac{a_{21}x_1^0}{1 + Bx_1^0} e^{-\lambda\tau} = 0. \tag{2.3}$$

Let $f(\lambda) = \lambda + r - (a_{21}x_1^0)/(1 + Bx_1^0)e^{-\lambda\tau}$. If (H2) holds, it follows that for λ real,

$$f(0) = \frac{a_{11}r(D_2 + r_2) - (a_{21} - Br)[r_1(D_2 + r_2) - D_1 r_2]}{a_{11}(D_2 + r_2)(1 + Bx_1^0)} < 0,$$
$$\lim_{\lambda \to +\infty} f(\lambda) = +\infty.$$

Hence, $f(\lambda) = 0$ has a positive real root. Accordingly, the equilibrium $E_1(x_1^0, x_2^0, 0)$ is unstable.

If the following holds:

$$\text{(H3)} \qquad (a_{21} - Br)\Big(r_1 - \frac{D_1 r_2}{D_2 + r_2}\Big) < a_{11}r,$$

we claim that the roots of $f(\lambda) = 0$ have only negative real parts. Suppose that $\operatorname{Re}\lambda \geq 0$. Then we derive from (2.3) that

$$\begin{aligned}\operatorname{Re}\lambda &= -r + \frac{a_{21}x_1^0}{1 + Bx_1^0}e^{-\tau \operatorname{Re}\lambda}\cos(\tau \operatorname{Im}\lambda) \leq -r + \frac{a_{21}x_1^0}{1 + Bx_1^0} \\ &= -\frac{a_{11}r(D_2 + r_2) - (a_{21} - Br)[r_1(D_2 + r_2) - D_1 r_2]}{a_{11}(D_2 + r_2)(1 + Bx_1^0)} < 0,\end{aligned}$$

a contradiction. Hence, we have $\operatorname{Re}\lambda < 0$. Thus, if (H3) holds, then $E_1(x_1^0, x_2^0, 0)$ is locally asymptotically stable.

The characteristic equation of system (1.2) at the positive equilibrium E^* is of the form

$$\text{(2.4)} \qquad \lambda^3 + p_2\lambda^2 + p_1\lambda + p_0 + (q_2\lambda^2 + q_1\lambda + q_0)e^{-\lambda\tau} = 0,$$

where

$$\begin{aligned}p_0 &= r[K(D_2 + r_2) - D_1 D_2],\\ p_1 &= K(D_2 + r_2) - D_1 D_2 + r(D_2 + r_2 + K),\\ p_2 &= D_2 + r_2 + r + K,\end{aligned}$$

(2.5)

$$q_0 = -\frac{a_{21}Kx_1^*(D_2+r_2)(1+Bx_1^*)}{(1+Bx_1^*+Cy^*)^2} + \frac{a_{21}D_1D_2x_1^*(1+Bx_1^*)}{(1+Bx_1^*+Cy^*)^2} + \frac{a_{12}a_{21}(D_2+r_2)x_1^*y^*(1+Bx_1^*)(1+Cy^*)}{(1+Bx_1^*+Cy^*)^4},$$

$$q_1 = -\frac{a_{21}(D_2+r_2+K)x_1^*(1+Bx_1^*)}{(1+Bx_1^*+Cy^*)^2} + \frac{a_{12}a_{21}x_1^*y^*(1+Bx_1^*)(1+Cy^*)}{(1+Bx_1^*+Cy^*)^4},$$

$$q_2 = -\frac{a_{21}x_1^*(1+Bx_1^*)}{(1+Bx_1^*+Cy^*)^2},$$

$$K = a_{11}x_1^* - \frac{a_{12}Bx_1^*y^*}{(1+Bx_1^*+Cy^*)^2} + \frac{D_1D_2}{D_2+r_2}.$$

If $i\omega(\omega > 0)$ is a solution of (2.4), separating real and imaginary parts, we have the following:

$$\begin{aligned} -\omega^3 + p_1\omega &= (q_0 - q_2\omega^2)\sin\omega\tau - q_1\omega\cos\omega\tau, \\ p_2\omega^2 - p_0 &= (q_0 - q_2\omega^2)\cos\omega\tau + q_1\omega\sin\omega\tau. \end{aligned} \tag{2.6}$$

Squaring and adding the two equations of (2.6), it follows that

$$\omega^6 + (p_2^2 - 2p_1 - q_2^2)\omega^4 + (p_1^2 - 2p_0p_2 + 2q_0q_2 - q_1^2)\omega^2 + p_0^2 - q_0^2 = 0. \tag{2.7}$$

Letting $z = \omega^2$ and

$$p = p_2^2 - 2p_1 - q_2^2, \quad q = p_1^2 - 2p_0p_2 + 2q_0q_2 - q_1^2, \quad r = p_0^2 - q_0^2, \tag{2.8}$$

then equation (2.7) becomes

$$z^3 + pz^2 + qz + r = 0. \tag{2.9}$$

Define

$$h(z) = z^3 + pz^2 + qz + r. \tag{2.10}$$

When $\tau = 0$, equation (2.4) becomes

$$\lambda^3 + (p_2+q_2)\lambda^2 + (p_1+q_1)\lambda + p_0 + q_0 = 0. \tag{2.11}$$

Hence, if $p_2+q_2>0, (p_2+q_2)(p_1+q_1)>p_0+q_0>0$, then the positive equilibrium E^* of system (1.2) is locally stable when $\tau=0$.

In order to discuss the local stability of the positive equilibrium E^* of system (1.2), we introduce the following results developed by Ruan and Wei [**14**] and Song and Yuan [**16**], respectively.

Lemma 2.1. *Consider the exponential polynomial*

$$\begin{aligned}P(\lambda, e^{-\lambda\tau_1},\dots,e^{-\lambda\tau_m})\\ &= \lambda^n + p_1^{(0)}\lambda^{n-1}+\cdots+p_{n-1}^{(0)}\lambda+p_n^{(0)}\\ &\quad + [p_1^{(1)}\lambda^{n-1}+\cdots+p_{n-1}^{(1)}\lambda+p_n^{(1)}]e^{-\lambda\tau_1}+\cdots\\ &\quad + [p_1^{(m)}\lambda^{n-1}+\cdots+p_{n-1}^{(m)}\lambda+p_n^{(m)}]e^{-\lambda\tau_m}\end{aligned}$$

where $\tau_i\ge 0$, $i=1,2,\dots,m$, and $p_j^{(i)}$, $i=0,1,\dots,m$; $j=1,2,\dots,n$, are constants. As $(\tau_1,\tau_2,\dots,\tau_m)$ vary, the sum of the order of the zeroes of $P(\lambda,e^{-\lambda\tau_1},\dots,e^{-\lambda\tau_m})$ on the right half plane can change only if a zero appears on or crosses the imaginary axis.

Lemma 2.2. *For equation* (2.9)*, we have the following results.*

(i) *If $r<0$, then equation* (2.9) *admits at least one positive root.*

(ii) *If $r\ge 0$ and $\Delta=p^2-3q\le 0$, then equation* (2.9) *has no positive roots.*

(iii) *If $r\ge 0$ and $\Delta>0$, then equation* (2.9) *has positive roots if and only if $z_1^*=(-p+\sqrt{\Delta})/3>0$, and $h(z_1^*)\le 0$.*

Lemma 2.3. *For equation* (2.4)*, we have the following results.*

(i) *If $r\ge 0$ and $\Delta=p^2-3q\le 0$, then all roots of equation* (2.4) *with positive real parts have the same sum of the order as those of equation* (2.11) *for all $\tau\ge 0$;*

(iii) *If either $r<0$ or $r\ge 0$, $\Delta>0$, $z_1^*=(-p+\sqrt{\Delta})/3>0$ and $h(z_1^*)\le 0$, then all roots of equation* (2.4) *with positive real parts have the same sum of the order as those of equation* (2.11) *for $\tau\in[0,\tau_0)$.*

Clearly, If $p_0<q_0$, then equation (2.7) has positive roots. Without loss of generality, we assume that (2.7) has three positive roots, namely, $\omega_1,\omega_2,\omega_3$.

Define

$$\tau_{kn} = \frac{1}{\omega_k} \arccos \frac{(q_0 - q_2\omega_k^2)(p_2\omega_k^2 - p_0) + q_1\omega_k^2(\omega_k^2 - p_1)}{(q_0 - q_2\omega_k^2)^2 + q_1^2\omega_k^2} + \frac{2n\pi}{\omega_k}, \quad n = 0, 1, 2, \dots . \tag{2.12}$$

Then $\pm i\omega_k$ are a pair of purely imaginary roots of the characteristic equation (2.4) with a sequence of critical values τ_{kn}.

Denote

$$\tau_0 := \tau_{k_0 0} = \min_{k\in\{1,2,3\}} \{\tau_{k0}\}, \quad \omega_0 := \omega_{k_0}. \tag{2.13}$$

Let $\lambda(\tau) = \sigma(\tau) + i\omega(\tau)$ be a root of equation (2.4) near $\tau = \tau_{kn}$ satisfying

$$\sigma(\tau_{kn}) = 0, \quad \omega(\tau_{kn}) = \omega_k, \quad k = 1, 2, 3; \; n = 0, 1, 2, \dots .$$

In the following we claim that

$$\operatorname{sgn} \left\{ \left. \frac{d(\sigma(\tau))}{d\tau} \right|_{\tau=\tau_{kn}} \right\} = \operatorname{sgn} \{h'(\omega_k^2)\}.$$

If $h'(\omega_k^2) > 0$, then there exists at least one eigenvalue with positive real part for $\tau > \tau_0$. Moreover, the conditions for the existence of a Hopf bifurcation [**9**] are then satisfied yielding a periodic solution. To this end, differentiating (2.4) with respect to τ, it follows that

$$(3\lambda^2 + 2p_2\lambda + p_1)\frac{d\lambda}{d\tau} + (2q_2\lambda + q_1)e^{-\lambda\tau}\frac{d\lambda}{d\tau} - \tau(q_2\lambda^2 + q_1\lambda + q_0)e^{-\lambda\tau}\frac{d\lambda}{d\tau} = \lambda(q_2\lambda^2 + q_1\lambda + q_0)e^{-\lambda\tau}.$$

Hence, we derive that

$$\left(\frac{d\lambda}{d\tau}\right)^{-1} = \frac{3\lambda^2 + 2p_2\lambda + p_1}{-\lambda(\lambda^3 + p_2\lambda^2 + p_1\lambda + p_0)} + \frac{2q_2\lambda + q_1}{\lambda(q_2\lambda^2 + q_1\lambda + q_0)} - \frac{\tau}{\lambda}.$$

Letting $\lambda = i\omega_k$, it follows that

$$\begin{aligned}\operatorname{Re}\left(\frac{d\lambda}{d\tau}\right)^{-1}\Bigg|_{\tau=\tau_{kn}} &= \frac{\omega_k^2}{(q_0 - q_2\omega_k^2)^2 + q_1^2\omega_k^2}[3\omega_k^4 + 2(p_2^2 - 2p_1 - q_2^2)\omega_k^2 \\ &\qquad + p_1^2 - 2p_0p_2 + 2q_0q_2 - q_1^2] \\ &= \frac{\omega_k^2 h'(\omega_k^2)}{(q_0 - q_2\omega_k^2)^2 + q_1^2\omega_k^2}.\end{aligned}$$

Therefore, we obtain that

$$\operatorname{sgn}\left\{\frac{d(\sigma(\tau))}{d\tau}\Bigg|_{\tau=\tau_{kn}}\right\} = \operatorname{sgn}\left\{\operatorname{Re}\left(\frac{d\lambda}{d\tau}\right)^{-1}\Bigg|_{\tau=\tau_{kn}}\right\} = \operatorname{sgn}\{h'(\omega_k^2)\}.$$

By the general theory on characteristic equations of delay differential equations from [**11**], we obtain the following result.

Theorem 2.1. *Let* (H1) *hold and* $\tau_{kn}, \omega_0, \tau_0$ *be defined by* (2.12) *and* (2.13), *respectively. Assume further that* $p_2+q_2 > 0$, $(p_2+q_2)(p_1+q_1) > p_0 + q_0 > 0$ *hold.*

(i) *If* $r \geq 0$ *and* $\Delta = p^2 - 3q \leq 0$, *then the positive equilibrium* E^* *of system* (1.2) *is asymptotically stable for all* $\tau \geq 0$.

(ii) *If either* $r < 0$ *or* $r \geq 0$, $\Delta > 0$, $z_1^* > 0$ *and* $h(z_1^*) \leq 0$, *then the positive equilibrium* E^* *of system* (1.2) *are asymptotically stable for* $\tau \in [0, \tau_0)$.

(iii) *If the conditions in* (ii) *are satisfied, and* $h'(\omega_k^2) > 0$, *then system* (1.2) *undergoes a Hopf bifurcation at the positive equilibrium* E^* *when* $\tau = \tau_{kn}$.

We now give an example to illustrate the above results.

Example. In system (1.2), let $a_{11} = 1$, $a_{12} = 1.5$, $a_{21} = 1.2$, $r_1 = 3$, $r_2 = 0.5$, $r = 0.5$, $B = 1$, $C = 0.1$, $D_1 = D_2 = 1$. It is easy to show that system (1.2) admits a unique positive equilibrium $E^*(0.8956, 0.5970, 2.5378)$. Let $\tau_0 = 1.0638$. By Theorem 2.1, we see that if $\tau < \tau_0$, the positive equilibrium E^* is locally asymptotically stable; if $\tau > \tau_0$, E^* becomes unstable; system (1.2) undergoes a Hopf bifurcation at E^* when $\tau = \tau_0$. We take $\tau = 0.95$ and $\tau = 1.1$,

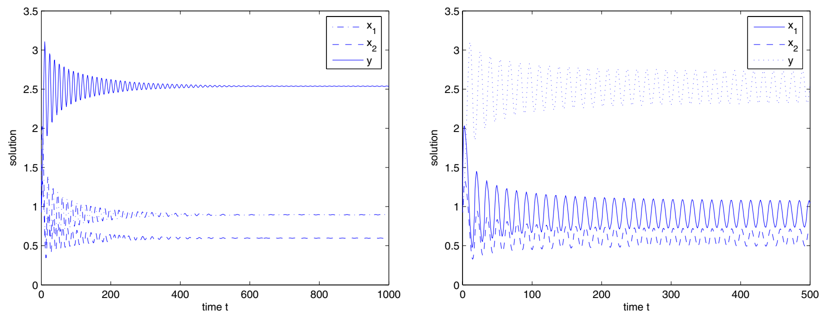

FIGURE 1. The temporal solution found by numerical integration of system (1.2) with $a_{11} = 1$, $a_{12} = 1.5$, $a_{21} = 1.2$, $r_1 = 3$, $r_2 = 0.5$, $r = 0.5$, $B = 1$, $C = 0.1$, $D_1 = D_2 = 1$, $\tau = 0.95$ and $\tau = 1.1$, respectively, $(\phi_1, \phi_2, \psi) \equiv (1, 1, 1)$.

respectively. Numerical simulations illustrate the observations above, see Figure 1.

3. Global stability. In this section, we discuss global attractiveness of the positive equilibrium and each of the boundary equilibria of system (1.2). The technique of proofs is to use an iteration scheme and comparison arguments, respectively. To this end, we need the following lemmas.

Consider the following differential equations

$$\begin{aligned} \dot{u}_1(t) &= u_1(t)(a - a_{11}u_1(t)) + D_2u_2(t) - D_1u_1(t), \\ \dot{u}_2(t) &= -r_2u_2(t) + D_1u_1(t) - D_2u_2(t). \end{aligned} \tag{3.1}$$

System (3.1) always has a trivial equilibrium $(0, 0)$. If $a(D_2 + r_2) - D_1r_2 > 0$, then (3.1) has a unique positive equilibrium (u_1^*, u_2^*), where

$$u_1^* = \frac{a(D_2 + r_2) - D_1r_2}{a_{11}(D_2 + r_2)}, \qquad u_2^* = \frac{D_1[a(D_2 + r_2) - D_1r_2]}{a_{11}(D_2 + r_2)^2}.$$

The following result for system (3.1) was developed by Xu and Ma in [**19**].

Lemma 3.1. *If $a(D_2 + r_2) > D_1r_2$, then the positive equilibrium (u_1^*, u_2^*) is globally stable; if $a(D_2 + r_2) < D_1r_2$, then the trivial equilibrium $(0, 0)$ is globally stable.*

We now consider the following equation with time delay

$$\begin{aligned} \dot{u}(t) &= \frac{a_{21}A_1 u(t-\tau)}{1+BA_1+Cu(t-\tau)} - ru(t), \\ u(\theta) &= \phi(\theta) \geq 0, \quad \theta \in [-\tau, 0), \quad \phi(0) > 0, \end{aligned} \tag{3.2}$$

where a_{21}, A_1, B, C, r are positive constants and $\tau \geq 0$. Using similar arguments as those in the proof of Lemma 3.1 in Song and Chen [**15**], we can prove the following result.

Lemma 3.2. *If $A_1(a_{21} - Br) > r$, then equation* (3.2) *admits a unique positive equilibrium $u^* = [A_1(a_{21} - Br) - r]/(Cr)$ which is globally asymptotically stable. If $A_1(a_{21} - Br) < r$, then the equilibrium $u_0 = 0$ is globally stable.*

We are now in a position to state and prove a result on the global attractiveness of the positive equilibrium E^* of system (1.2).

Theorem 3.1. *Let* (H1)–(H2) *hold. Then the positive equilibrium $E^*(x_1^*, x_2^*, y^*)$ is globally attractive provided that one of the following assumptions holds*:

(H4) $C(a_{21} - 2Br)(r_1 - D_1 r_2/(D_2 + r_2)) > a_{12}(a_{21} - Br)$,

(H5) $a_{21} < 2Br$, $C(r_1 - D_1 r_2/(D_2 + r_2)) > a_{12}$.

Proof. Let $(x_1(t), x_2(t), y(t))$ be any positive solution of system (1.2) with initial conditions (1.3).

Let

$$U_i = \limsup_{t\to+\infty} x_i(t), \qquad V_i = \liminf_{t\to+\infty} x_i(t), \quad i = 1, 2,$$
$$U = \limsup_{t\to+\infty} y(t), \qquad V = \liminf_{t\to+\infty} y(t).$$

We now claim that $U_1 = V_1 = x_1^*$, $U_2 = V_2 = x_2^*$, $U = V = y^*$.

It follows from the first and the second equations of system (1.2) that

$$\begin{aligned} \dot{x}_1(t) &\leq x_1(t)(r_1 - a_{11}x_1(t)) + D_2 x_2(t) - D_1 x_1(t), \\ \dot{x}_2(t) &= -r_2 x_2(t) + D_1 x_1(t) - D_2 x_2(t). \end{aligned} \tag{3.3}$$

Consider the following auxiliary equations

$$\begin{aligned}\dot{u}_1(t) &= u_1(t)(r_1 - a_{11}u_1(t)) + D_2u_2(t) - D_1u_1(t),\\ \dot{u}_2(t) &= -r_2u_2(t) + D_1u_1(t) - D_2u_2(t).\end{aligned} \tag{3.4}$$

By Lemma 3.1 we derive from (3.4) that

$$\begin{aligned}\lim_{t\to+\infty} u_1(t) &= \frac{r_1(D_2+r_2) - D_1r_2}{a_{11}(D_2+r_2)},\\ \lim_{t\to+\infty} u_2(t) &= \frac{D_1[r_1(D_2+r_2) - D_1r_2]}{a_{11}(D_2+r_2)^2}.\end{aligned}$$

By comparison it follows that

$$\begin{aligned}\limsup_{t\to+\infty} x_1(t) &\le \frac{r_1(D_2+r_2) - D_1r_2}{a_{11}(D_2+r_2)} := M_1^{x_1},\\ \limsup_{t\to+\infty} x_2(t) &\le \frac{D_1[r_1(D_2+r_2) - D_1r_2]}{a_{11}(D_2+r_2)^2} := M_1^{x_2}.\end{aligned}$$

Hence, for $\varepsilon > 0$ sufficiently small, there is a $T_1 > 0$ such that if $t > T_1$, $x_i(t) \le M_1^{x_i} + \varepsilon$, $i = 1, 2$. We therefore derive from the third equation of system (1.2) that, for $t > T_1 + \tau$,

$$\dot{y}(t) \le \frac{a_{21}(M_1^{x_1}+\varepsilon)y(t-\tau)}{1 + B(M_1^{x_1}+\varepsilon) + Cy(t-\tau)} - ry(t). \tag{3.5}$$

Consider the following auxiliary equation

$$\dot{u}(t) = \frac{a_{21}(M_1^{x_1}+\varepsilon)u(t-\tau)}{1 + B(M_1^{x_1}+\varepsilon) + Cu(t-\tau)} - ru(t). \tag{3.6}$$

By Lemma 3.2 it follows from (3.6) that

$$\lim_{t\to+\infty} u(t) = \frac{(a_{21} - Br)(M_1^{x_1}+\varepsilon) - r}{Cr}.$$

By comparison, we obtain that

$$U = \limsup_{t\to+\infty} y(t) \le \frac{(a_{21} - Br)(M_1^{x_1}+\varepsilon) - r}{Cr}.$$

Since this is true for arbitrary $\varepsilon > 0$ sufficiently small, it follows that $U \le M_1^y$, where

$$M_1^y = \frac{(a_{21} - Br)M_1^{x_1} - r}{Cr}.$$

Hence, for $\varepsilon > 0$ sufficiently small, there is a $T_2 \ge T_1 + \tau$ such that, if $t > T_2$, $y(t) \le M_1^y + \varepsilon$.

For $\varepsilon > 0$ sufficiently small, we derive from the first and the second equations of system (1.2) that, for $t > T_2$,

$$\begin{aligned} \dot{x}_1(t) &\ge x_1(t)\Big(r_1 - a_{11}x_1(t) - \frac{a_{12}(M_1^y + \varepsilon)}{1 + C(M_1^y + \varepsilon)}\Big) + D_2x_2(t) - D_1x_1(t), \\ \dot{x}_2(t) &= -r_2x_2(t) + D_1x_1(t) - D_2x_2(t). \end{aligned}$$

By Lemma 3.1 and a comparison argument, it follows that

$$\begin{aligned} \liminf_{t\to+\infty} x_1(t) &\ge \frac{1}{a_{11}}\left[r_1 - \frac{a_{12}(M_1^y + \varepsilon)}{1 + C(M_1^y + \varepsilon)} - \frac{D_1r_2}{D_2 + r_2}\right], \\ \liminf_{t\to+\infty} x_2(t) &\ge \frac{D_1}{a_{11}(D_2 + r_2)}\left[r_1 - \frac{a_{12}(M_1^y + \varepsilon)}{1 + C(M_1^y + \varepsilon)} - \frac{D_1r_2}{D_2 + r_2}\right]. \end{aligned}$$

Since these inequalities hold true for arbitrary $\varepsilon > 0$ sufficiently small, we conclude that $V_1 \ge N_1^{x_1}$, $V_2 \ge N_1^{x_2}$, where

$$\begin{aligned} N_1^{x_1} &= \frac{1}{a_{11}}\left[r_1 - \frac{a_{12}M_1^y}{1 + CM_1^y} - \frac{D_1r_2}{D_2 + r_2}\right], \\ N_1^{x_2} &= \frac{D_1}{a_{11}(D_2 + r_2)}\left[r_1 - \frac{a_{12}M_1^y}{1 + CM_1^y} - \frac{D_1r_2}{D_2 + r_2}\right]. \end{aligned}$$

Therefore, for $\varepsilon > 0$ sufficiently small, there is a $T_3 \ge T_2$ such that if $t > T_3$, $x_i(t) \ge N_1^{x_i} - \varepsilon$, $i = 1, 2$.

For $\varepsilon > 0$ sufficiently small, it follows from the third equation of system (1.2) that for $t > T_3 + \tau$,

$$\dot{y}(t) \ge \frac{a_{21}(N_1^{x_1} - \varepsilon)y(t - \tau)}{1 + B(N_1^{x_1} - \varepsilon) + Cy(t - \tau)} - ry(t).$$

By Lemma 3.2, a comparison argument shows that

$$V = \liminf_{t\to+\infty} y(t) \ge \frac{(a_{21} - Br)(N_1^{x_1} - \varepsilon) - r}{Cr}.$$

Since this is true for arbitrary $\varepsilon > 0$ sufficiently small, we conclude that $V \geq N_1^y$, where

$$N_1^y = \frac{(a_{21} - Br)N_1^{x_1} - r}{Cr}.$$

Therefore, for $\varepsilon > 0$ sufficiently small, there exists a $T_4 \geq T_3 + \tau$ such that if $t > T_4$, $y(t) \geq N_1^y - \varepsilon$.

Again, for $\varepsilon > 0$ sufficiently small, it follows from the first and the second equations of system (1.2) that, for $t > T_4$,

$$\begin{aligned}
\dot{x}_1(t) &\leq x_1(t)\left(r_1 - a_{11}x_1(t) - \frac{a_{12}(N_1^y - \varepsilon)}{1 + B(M_1^{x_1} + \varepsilon) + C(N_1^y - \varepsilon)}\right) \\
&\quad + D_2x_2(t) - D_1x_1(t), \\
\dot{x}_2(t) &= -r_2x_2(t) + D_1x_1(t) - D_2x_2(t).
\end{aligned}$$

By Lemma 3.1 and a comparison argument, we derive that

$$\begin{aligned}
\limsup_{t\to+\infty} x_1(t) &\leq \frac{1}{a_{11}}\left[r_1 - \frac{a_{12}(N_1^y - \varepsilon)}{1 + B(M_1^{x_1} + \varepsilon) + C(N_1^y - \varepsilon)} - \frac{D_1r_2}{D_2 + r_2}\right], \\
\limsup_{t\to+\infty} x_2(t) &\leq \frac{D_1}{a_{11}(D_2 + r_2)}\left[r_1 - \frac{a_{12}(N_1^y - \varepsilon)}{1 + B(M_1^{x_1} + \varepsilon) + C(N_1^y - \varepsilon)}\right. \\
&\qquad\qquad \left. - \frac{D_1r_2}{D_2 + r_2}\right].
\end{aligned}$$

Since these inequalities are true for arbitrary $\varepsilon > 0$ sufficiently small, it follows that $U_1 \leq M_2^{x_1}$ and $U_2 \leq M_2^{x_2}$, where

$$\begin{aligned}
M_2^{x_1} &= \frac{1}{a_{11}}\left[r_1 - \frac{a_{12}N_1^y}{1 + BM_1^{x_1} + CN_1^y} - \frac{D_1r_2}{D_2 + r_2}\right], \\
M_2^{x_2} &= \frac{D_1}{a_{11}(D_2 + r_2)}\left[r_1 - \frac{a_{12}N_1^y}{1 + BM_1^{x_1} + CN_1^y} - \frac{D_1r_2}{D_2 + r_2}\right].
\end{aligned}$$

Hence, for $\varepsilon > 0$ sufficiently small, there is a $T_5 \geq T_4$ such that if $t > T_5$, $x_i(t) \leq M_2^{x_i} + \varepsilon$, $i = 1, 2$. We derive from the third equation of system (1.2) that, for $t > T_5 + \tau$,

$$\dot{y}(t) \leq \frac{a_{21}(M_2^{x_1} + \varepsilon)y(t - \tau)}{1 + B(M_2^{x_1} + \varepsilon) + Cy(t - \tau)} - ry(t). \tag{3.7}$$

By Lemma 3.2 and a standard comparison argument, it follows that

$$U = \limsup_{t\to+\infty} y(t) \le \frac{(a_{21} - Br)(M_2^{x_1} + \varepsilon) - r}{Cr}.$$

Since this is true for arbitrary $\varepsilon > 0$ sufficiently small, we conclude that $U \le M_2^y$, where

$$M_2^y = \frac{(a_{21} - Br)M_2^{x_1} - r}{Cr}.$$

Hence, for $\varepsilon > 0$ sufficiently small, there is a $T_6 \ge T_5 + \tau$ such that if $t > T_6$, $y(t) \le M_2^y + \varepsilon$.

For $\varepsilon > 0$ sufficiently small, it follows from the first and the second equations of system (1.2) that, for $t > T_6$,

$$\begin{aligned}
\dot{x}_1(t) &\ge x_1(t)\left(r_1 - a_{11}x_1(t) - \frac{a_{12}(M_2^y + \varepsilon)}{1 + B(N_1^{x_1} - \varepsilon) + C(M_2^y + \varepsilon)}\right) \\
&\quad + D_2x_2(t) - D_1x_1(t), \\
\dot{x}_2(t) &= -r_2x_2(t) + D_1x_1(t) - D_2x_2(t).
\end{aligned}$$

By Lemma 3.1 and a comparison argument, we derive that, for $t > T_6$,

$$\begin{aligned}
\liminf_{t\to+\infty} x_1(t) &\ge \frac{1}{a_{11}}\left[r_1 - \frac{a_{12}(M_2^y + \varepsilon)}{1 + B(N_1^{x_1} - \varepsilon) + C(M_2^y + \varepsilon)} - \frac{D_1r_2}{D_2 + r_2}\right], \\
\liminf_{t\to+\infty} x_2(t) &\ge \frac{D_1}{a_{11}(D_2 + r_2)}\left[r_1 - \frac{a_{12}(M_2^y + \varepsilon)}{1 + B(N_1^{x_1} - \varepsilon) + C(M_2^y + \varepsilon)} \right.\\
&\qquad \left. - \frac{D_1r_2}{D_2 + r_2}\right].
\end{aligned}$$

Since these inequalities hold true for arbitrary $\varepsilon > 0$ sufficiently small, we conclude that $V_1 \ge N_2^{x_1}$, $V_2 \ge N_2^{x_2}$, where

$$\begin{aligned}
N_2^{x_1} &= \frac{1}{a_{11}}\left[r_1 - \frac{a_{12}M_2^y}{1 + BN_1^{x_1} + CM_2^y} - \frac{D_1r_2}{D_2 + r_2}\right], \\
N_2^{x_2} &= \frac{D_1}{a_{11}(D_2 + r_2)}\left[r_1 - \frac{a_{12}M_2^y}{1 + BN_1^{x_1} + CM_2^y} - \frac{D_1r_2}{D_2 + r_2}\right].
\end{aligned}$$

Therefore, for $\varepsilon > 0$ sufficiently small, there is a $T_7 \ge T_6$ such that, if $t > T_7$, $x_i(t) \ge N_2^{x_i} - \varepsilon$, $i = 1, 2$.

For $\varepsilon > 0$ sufficiently small, it follows from the third equation of system (1.2) that, for $t > T_7 + \tau$

$$\dot{y}(t) \geq \frac{a_{21}(N_2^{x_1} - \varepsilon)y(t-\tau)}{1 + B(N_2^{x_1} - \varepsilon) + Cy(t-\tau)} - ry(t). \tag{3.8}$$

By Lemma 3.2 and a standard comparison argument, we obtain from (3.8) that

$$V = \liminf_{t\to+\infty} y(t) \geq \frac{(a_{21} - Br)(N_2^{x_1} - \varepsilon) - r}{Cr}.$$

Since this holds true for arbitrary $\varepsilon > 0$ sufficiently small, we conclude that $V \geq N_2^y$, where

$$N_2^y = \frac{(a_{21} - Br)N_2^{x_1} - r}{Cr}.$$

Continuing this process, we derive six sequences $M_n^{x_1}$, $M_n^{x_2}$, M_n^y, $N_n^{x_1}$, $N_n^{x_2}$, N_n^y, $n = 1, 2, \ldots$, such that for $n \geq 2$,

$$\begin{aligned}
M_n^{x_1} &= \frac{1}{a_{11}}\left[r_1 - \frac{a_{12}N_{n-1}^y}{1 + BM_{n-1}^{x_1} + CN_{n-1}^y} - \frac{D_1 r_2}{D_2 + r_2}\right],\\
M_n^{x_2} &= \frac{D_1}{a_{11}(D_2 + r_2)}\left[r_1 - \frac{a_{12}N_{n-1}^y}{1 + BM_{n-1}^{x_1} + CN_{n-1}^y} - \frac{D_1 r_2}{D_2 + r_2}\right],\\
M_n^y &= \frac{(a_{21} - Br)M_n^{x_1} - r}{Cr},\\
N_n^{x_1} &= \frac{1}{a_{11}}\left[r_1 - \frac{a_{12}M_n^y}{1 + BN_{n-1}^{x_1} + CM_n^y} - \frac{D_1 r_2}{D_2 + r_2}\right],\\
N_n^{x_2} &= \frac{D_1}{a_{11}(D_2 + r_2)}\left[r_1 - \frac{a_{12}M_n^y}{1 + BN_{n-1}^{x_1} + CM_n^y} - \frac{D_1 r_2}{D_2 + r_2}\right],\\
N_n^y &= \frac{(a_{21} - Br)N_n^{x_1} - r}{Cr}.
\end{aligned} \tag{3.9}$$

Clearly, we have

$$N_n^{x_i} \leq V_i \leq U_i \leq M_n^{x_i}, \quad i = 1, 2, \qquad N_n^y \leq V \leq U \leq M_n^y. \tag{3.10}$$

It is easy to see that the sequences $M_n^{x_1}$, $M_n^{x_2}$ and M_n^y are nonincreasing, $N_n^{x_1}$, $N_n^{x_2}$ and N_n^y are nondecreasing. Hence, we let

$$\begin{aligned} &\lim_{n\to+\infty} M_n^{x_1} = \bar{x}_1, \qquad \lim_{n\to+\infty} M_n^{x_2} = \bar{x}_2, \\ &\lim_{n\to+\infty} M_n^{y} = \bar{y}, \qquad \lim_{n\to+\infty} N_n^{x_1} = \underline{x}_1, \\ &\lim_{n\to+\infty} N_n^{x_2} = \underline{x}_2, \qquad \lim_{n\to+\infty} N_n^{y} = \underline{y}. \end{aligned} \tag{3.11}$$

We therefore derive from (3.9) and (3.11) that

$$\begin{aligned} \bar{x}_1 &= \frac{1}{a_{11}}\left[r_1 - \frac{a_{12}\underline{y}}{1+B\bar{x}_1+C\underline{y}} - \frac{D_1 r_2}{D_2+r_2}\right], \\ \underline{x}_1 &= \frac{1}{a_{11}}\left[r_1 - \frac{a_{12}\bar{y}}{1+B\underline{x}_1+C\bar{y}} - \frac{D_1 r_2}{D_2+r_2}\right], \\ \bar{x}_2 &= \frac{D_1}{D_2+r_2}\bar{x}_1, \qquad \bar{y} = \frac{(a_{21}-Br)\bar{x}_1 - r}{Cr}, \\ \underline{x}_2 &= \frac{D_1}{D_2+r_2}\underline{x}_1, \qquad \underline{y} = \frac{(a_{21}-Br)\underline{x}_1 - r}{Cr}. \end{aligned} \tag{3.12}$$

It follows from (3.12) that

$$\begin{aligned} &a_{11}BCr\bar{x}_1^2 + a_{11}C(a_{21}-Br)\bar{x}_1\underline{x}_1 \\ &\quad = Cr\left(r_1 - \frac{D_1 r_2}{D_2+r_2}\right)(1+B\bar{x}_1) \\ &\qquad + \left[C\left(r_1 - \frac{D_1 r_2}{D_2+r_2}\right) - a_{12}\right][(a_{21}-Br)\underline{x}_1 - r], \end{aligned} \tag{3.13}$$

and

$$\begin{aligned} &a_{11}BCr\underline{x}_1^2 + a_{11}C(a_{21}-Br)\bar{x}_1\underline{x}_1 \\ &\quad = Cr\left(r_1 - \frac{D_1 r_2}{D_2+r_2}\right)(1+B\underline{x}_1) \\ &\qquad + \left[C\left(r_1 - \frac{D_1 r_2}{D_2+r_2}\right) - a_{12}\right][(a_{21}-Br)\bar{x}_1 - r]. \end{aligned} \tag{3.14}$$

(3.13) minus (3.14),

$$\begin{aligned} a_{11}BCr(\bar{x}_1^2 - \underline{x}_1^2) &= BCr\left(r_1 - \frac{D_1 r_2}{D_2+r_2}\right)(\bar{x}_1 - \underline{x}_1) \\ &\quad + (a_{21}-Br)\left[C\left(r_1 - \frac{D_1 r_2}{D_2+r_2}\right) - a_{12}\right](\underline{x}_1 - \bar{x}_1). \end{aligned} \tag{3.15}$$

If $\bar{x}_1 \neq \underline{x}_1$, then we derive from (3.15) that
(3.16)

$$a_{11}BCr(\bar{x}_1 + \underline{x}_1) = C(2Br - a_{21})\left(r_1 - \frac{D_1 r_2}{D_2 + r_2}\right) + a_{12}(a_{21} - Br).$$

Hence, if (H4) holds, a contradiction occurs. Therefore, If (H4) holds, then $\bar{x}_1 = \underline{x}_1$.

Further, (3.13) plus (3.14),

$$\begin{aligned}(3.17)\quad a_{11}BCr(\bar{x}_1^2 + \underline{x}_1^2) &+ 2a_{11}C(a_{21} - Br)\bar{x}_1\underline{x}_1 \\ &= BCr\left(r_1 - \frac{D_1 r_2}{D_2 + r_2}\right)(\bar{x}_1 + \underline{x}_1) + 2a_{12}r \\ &\quad + (a_{21} - Br)\left[C\left(r_1 - \frac{D_1 r_2}{D_2 + r_2}\right) - a_{12}\right](\bar{x}_1 + \underline{x}_1).\end{aligned}$$

If (H5) holds, on substituting (3.16) into (3.17), we derive that

$$\begin{aligned}a_{11}C(a_{21} - 2Br)\bar{x}_1\underline{x}_1 = &\left[C\left(r_1 - \frac{D_1 r_2}{D_2 + r_2}\right) - a_{12}\right] \\ &\cdot [(a_{21} - Br)(\bar{x}_1 + \underline{x}_1)] + a_{12}r.\end{aligned}$$

Hence, if $Br < a_{21} < 2Br$, $C(r_1 - D_1 r_2/(D_2 + r_2)) > a_{12}$, a contradiction occurs. Hence, if (H5) holds, $\bar{x}_1 = \underline{x}_1$.

We therefore derive from (3.12) that if (H1), (H2), (H4) or (H1), (H2), (H5) hold, then $\bar{x}_2 = \underline{x}_2, \bar{y} = \underline{y}$. Accordingly, we have that

$$U_1 = V_1 = x_1^*, \qquad U_2 = V_2 = x_2^*, \qquad U = V = y^*.$$

It therefore follows that

$$\lim_{t\to+\infty} x_1(t) = x_1^*, \qquad \lim_{t\to+\infty} x_2(t) = x_2^*, \qquad \lim_{t\to+\infty} y(t) = y^*.$$

The proof is complete. □

Theorem 3.2. *If* (H1) *and* (H3) *hold, then the boundary equilibrium* $E_1(x_1^0, x_2^0, 0)$ *is globally stable, i.e., the prey species is permanent and the predator species goes to extinction.*

Proof. Let $(x_1(t), x_2(t), y(t))$ be any positive solution of system (1.2) with initial conditions (1.3). It follows from the first and the second equations of system (1.2) that

$$\begin{aligned} &\dot{x}_1(t) \le x_1(t)(r_1 - a_{11}x_1(t)) + D_2x_2(t) - D_1x_1(t), \\ &\dot{x}_2(t) = -r_2x_2(t) + D_1x_1(t) - D_2x_2(t). \end{aligned} \tag{3.18}$$

By comparison we derive that

$$\begin{aligned} &\limsup_{t\to+\infty} x_1(t) \le \frac{r_1(D_2 + r_2) - D_1r_2}{a_{11}(D_2 + r_2)} := M_1^{x_1}, \\ &\limsup_{t\to+\infty} x_2(t) \le \frac{D_1[r_1(D_2 + r_2) - D_1r_2]}{a_{11}(D_2 + r_2)^2} := M_1^{x_2}. \end{aligned} \tag{3.19}$$

Hence, for $\varepsilon > 0$ sufficiently small satisfying

$$(a_{21} - Br)\left(\frac{r_1(D_2 + r_2) - D_1r_2}{a_{11}(D_2 + r_2)} + \varepsilon\right) - r < 0, \tag{3.20}$$

there is a $T_1 > 0$ such that if $t > T_1$, $x_i(t) \le M_1^{x_i} + \varepsilon$, $i = 1, 2$.

It follows from the third equation of system (1.2) that, for $t > T_1 + \tau$,

$$\dot{y}(t) \le \frac{a_{21}(M_1^{x_1} + \varepsilon)y(t - \tau)}{1 + B(M_1^{x_1} + \varepsilon) + Cy(t - \tau)} - ry(t). \tag{3.21}$$

Consider the following auxiliary equation

$$\dot{u}(t) = \frac{a_{21}(M_1^{x_1} + \varepsilon)u(t - \tau)}{1 + B(M_1^{x_1} + \varepsilon) + Cu(t - \tau)} - ru(t). \tag{3.22}$$

By Lemma 3.2 it follows from (3.20) and (3.22) that

$$\lim_{t\to+\infty} u(t) = 0.$$

By comparison, we have

$$\lim_{t\to+\infty} y(t) = 0.$$

Hence, for $\varepsilon > 0$ sufficiently small satisfying (3.20), there is a $T_2 > T_1$ such that if $t > T_2$, $y(t) < \varepsilon$.

For $\varepsilon > 0$ sufficiently small satisfying (3.20), it follows from the first and the second equations of system (1.2) that, for $t > T_2$,

$$\dot{x}_1(t) \geq x_1(t)\left(r_1 - a_{11}x_1(t) - \frac{a_{12}\varepsilon}{1+C\varepsilon}\right) + D_2x_2(t) - D_1x_1(t),$$
$$\dot{x}_2(t) = -r_2x_2(t) + D_1x_1(t) - D_2x_2(t).$$

By comparison we derive that

$$\liminf_{t\to+\infty} x_1(t) \geq \frac{1}{a_{11}}\left[r_1 - \frac{a_{12}\varepsilon}{1+C\varepsilon} - \frac{D_1r_2}{D_2+r_2}\right],$$
$$\liminf_{t\to+\infty} x_2(t) \geq \frac{D_1}{a_{11}(D_2+r_2)}\left[r_1 - \frac{a_{12}\varepsilon}{1+C\varepsilon} - \frac{D_1r_2}{D_2+r_2}\right].$$

Since $\varepsilon > 0$ is arbitrarily small, we conclude that

$$\liminf_{t\to+\infty} x_1(t) \geq \frac{r_1(D_2+r_2) - D_1r_2}{a_{11}(D_2+r_2)},$$
$$\liminf_{t\to+\infty} x_2(t) \geq \frac{D_1[r_1(D_2+r_2) - D_1r_2]}{a_{11}(D_2+r_2)^2},$$

which, together with (3.19), yields

$$\lim_{t\to+\infty} x_1(t) = \frac{r_1(D_2+r_2) - D_1r_2}{a_{11}(D_2+r_2)} = x_1^0,$$
$$\lim_{t\to+\infty} x_2(t) = \frac{D_1[r_1(D_2+r_2) - D_1r_2]}{a_{11}(D_2+r_2)^2} = x_2^0.$$

This completes the proof. □

Using similar arguments as those in the proof of Theorem 3.2, we can obtain the following result.

Theorem 3.3. *If $r_1(D_2+r_2) < D_1r_2$, then the equilibrium $E_0(0,0,0)$ is globally stable, i.e., both the prey and the predator population go to extinction.*

4. Discussion. In this paper, we discussed the global dynamics of a two species predator-prey model with Beddington-DeAngelis type functional response and prey dispersal between two patches in which the breeding area in one of the patches is damaged. By using the iteration technique and by comparison arguments, respectively, we established sufficient conditions for the global attractiveness of the positive equilibrium and the global stability of each of the boundary equilibria of system (1.2). By Theorems 3.1, 3.2 and 3.3, we see that, if (H1)–(H2) hold, then both the prey and the predator of system (1.2) are permanent if (H4) or (H5) holds; the predator species is extinct and the prey species is permanent if (H3) holds; both the prey and the predator of system (1.2) go to extinction if $r_1(D_2 + r_2) - D_1 r_2 < 0$.

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