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A CLASS OF NONLINEAR MULTISTAGE DYNAMICAL SYSTEM AND ITS OPTIMAL CONTROL

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ABSTRACT. In this paper we study a nonlinear multi-stage dynamical system as well as its optimal control. Specifically, based on the batch fermentation including three different phases of bio-dissimilation of glycerol to 1,3-Propanediol by *Klebsiella pneumonicae*, the nonlinear multi-stage dynamical system is proposed. Then we discuss several properties of this nonlinear system. In order to optimize the initial state such that the concentration of 1,3-Propanediol at terminal time is as large as possible, an optimal control model is established. We investigate the existence of the local maximizer. Furthermore, by using the infinite-dimensional optimization principle, the necessary condition for the optimal control problem is obtained. Finally, employing some properties of the feasible region, infinite-dimensional constraints can be transformed into finite-dimensional constraints.

1. Introduction. In nature, kinestate in many problems, such as the control of modifying 3D horizontal wells trajectory while drilling [**9**], biotechnology, macroscopical or microcosmic control of economy and so on, has some characteristics, such as jump or speed change, for example. For this kinestate the common continuous differential dynamical system is not so valid that a new dynamical system, the nonlinear multistage dynamical system, has to be adopted. In this paper, we investigate both properties and optimal control of a nonlinear multistage dynamical system developed from a practical problem. Specifically, the system is developed from the batch fermentation of the bio-dissimilation of glycerol to 1,3-Propanediol by *Klebsiella pneumonicae* which is a popular subject.

Keywords and phrases. Bio-dissimilation, nonlinear multistage dynamical system, batch fermentation, optimal control.
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1,3-Propanediol (1,3-PD) is a very important chemical material. Polyesters which use 1,3-PD as a monomer have some excellent characteristics, such as strong capacity of pigmentation, weak capacity of adsorption water and so on. Hence, it attracts the increasing attention of many big companies all over the world; DuPont, Shell and Degussa, for example. At the present, the main technique to produce 1,3-PD is chemical synthesis. In the course of production silver is used as catalyst, so that this technique has some drawbacks such as high cost, severe pollution, etc. Since a new technique, the bio-dissimilation of glycerol to 1,3-Propanediol by *Klebsiella pneumonicae*, was proposed in the 1990s, the research on this new technique is becoming more and more popular. Compared with the traditional technique, the new technique has many distinct merits: soft production condition, being easy to operate and few byproducts, for example. However, up until now the technique has just been used in the laboratory.

With the exception of a lot of fermentation experiments, there has been much theoretical research on this new technique. In 1995, Zeng and Deckwer provided a kinetic model of the bio-dissimilation of glycerol to 1,3-PD [**10**]. The phenomena and characteristic of oscillation and hysteresis were studied in [**1, 6**]. In 2005, Xiu modified Zeng's kinetic model and used the excess kinetic model to describe the continuous and batch fermentation of the bio-dissimilation of glycerol to 1,3-PD. Based on Xiu's model, Gao et al. [**2**] and Li et al. [**5**] investigated the optimal control and stability of equilibrium of the continuous fermentation, respectively. With the development of this technique, researchers proposed the fed-batch fermentation of bio-dissimilation of glycerol to 1,3-PD. Gao established a nonlinear impulsive dynamical system in 2005 [**3**] to describe the process of fed-batch fermentation. The optimal control and some properties of this system were studied by Wang et al. [**8**] and Gao and Li [**4**].

In this paper, we briefly discuss the system of batch fermentation. According to the specific formation rate of 1,3-PD, the whole process of batch fermentation can be divided into three different phases which are: developmental, growth and stationary. Since the common continuous differential system cannot describe the three phases as good as well, we establish a nonlinear multi-stage dynamical system. Then the optimal control model of this system is established as well, in which the initial state is a control variable and the objective is to maximize the concentration of 1,3-PD at terminal moment. By the compactness of the festival region and the continuousness of the optimal function, the existence of the local maximizer is proved. Finally, in light of some properties of a closed set, infinite-dimensional constraint conditions are transformed into finite-dimensional constraint conditions. We get the necessary condition for the optimal control.

This paper is organized as follows: in Section 2, the nonlinear multistage dynamical system is proposed. In Section 3, we establish the optimal control model of the nonlinear system, prove existence of the local maximizer and present the necessary condition. Some conclusions are presented at the end of the paper.

2. The nonlinear multistage dynamical system and its properties. Considering the nonlinear dynamical system as below [**11**]:

$$\begin{cases} \dot{x}(t) = h(x(t), u_p) & t \in I = [t_0, t_f] \\ x(t_0) = x_0, \end{cases} \tag{1}$$

where $x(t) \in R^5$ is a state vector, the components of which denote the concentrations of biomass, glycerol, 1,3-PD, acetate and ethanol at t in the reactor respectively, I is the time-interval of batch fermentation and $h : R^5 \times R^{10} \to R^5$ is continuously differentiable.

$$h(x, u_p) = (\mu x_1(t), -q_2 x_1(t), q_3 x_1(t), q_4 x_1(t), q_5 x_1(t)) \in R^5 \tag{2}$$

$$\mu = u_p(1) \frac{x_2(t)}{x_2(t) + 0.28} \prod_{j=2}^{5} \left(1 - \frac{x_j(t)}{x_j^*}\right) \tag{3}$$

$$q_2 = u_p(2) + \frac{\mu}{u_p(3)} + u_p(4) \frac{x_2(t)}{x_2(t) + 11.43} \tag{4}$$

$$q_3 = u_p(5) + \mu u_p(6) + u_p(7) \frac{x_2(t)}{x_2(t) + 15.5} \tag{5}$$

$$q_4 = u_p(8) + \mu u_p(9) + u_p(10) \frac{x_2(t)}{x_2(t) + 85.71} \tag{6}$$

$$q_5 = q_2 \left(\frac{0.0025}{0.06 + \mu x_2(t)} + \frac{5.18}{50.45 + \mu^i x_2(t)} \right) \tag{7}$$

where μ denotes the specific growth rate of biomass, q_2 denotes the specific consumption rate of glycerol and q_3, q_4 and q_5 denote the specific formation rates of three products, respectively. x_i^* is the maximum of x_i, for all $i \in Q_5 := \{1,2,3,4,5\}$, which are positive numbers under some conditions. $u_p = (u_p(1), \dots, u_p(10)) \in R^{10}$ is a vector, the components of which are parameters in the model. The value of it has already been given.

In fact, the real process of fermentation contains three different phases. However system (1) cannot describe these phases sufficiently. Therefore, we will establish a new system, that is, a nonlinear multi-stage dynamical system:

$$\begin{cases} \dot{x}^1(t) = h^1(x^1(t), u_p^1) & t \in I_1 = [t_0, t_{f_1}], \\ x^1(t_0) = u, \end{cases} \tag{8}$$

$$\begin{cases} \dot{x}^2(t) = h^2(x^2(t), u_p^2) & t \in I_2 = [t_{f_1}, t_{f_2}], \\ x^2(t_{f_1}) = x^1(t_{f_1}), \end{cases} \tag{9}$$

$$\begin{cases} \dot{x}^3(t) = h^3(x^3(t), u_p^3) & t \in I_3 = [t_{f_2}, t_{f_3}], \\ x^3(t_{f_2}) = x^2(t_{f_2}), \end{cases} \tag{10}$$

where $x^1(t)$, $x^2(t)$ and $x^3(t)$ are state vectors of the developmental, growth and stationary phases, respectively. The meanings of components are the same as system (1). The initial state $u \in R^5$ is the element of the following set

$$U_{ad} := [0.001, 10] \times [200, 2500] \times \{0\} \times \{0\} \times \{0\} \subset R^5, \tag{11}$$

$t_{f_1}, t_{f_2}, t_{f_3}$ are terminal times of the developmental phase, growth phase and stationary phase, respectively, which satisfy

$$0 \le t_0 = t_{f_0} < t_{f_1} < t_{f_2} < t_{f_3} = t_f < \infty.$$

$h^i : R^5 \times R^{10} \to R^5$ is continuously differentiable on $I_i, i \in Q_3$.

$$h^i(x^i(t), u_p^i) = (\mu^i x_1^i(t), -q_2^i x_1^i(t), q_3^i x_1^i(t), q_4^i x_1^i(t), q_5^i x_1^i(t)) \in R^5 \tag{12}$$

$$\mu^i = u_p^i(1) \frac{x_2^i(t)}{x_2^i(t) + 0.28} \prod_{j=2}^{5} \left(1 - \frac{x_j^i(t)}{x_j^*} \right) \tag{13}$$

$$q_2^i = u_p^i(2) + \frac{\mu^i}{u_p^i(3)} + u_p^i(4)\frac{x_2^i(t)}{x_2^i(t) + 11.43} \tag{14}$$

$$q_3^i = u_p^i(5) + \mu^i u_p^i(6) + u_p^i(7)\frac{x_2^i(t)}{x_2^i(t) + 15.5} \tag{15}$$

$$q_4^i = u_p^i(8) + \mu^i u_p^i(9) + u_p^i(10)\frac{x_2^i(t)}{x_2^i(t) + 85.71} \tag{16}$$

$$q_5^i = q_2^i\left(\frac{0.0025}{0.06 + \mu^i x_2^i(t)} + \frac{5.18}{50.45 + \mu^i x_2^i(t)}\right) \tag{17}$$

where μ^i, q_2^i, q_3^i, q_4^i and q_5^i denote the specific growth rate of biomass, the specific consumption rate of glycerol and the specific formation rates of three products in different phases, respectively. $u_p^i = (u_p^i(1), \ldots, u_p^i(10)) \in R^{10}$ is a vector, the components of which are parameters of the ith phase, $i \in Q_3 := \{1, 2, 3\}$. x_j^* is defined by system (1), x_{*j} is the minimizer of x_j, for all $j \in Q_5$, which satisfy $x_{*j} < x_j^*$. Setting

$$S_0 := \prod_{j=1}^{5}[x_{*j}, x_j^*] \subset R^5. \tag{18}$$

It is obvious that $S_0 \subset R^5$ is a nonempty, bounded and closed set. The norm of y in R^5 is denoted by $\|\cdot\|$ and is defined by $\|y\| := \max_{i\in Q_5} |y_i|$. The norm of the function $x^i(t)$ is defined by $\|x^i\| = \sup\{\|x^i(t)\|, t \in I_i\}$, $i \in Q_3$.

Property 1. *Suppose that $h^i(x^i(t), u_p^i)$, for all $i \in Q_3$, are defined by (13)–(17), and for all $t \in [t_0, t_f], x^i(t) \in S_0$, then the following statements hold.*

(i) *$h^i(x^i(t), u_p^i)$ are Borel measurable on I, for all $i \in Q_3$.*

(ii) *There exists a constant $K > 0$, such that*

$$\|h^i(x^i(t), u_p^i)\| \le K(1 + \|x^i(t)\|), \text{ for all } x^i(t) \in S_0, i \in Q_3. \tag{19}$$

Furthermore, $h^i(x^i(t), u_p^i)$ and their gradients $\nabla_x h^i(x^i(t), u_p^i)$ are Lipschitz continuous on the bounded set S_0.

Proof. It is obvious that $h^i(x^i(t), u_p^i)$ are Borel measurable on $I, i \in Q_3$.

Next we will show that there exists a constant K such that $\|h^i(x^i(t), u_p^i)\| \le K(1 + \|x^i(t)\|)$ for all $x^i(t) \in S_0$, $i \in Q_3$.

Clearly, for any $t \in [t_0, t_f]$, there exists an $i \in Q_3$ such that $t \in I_i = [t_{f_{i-1}}, t_{f_i}]$. Now we will establish the Lipschitz continuity of each component of $h^i(x^i(t), u_p^i)$.

$$\begin{aligned}|h_1^i(x^i(t), u_p^i)| &= |\mu^i x_1^i(t)| \\ &= \left| u_p^i(1) \frac{x_2^i(t)}{x_2^i(t) + 0.28} \prod_{j=2}^{5} \left(1 - \frac{x_j^i(t)}{x_j^*}\right) x_1^i(t) \right| \\ &\le |u_p^i(1)||x_1^i(t)| \le |u_p^i(1)|\|x^i(t)\|.\end{aligned}$$

We conclude that $|h_1^i(x^i(t), u_p^i)| \le K_1^i \|x^i(t)\|$, where $K_1^i = |u_p^i(1)|$.

$$\begin{aligned}|h_2^i(x^i(t), u_p^i)| &= |-q_2^i x_1^i(t)| \\ &= \left| -(u_p^i(2) + \frac{\mu^i}{u_p^i(3)} + u_p^i(4) \frac{x_2^i(t)}{x_2^i(t) + 11.43}) x_1^i(t) \right| \\ &\le \left| u_p^i(2) + \frac{u_p^i(1)}{u_p^i(3)} + u_p^i(4) \right| |x_1^i(t)| \\ &\le \left| u_p^i(2) + \frac{u_p^i(1)}{u_p^i(3)} + u_p^i(4) \right| \|x^i(t)\|.\end{aligned}$$

We conclude that$|h_2^i(x^i(t), u_p^i)| \le K_2^i \|x^i(t)\|$, where $K_2^i = |u_p^i(2) + (u_p^i(1)/u_p^i(3)) + u_p^i(4)|$.

$$\begin{aligned}|h_3^i(x^i(t), u_p^i)| &= |q_3^i x_1^i(t)| \\ &\le |u_p^i(5) + u_p^i(1) u_p^i(6) + u_p^i(7)| \|x^i(t)\|.\end{aligned}$$

We conclude that $|h_3^i(x^i(t), u_p^i)| \le K_3^i \|x^i(t)\|$, where $K_3^i = |u_p^i(5) + u_p^i(1) u_p^i(6) + u_p^i(7)|$.

$$\begin{aligned}|h_4^i(x^i(t), u_p^i)| &= |q_4^i x_1^i(t)| \\ &\le |u_p^i(8) + u_p^i(1) u_p^i(9) + u_p^i(10)| \|x^i(t)\|.\end{aligned}$$

We conclude that $|h_4^i(x^i(t), u_p^i)| \le K_4^i \|x^i(t)\|$, where $K_4^i = |u_p^i(8) + u_p^i(1)u_p^i(9) + u_p^i(10)|$.

$$
\begin{aligned}
|h_5^i(x^i(t), u_p^i)| &= |q_5^i x_1^i(t)| \\
&= \left| q_2^i \left(\frac{0.0025}{0.06 + \mu^i x_2^i(t)} + \frac{5.18}{50.45 + \mu^i x_2^i(t)} \right) x_1^i(t) \right| \\
&\le \left| u_p^i(2) + \frac{u_p^i(1)}{u_p^i(3)} + u_p^i(4) \right| \left| \frac{0.0025}{0.06} + \frac{5.18}{50.45} \right| \|x^i(t)\|.
\end{aligned}
$$

We conclude that $|h_5^i(x^i(t), u_p^i)| \le K_5^i \|x^i(t)\|$, where $K_5^i = |u_p^i(2) + (u_p^i(1)/u_p^i(3)) + u_p^i(4)||(0.0025/0.06) + (5.18/50.45)|$.

Consequently, in terms of the definition of the norm, we know that for all $t \in [t_0, t_f]$, (19) holds, where $K = \max\{K_j^i \mid i \in Q_3, j \in Q_5\}$.

Finally, we will show that $h^i(x^i(t), u_p^i)$ and their gradients $\nabla_x h^i(x^i(t), u_p^i)$ are Lipschitz continuous on the bounded set S_0.

Suppose that $x^i(t), x^{i'}(t) \in S_0$, and $x^{i'}(t) = x^i(t) + \triangle x^i(t)$; by the mean-value theorem, we see that

$$
\begin{aligned}
&\|h^i(x^{i'}(t), u_p^i) - h^i(x^i(t), u_p^i)\| \\
&\qquad = \|h^i(x^i(t) + \triangle x^i(t), u_p^i) - h^i(x^i(t), u_p^i)\| \\
&\qquad = \left\| \frac{dh^i}{dx}(x^i(t) + \theta \triangle x^i(t), u_p^i) \triangle x^i \right\| \\
&\qquad \le \left\| \frac{dh^i}{dx}(x^i(t) + \theta \triangle x^i, u_p^i) \right\| \|\triangle x^i(t)\|,
\end{aligned}
$$

where $\theta \in (0, 1)$.

Making use of the definition of $h^i(x^i(t), u_p^i)$ and the boundedness of $x^i(t)$, we get the result that $\|(dh^i/dx)(x^i(t) + \theta \triangle x^i, u_p^i)\|$ is bounded, i.e., there exists an $L^i > 0$, such that $\|(dh^i/dx)(x^i(t) + \theta \triangle x, u_p^i)\| \le L^i$.

Consequently,

$$
\|h^i(x^{i'}(t), u_p^i) - h^i(x^i(t), u_p^i)\| \le L^i \|x^{i'}(t) - x^i(t)\|.
$$

In the same way, we get that $\|\nabla_x h^i(x^{i'}(t), u_p^i) - \nabla_x h^i(x^i(t), u_p^i)\| \le L^i \|x^{i'}(t) - x^i(t)\|$, where L^i satisfies $\|\nabla_{xx} h^i(x^i(t) + \theta_1 \triangle x, u_p^i)\| \le L^i$, $\theta_1 \in (0, 1)$. Our proof is complete. □

By Property 1 and the existence theorem for the solution to the differential equation, the following conclusion is easy to obtain.

Property 2. *Consider system* (8), (9), (10). *Then, for any* $u \in U_{ad}$, *there exists a unique solution to system* (8), (9), (10), *which is denoted as* $x^i(t) = x^i(t; x^{i-1}(t_{f_{i-1}}), u) \in R^5$, $i \in Q_3$.

The solution of system (8)–(10) can be written as below.

$$x^i(t; x^{i-1}(t_{f_{i-1}}), u) = x^{i-1}(t_{f_{i-1}}) + \int_{t_{f_{i-1}}}^{t} h^i(x^i(\tau), u_p^i)\, d\tau, \quad t \in [t_{f_{i-1}}, t_{f_i}], \tag{20}$$

where $x^0(t_{f_0}) = u$, $i \in Q_3$. □

Property 3. *Consider system* (8)–(10). *Then, for any* $u \in U_{ad}$, *there exists a constant* $M > 0$ *such that* $\|x^i(t; x^{i-1}(t_{f_{i-1}}), u)\| \le M$, *for all* $i \in Q_3$.

Proof. If $t \in I_1$, the solution of the system can be denoted as

$$x^1(t; u, u) = u + \int_{t_0}^{t} h^1(x^1(\tau), u_p^1)\, d\tau.$$

Hence,

$$\|x^1(t; u, u)\| \le \|u\| + K \int_{t_0}^{t} (1 + \|x^1(\tau; u, u)\|)\, d\tau.$$

The above inequality is equal to $y(t) \le y(t_0) + K \int_{t_0}^{t} y(\tau)\, d\tau$, where $y(t) = \|x^1(t; u, u)\| + 1$.

Making use of the Bellman-Gronwall lemma, we get the result that $y(t) \le y(t_0) \exp K$. Therefore,

$$\|x^1(t; u, u)\| \le (1 + \|u\|) \exp K, \quad \text{for all } t \in I_1.$$

Then, for any $t \in I_1$, there exists $M^1 = (1 + \|u\|) \exp K$ such that $\|x^1(t; u, u)\| \le M^1$. If $t \in I_2$, the solution of the system can be denoted as

$$x^2(t; x^1(t_{f_1}), u) = x^1(t_{f_1}) + \int_{t_{f_1}}^{t} h^2(x^2(\tau), u_p^2)\, d\tau.$$

Hence,

$$\|x^2(t; x^1(t_{f_1}), u)\| \le \|x^1(t_{f_1})\| + K \int_{t_{f_1}}^{t} (1 + \|x^2(t; x^1(t_{f_1}), u))\|\, d\tau.$$

The above inequality is equal to $y(t) \le y(t_{f_1}) + K \int_{t_{f_1}}^{t} y(\tau)\, d\tau$, where $y(t) = \|x^2(t; x^1(t_{f_1}), u)\| + 1$.

Making use of the Bellman-Gronwall lemma, we get the result that $y(t) \le y(t_{f_1}) \exp K$. Therefore,

$$\|x^2(t; x^1(t_{f_1}), u)\| \le (1 + \|x^1(t_{f_1})\|) \exp K \le (1 + M^1) \exp K, \quad \text{for all } t \in I_2.$$

Then, for any $t \in I_2$, there exists $M^2 = (1 + M^1) \exp K$ such that $\|x^2(t; x^1(t_{f_1}), u)\| \le M^2$.

In the same way, we can conclude that, for any $t \in I_3$, there exists an M^3 such that $\|x^3(t; x^2(t_{f_2}), u)\| \le M^3$. Consequently, for any $t \in I$, there exists an M such that $\|x^i(t; x^{i-1}(t_{f_{i-1}}), u)\| \le M$, $i \in Q_3$, where $M = \max\{M^1, M^2, M^3\}$. □

Property 4. *For any $t \in I$, the solution of system* (8)–(10) *is Lipschitz continuous on U_{ad}.*

Proof. Suppose $u, v \in U_{ad}$ and $u = v + \triangle v$. If $t \in I_1$, then

(21)

$$\begin{aligned} &\|x^1(t; u, u) - x^1(t; v, v)\| \\ &\quad = \left\| u + \int_{t_{f_0}}^{t} h^1(x^1(\tau; u, u), u_p^1)\, d\tau - v - \int_{t_{f_0}}^{t} h^1(x^1(\tau; v, v), u_p^1)\, d\tau \right\| \end{aligned}$$

$$\leq \|u - v\| + \int_{t_{f_0}}^{t} \|h^1(x^1(\tau; u, u), u_p^1) - h^1(x^1(\tau; v, v), u_p^1)\| \, d\tau$$
$$\leq \|u - v\| + L^1 \int_{t_{f_0}}^{t} \|x^1(\tau; u, u) - x^1(\tau; v, v)\| \, d\tau.$$

Formula (21) is equal to

$$y(t) \leq \|u - v\| + L^1 \int_{t_{f_0}}^{t} y(\tau) \, d\tau,$$

where $y(t) = \|x^1(t; u, u) - x^1(t; v, v)\|$. By the Bellman-Gronwall lemma, we see that, for any $t \in I_1$, $y(t) \leq \|u - v\| \exp L^1$. Therefore,

$$\|x^1(t; u, u) - x^1(t; v, v)\| \leq \|u - v\| \exp L^1 = l_1 \|u - v\|,$$

where $l_1 = \exp L^1$.

If $t \in I_2$, then

$$\begin{aligned} (22)\quad & \|x^2(t; x^1(t_{f_1}), u) - x^2(t; x^1(t_{f_1}), v)\| \\ &= \Big\| x^1(t_{f_1}; u, u) + \int_{t_{f_1}}^{t} h^2(x^2(\tau; x^1(t_{f_1}), u), u_p^2) \, d\tau \\ &\quad - x^1(t_{f_1}; v, v) - \int_{t_{f_1}}^{t} h^2(x^2(\tau; x^1(t_{f_1}), v), u_p^2) \, d\tau \Big\| \\ &\leq \|x^1(t_{f_1}; u, u) - x^1(t_{f_1}; v, v)\| \\ &\quad + \int_{t_{f_1}}^{t} \|h^2(x^2(\tau; x^1(t_{f_1}), u), u_p^2) - h^2(x^2(\tau; x^1(t_{f_1}), v), u_p^2)\| \, d\tau \\ &\leq \|x^1(t_{f_1}; u, u) - x^1(t_{f_1}; v, v)\| \\ &\quad + L^2 \int_{t_{f_1}}^{t} \|x^2(\tau; x^1(t_{f_1}), u) - x^2(\tau; x^1(t_{f_1}), v)\| \, d\tau \end{aligned}$$

Formula (22) is equal to

$$y(t) \leq \|x^1(t_{f_1}; u, u) - x^1(t_{f_1}; v, v)\| + L^2 \int_{t_{f_1}}^{t} y(\tau) \, d\tau,$$

where $y(t) = \|x^2(t; x^1(t_{f_1}), u) - x^2(t; x^1(t_{f_1}), v)\|$.

By the Bellman-Gronwall lemma, we see that, for any $t \in I_2$, $y(t) \leq \|x^1(t_{f_1}; u, u) - x^1(t_{f_1}; v, v)\| \exp L^2$. It follows from $\|x^1(t_{f_1}; u, u) - x^1(t_{f_1}; v, v)\| \leq l_1 \|u - v\|$ that

$$\|x^2(t; x^1(t_{f_1}), u) - x^2(t; x^1(t_{f_1}), v)\| \leq l_2 \|u - v\|,$$

where $l_2 = l_1 \exp L^2$.

In the same way, we conclude that, for any $t \in I_3$, $\|x^3(t; x^2(t_{f_2}), u) - x^3(t; x^2(t_{f_2}), v)\| \leq l_3 \|u - v\|$, where $l_3 = l_2 \exp L^3$, which completes our proof. □

Let the set of solutions of system $(7 + i)$ be denoted by S_i, $i \in Q_3$, i.e.,

$$S_i := \{x^i(t; x^{i-1}(t_{f_{i-1}}), u) \in C^1(I_i; R^5) \mid x^i(t; x^{i-1}(t_{f_{i-1}}), u) \text{ is the solution of system } (7+i) \text{ for any } u \in U_{ad}\}. \tag{23}$$

Now we define a mapping from U_{ad} into $S_1 \times S_2 \times S_3$, as below

$$A(u) = (x^1(t; u, u), x^2(t; x^1(t_{f_1}), u), x^3(t; x^2(t_{f_2}), u)) \in S_1 \times S_2 \times S_3.$$

Property 5. *Suppose that the sets S_i are defined by* (23). *Then the sets S_i are compact on $C^1(I_i; R^5)$, for all $i \in Q_3$.*

Proof. By Property 4, we see that $A(u)$ is continuous on U_{ad}. Moreover, since U_{ad} is compact in R^5, our result is completed. □

3. The optimal control of the nonlinear multistage dynamical system. The problem, to optimize the initial state such that the concentration of 1,3-Propanediol at the terminal moment is as large as possible, can be described as follows.

$$\begin{aligned}
\text{MOP}: \quad \min \quad & J(u) = -x_3^3(t_f; x^2(t_{f_2}), u) \\
\text{s.t.} \quad & A(u) = (x^1(t; u, u), x^2(t; x^1(t_{f_1}), u), x^3(t; x^2(t_{f_2}), u)) \\
& \in S_1 \times S_2 \times S_3 \\
& u \in U_{ad} \subset R^5 \\
& x^i(t; x^{i-1}(t_{f_{i-1}}), u) \in S_0 \quad \text{for all } t \in I_i, \quad i \in Q_3.
\end{aligned}$$

Theorem 1. *Consider the problem* MOP*. Then there exists a* $u^* \in U_{ad}$*, such that* $J(u^*) \le J(u)$*, for all* $u \in U_{ad}$*.*

Proof. By Property 4, we see that the performance function $J(u)$ is continuous on U_{ad}. Furthermore, since $U_{ad} \subset R^5$ is a nonempty, bounded and closed set, there exists a $u^* \in U_{ad}$ such that $J(u^*) \le J(u)$, for all $u \in U_{ad}$. □

The problem MOP has some infinite-dimensional constraints:

$$x^i(t; x^{i-1}(t_{f_{i-1}}), u) \in S_0, \quad \text{for all } t \in I_i = [t_{f_{i-1}}, t_{f_i}], \quad i \in Q_3. \tag{24}$$

In terms of (18), (24) is equal to the following inequalities:

$$x_{*j} \le x_j^i(t; x^{i-1}(t_{f_{i-1}}), u) \le x_j^* \quad \text{for all } t \in I_i, \quad (i,j) \in Q_3 \times Q_5. \tag{25}$$

For the sake of studying the infinite-dimensional constraints, we define the following functions

(26)
$$\begin{aligned} \varphi_j^i(t,u) &:= \widehat{\varphi}_j^i(t; x^i(t; x^{i-1}(t_{f_{i-1}}), u), u) \\ &:= x_j^i(t; x^{i-1}(t_{f_{i-1}}), u) - x_j^* \end{aligned}$$

(27)
$$\begin{aligned} \varphi_{j+5}^i(t,u) &:= \widehat{\varphi}_{j+5}^i(t; x^i(t; x^{i-1}(t_{f_{i-1}}), u), u) \\ &:= -x_j^i(t; x^{i-1}(t_{f_{i-1}}), u) + x_{*j}, \quad (i,j) \in Q_3 \times Q_5 \end{aligned}$$

$$f^0(u) := J(u) \tag{28}$$

$$f_j^i(u) := \max_{t \in I_i} \{\varphi_j^i(t,u)\}, \quad (i,j) \in Q_3 \times Q_{10}. \tag{29}$$

Since, for any $u \in U_{ad}$, $x^i(t; x^{i-1}(t_{f_{i-1}}), u)$ is continuously differential with respect to t on the interval $I_i \subset R_+$, the max function $f_j^i(u)$ is well defined on U_{ad}.

It follows from the theorem of variational analysis in Banach space [**7**] that the following property holds.

Property 6. *Suppose that* $\varphi_j^i(t,u)$ *are defined by* (26), (27) *and* $(i,j) \in Q_3 \times Q_{10}$*. Then the following statements hold*:

(a) *For any* $u \in U_{ad}$, $\varphi_j^i(t,u)$ *are continuous differential on* I_i, *for all* $i \in Q_3$.

(b) *For any* $t \in I_i$, $\varphi_j^i(t,u)$ *are continuous differential on* U_{ad} *and*

$$d_u \varphi_j^i(t,u) = \nabla_u \varphi_j^i(t,u)^T \, du, \quad (i,j) \in Q_3 \times Q_{10},$$

where $\nabla_u \varphi_j^i(t,u)$ *are defined by*

(30)
$$\nabla_u \varphi_j^i(t,u) = \nabla_u \hat{\varphi}_j^i(t; x^i(t; x^{i-1}(t_{f_{i-1}}), u), u) P_{tj}^i(t_{f_{i-1}}, u), \quad t \in I_i,$$

and $P_{tj}^i(\tau, u) \in R$ *is the solution of the adjoint equation*

$$\begin{cases} \dot{P}(\tau,u) = -\nabla_x h_j^i(x^i(\tau; x^{i-1}(t_{f_{i-1}}, u), u_p^i)^T P(\tau,u) & \tau \in [t_{f_{i-1}}, t] \\ P(t,u) = \nabla_x \hat{\varphi}_j^i(t; x^i(t; x^{i-1}(t_{f_{i-1}}), u), u). \end{cases} \quad \square$$

Property 7. *Suppose that* $f_j^i(u)$ *is defined by* (29) *and* $(i,j) \in Q_3 \times Q_{10}$. *Then the following statements hold*:

(a) *The function* $f_j^i(u)$ *is Lipschitz continuous on the bounded set* U_{ad}.

(b) (i) *For any* $u \in U_{ad}$ *and any* $\delta u \in U_{ad}$, *the directional derivative* $df_j^i(u; \delta u)$ *exists and is given by*

$$df_j^i(u;\delta u) = \max_{t \in T_j^i(u)} \{\nabla_u \varphi_j^i(t,u)^T \delta u\},$$

where $T_j^i(u) := \{t \in I_i \mid \varphi_j^i(t,u) = f_j^i(u)\}$.

(ii) *The directional derivative* $df_j^i(u;\delta u)$ *is upper semi-continuous on* $U_{ad} \times U_{ad}$ *and, for every* $u \in U_{ad}$, $df_j^i(u;\cdot)$ *is positively homogeneous, subadditive and Lipschitz continuous.* $\square$

By Property 7, the infinite-dimensional constraint conditions can be transformed into the finite-dimensional constraint conditions as below:

$$f_j^i(u) \le 0, \quad (i,j) \in Q_3 \times Q_{10}.$$

Therefore, the optimal control problem MOP is equivalent to MOP1.

$$\text{MOP1:} \quad \min\{f^0(u) \mid f_j^i(u) \le 0, (i,j) \in Q_3 \times Q_{10}, A(u) \in S_1 \times S_2 \times S_3, u \in U_{ad}\}.$$

It follows from $f^0(u) = J(u)$ that $f^0(u)$ is continuous differentiable on U_{ad}.

Let $u^* \in U_{ad}$ be a local minimizer for MOP, setting

$$F(u) := \max\{f^0(u^*) - f^0(u), \psi(u)\}, \tag{31}$$

where $\psi(u) := \max\{f_j^i(u) \mid (i,j) \in Q_3 \times Q_{10}\}$ and

$$d\psi(u^*; u - u^*) = \max_{(i,j)\in \mathbf{q}(u^*)} \{df_j^i(u^*; u - u^*)\},$$

where $\mathbf{q}(u^*) := \{(i,j) \in Q_3 \times Q_{10} \mid f_j^i(u^*) = \psi(u^*)\}$.

Theorem 2 (Optimal condition). *Suppose that u^* is the local minimizer of* MOP*. Then u^* is also the local minimizer of $F(u)$ and $dF(u^*; u - u^*) \ge 0$ for all $u \in U_{ad}$.*

Proof. Since u^* is the local minimizer of MOP, $f_j^i(u^*) \le 0$, $(i,j) \in Q_3 \times Q_{10}$ and $f^0(u^*) \le f^0(u)$, for all $u \in U_{ad}$.

It follows from $\psi(u) = \max\{f_j^i(u) \mid (i,j) \in Q_3 \times Q_{10}\}$ that $\psi(u^*) \le 0$. By (31),we see that $F(u^*) = \max\{f^0(u^*) - f^0(u), \psi(u^*)\} = 0$. For any $u \in U_{ad}$, if $\psi(u) > 0$, then $F(u) > 0$. And if $\psi(u) \le 0$, then $f^0(u^*) - f^0(u) \ge 0$, which also implies that $F(u) \ge 0$. Hence, u^* is a local minimizer for $F(u)$.

Assume that there exists an $u \in U_{ad}$ such that $dF(u^*; u - u^*) < 0, u \ne u^*$. By the definition of the directional derivative, we conclude that

$$\lim_{t \downarrow 0} \left[\frac{F(u^* + t(u - u^*)) - F(u^*)}{t} - dF(u^*; u - u^*) \right] = 0.$$

Then there exists a $t^* \in (0, \hat{\rho}/\|u - u^*\|)$ such that

$$\frac{F(u_1) - F(u^*)}{t^*} - dF(u^*; u - u^*) \le -\frac{1}{2} dF(u^*; u - u^*),$$

which is equal to

$$\begin{aligned} F(u_1) - F(u^*) &= F(u^* + t^*(u - u^*)) - F(u^*) \\ &\leq \frac{1}{2} t^* dF(u^*; u - u^*) < 0, \end{aligned}$$

where $u_1 = u^* + t^*(u - u^*) \in B(u^*, \widehat{\rho}) \cap U_{ad}$.

The above conclusion contradicts the fact that u^* is the local minimizer of $F(u)$. Hence, our result is true. □

4. Conclusions. In this paper, the properties of both the nonlinear multi-stage dynamical system and its optimal control were discussed. Then we investigated the existence of the local minimizer as well as the necessary conditions for the optimal control problem. The construction of the optimization algorithm of this nonlinear multistage dynamical system will be the main work in the future.

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