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GLOBAL STABILITY AND HOPF BIFURCATION ON A PREDATOR-PREY SYSTEM WITH DIFFUSION AND DELAYS

YUQUAN WANG

ABSTRACT. In this paper a predator-prey system with diffusion and two delays is considered, where the time delays are regarded as parameters. Its dynamics are studied in terms of permanence analysis and Hopf bifurcation analysis. By constructing a suitable Lyapunov function, sufficient conditions are obtained for both local and global stability of the positive equilibrium. An example is presented to show the main conclusion.

1. Introduction. In this paper we consider a system composed of two patches. The system has the predator species and the prey species. The prey species can diffuse between two patches, and the predator species is confined to one of the patches. Several authors established the persistence for predator-prey system with diffusion [**1, 2, 5–7, 9, 10, 13**].

Now we consider the following predator-prey system with diffusion and two discrete delays

$$\begin{aligned}
&\dot{x}_1(t) = x_1(t)\bigg(r_1 - a_1x_1(t) - \frac{c_1y(t-\tau_1)}{1+kx_1(t)}\bigg) + \delta(x_2(t)-x_1(t)),\\
(1.1)\quad &\dot{x}_2(t) = x_2(t)\Big(r_2 - a_2x_2(t-\tau_2)\Big) + \delta(x_1(t)-x_2(t)),\\
&\dot{y}(t) = y(t)\bigg(-d_1 + \frac{c_2x_1(t)}{1+kx_1(t)} - d_2y(t)\bigg),
\end{aligned}$$

where $x_1(t)$ and $y(t)$ are the numbers of prey and predator species in patch 1; $x_2(t)$ is the number of prey species in patch 2. The term

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$(x_1(t))/(1+kx_1(t))$ is the functional response of predator in patch 1. $\delta > 0$ is the diffusion coefficient; $\tau_1(>0)$ denotes the hunting time, see [**8**]; $\tau_2(>0)$ denotes the autumn of prey species in patch 2; the other parameters k, r_i, a_i, c_i, d_i, $i=1,2$, are all positive constants.

When $k=0$ and $\tau_1=\tau_2$ in (1.1), Gui and Ge [**4**] discussed the persistence and global stability. When $k=0$, $x_2(t)\equiv 0$, Song, Han and Wei [**8**] discussed local and global bifurcations. When $\delta=0$, $x_2(t)\equiv 0$ and $\tau_1=\tau_2=0$, a predator-prey system without diffusion and time delay is obtained (see, for example, [**11, 12**]).

The goal of this paper is to investigate the persistence and Hopf bifurcation of the system. By constructing a suitable Lyapunov function, we obtain sufficient conditions for both local and global stability of a positive equilibrium.

Let $X=(x_1,x_2,y)\in R^3_+=\{(x_1,x_2,y)\mid x_1\geq 0, x_2\geq 0, y\geq 0\}$. $X>0$ denotes that $X\in \operatorname{int} R^3_+$. By the practical meaning of the variables in (1.1), we discuss only $\operatorname{int} R^3_+$. The initial condition of (1.1) is given as

$$\phi(t)=(\phi_1(t),\phi_2(t),\phi_3(t))\in C^+,\quad \phi(0)>0, \tag{1.2}$$

where $C^+=([-\tau,0];R^3_+)$, $\tau=\max\{\tau_1,\tau_2\}$.

We call system (1.1) persistent if all solutions $\phi(t)$ of (1.1) with positive initial values satisfy

$$\lim_{t\to+\infty}\inf \phi(t)>0.$$

Clearly, the positive equilibrium $E^*(x_1^*,x_2^*,y^*)$ of (1.1) satisfies the following equations:

$$\begin{aligned} x_1\left(r_1-a_1x_1-\frac{c_1y}{1+kx_1}\right)+\delta(x_2-x_1)&=0,\\ x_2(r_2-a_2x_2)+\delta(x_1-x_2)&=0,\\ -d_1+\frac{c_2x_1}{1+kx_1}-d_2y&=0. \end{aligned} \tag{1.3}$$

To discuss the existence of positive equilibria of (1.1), we suppose that

$$(W_1)\qquad \delta<\min\{r_1,r_2\},\qquad c_2-kd_1>0.$$

From (1.3), we consider the following functions:

(1.4)
$$x_2 = \frac{a_1}{\delta} x_1 \left(x_1 - \frac{r_1 - \delta}{a_1} + \frac{c_1(c_2 - kd_1)}{a_1 d_2 (1 + kx_1)^2} \left(x_1 - \frac{d_1}{c_2 - kd_1} \right) \right) \equiv f(x_1),$$
$$x_2 = \frac{r_2 - \delta}{2a_2} + \sqrt{\left(\frac{r_2 - \delta}{2a_2} \right)^2 + \frac{\delta}{a_2} x_1} \equiv g(x_1).$$

Here, $g(x_1)$ is monotone increasing and convex. If x_1 sufficiently large, then $f(x_1)$ is monotone increasing and concave. From (1.4), we have $f(x_1) < 0$ as $0 < x_1 < \min\{(r_1 - \delta)/a_1, d_1/(c_2 - kd_1)\}$ and $f(x_1) > 0$ as $x_1 > \max\{(r_1 - \delta)/a_1, d_1/(c_2 - kd_1)\}$. This shows that $f(x_1)$ has at least a positive root. Clearly, $f(x_1)$ has at most three positive roots. Let x_{10} be the maximum root of $f(x_1)$; then $f(x_1) > 0$ for all $x_1 > x_{10}$.

Let $h(x_1) = f(x_1) - g(x_1)$. Then $h(x_{10}) < 0$ and $\lim_{x_1 \to +\infty} h(x_1) = +\infty$, so that $h(x_1)$ has at least a positive root $x_1^* (x_1^* > x_{10})$. Define

$$x_2^* = f(x_1^*), \qquad y^* = \frac{1}{d_2} \left(-d_1 + \frac{c_2 x_1^*}{1 + kx_1^*} \right). \tag{1.5}$$

Then $x_2^* > 0$ and $y^* > 0$. This implies that system (1.1) has at least a positive equilibrium $E^*(x_1^*, x_2^*, y^*)$. From (1.4), we have

$$f'(x_1) = \frac{a_1}{\delta} \left(2x_1 - \frac{r_1 - \delta}{a_1} + \frac{c_1 d_1}{a_1 d_2 A_0} \cdot \frac{(2 + kA_0)x_1 - A_0}{(1 + kx_1)^3} \right),$$
$$f''(x_1) = \frac{a_1}{\delta} \left(2 + \frac{c_1 d_1}{a_1 d_2 A_0} \cdot \frac{2 + 4kA_0 - 2k(2 + kA_0)x_1}{(1 + kx_1)^4} \right),$$

where $A_0 = d_1/(c_2 - kd_1) > 0$. Thus, there exists a positive number $M_f > x_{10}$ such that $f'(x_1) > 0$ and $f''(x_1) > 0$ as $x_1 > M_f$. This implies that $f(x_1)$ is monotone increasing and concave. Suppose that

$$\frac{r_2 - \delta}{a_2} \geq \max_{0 \leq x_1 \leq M_f} f(x_1). \tag{W_2}$$

Then the functions $f(x_1)$ and $g(x_1)$ have a positive crossover point, which implies that system (1.1) only has a positive equilibrium. By the above analysis, we obtain the following lemma.

Lemma 1.1. *Assume that* (W_1) *holds. Then*

(i) *System* (1.1) *has at least a positive equilibrium.*

(ii) *System* (1.1) *has only a positive equilibrium if* (W_2) *still holds.*

2. Persistence theorem. In this section, we discuss the persistence of system (1.1). We now prove the following lemma.

Lemma 2.1. *Let* $\phi(t) = (x_1(t), x_2(t), y(t))$ *denote every positive solution of system* (1.1) *which satisfies the initial condition* (1.2). *If* (W_1) *holds and* $c_2M_1^* - d_1 > 0$, *then there exists a constant* $T > 0$ *such that for* $t > T$

$$x_1(t) \le M_1, \qquad x_2(t) \le M_1, \qquad y(t) \le M_2, \tag{2.1}$$

where

$$M_1 > M_1^* = \max\left\{\frac{r_1}{a_1},\ \frac{r_2}{a_2}e^{r_2\tau_2},\right\}, \qquad M_2 > M_2^* = \frac{c_2M_1 - d_1}{d_2}.$$

Proof. Let $u(t) = \max\{x_1(t), x_2(t)\}$. We now consider the Dini derivative of $u(t)$ along the positive solutions of (1.1). Denote $\tau = \max\{\tau_1, \tau_2\}$.

If $x_1(t) \ge x_2(t)$, then we have for $t > \tau$

$$D^+u(t) = \dot{x}_1(t) \le x_1(t)(r_1 - a_1x_1(t)),$$

which implies that

$$\lim_{t\to+\infty}\sup x_1(t) \le \frac{r_1}{a_1}.$$

Hence, there exists a $T_1 > \tau$ such that $x_1(t) \le M_1$ for $t > T_1$.

If $x_1(t) \le x_2(t)$, then we have

$$D^+u(t) = \dot{x}_2(t) \le x_2(t)(r_2 - a_2x_2(t - \tau_2)). \tag{2.2}$$

From (2.2) we obtain $\dot{x}_2(t) \le r_2x_2$, which implies that for $t > \tau$

$$\int_{t-\tau_2}^{t} \frac{\dot{x}_2(s)}{x_2(s)}\,ds \le r_2\tau_2.$$

That is, $x_2(t-\tau_2) \geq e^{-r_2\tau_2}x_2(t)$. Hence, we have by (2.2),

$$D^+u(t) = \dot{x}_2(t) \leq x_2(t)(r_2 - a_2e^{-r_2\tau_2}x_2(t)),$$

which shows

$$\lim_{t\to+\infty} \sup x_2(t) \leq \frac{r_2}{a_2}e^{r_2\tau_2}.$$

That is, there exists a $T_2 > \tau$ such that $x_2(t) \leq M_1$ for $t > T_2$.

Hence, we have for $t > T_{12} = \max\{T_1, T_2\}$,

$$D^+u(t) \leq M_1.$$

From (1.1), we have for $t > T_{12}$,

$$\dot{y}(t) \leq y(t)(-d_1 + c_2M_1^* - d_2y(t)),$$

which shows

$$\lim_{t\to+\infty} \sup y(t) \leq \frac{c_2M_1^* - d_1}{d_2}.$$

Hence, there exists a $T_3 > T_{12}$ such that $y(t) \leq M_2$ for $t > T_3$. Let $T = \max\{T_1, T_2, T_3\}$; then the inequalities (2.1) hold for $t > T$.

Define

$$\begin{aligned} &0 < m_1 \leq m_1^* = \min\left\{\frac{r_1 - c_1M_2}{a_1},\ \frac{r_2}{a_2}e^{(r_2 - a_2M_1)\tau_2}\right\}, \\ &0 < m_2 \leq m_2^* = \frac{1}{d_2}\left(\frac{c_2m_1}{1+kM_1} - d_1\right). \end{aligned} \tag{2.3}$$

We suppose that

$$r_1 > c_1M_2, \qquad \frac{c_2m_1}{1+kM_1} > d_1. \tag{W_3}$$

Then we obtain the following theorem.

Theorem 2.1. *Assume that* (W_1) *and* (W_3) *hold. Then system* (1.1) *is persistent.*

Proof. Since $(c_2m_1^*)/(1+kM_1^*) > d_1$, we obtain $c_2M_1^* - d_1 > (c_2m_1^*)/(1+kM_1^*) - d_1 > 0$. By Lemma 2.1, there exists $T_0 > \tau$ such that for $t > T_0$

$$x_i(t) \le M_1 (i=1,2), \qquad y(t) \le M_2.$$

Denote $v(t) = \min\{x_1(t), x_2(t)\}$. Then we now calculate the Dini derivative of $v(t)$ along the solutions of (1.1).

If $x_2(t) \ge x_1(t)$, then we have

$$D_+v(t) = \dot{x}_1(t) \ge x_1(t)(r_1 - c_1M_2 - a_1x_1(t)),$$

which implies that

$$\lim_{t\to+\infty} \inf x_1(t) \ge \frac{r_1 - c_1M_2}{a_1}.$$

That is, there exists a $T_1 > T_0$ such that $x_1(t) \ge m_1$ for $t > T_1$.

If $x_2(t) \le x_1(t)$, then we have

$$D_+v(t) = \dot{x}_2(t) \ge x_2(t)(r_2 - a_2x_2(t-\tau_2)). \tag{2.4}$$

From (2.4), we have $\dot{x}_2(t) \ge (r_2 - a_2M_1)x_2(t)$, so that for $t > T_0$,

$$\int_{t-\tau_2}^{t} \frac{\dot{x}_2(s)}{x_2(s)}\, ds \ge (r_2 - a_2M_1)\tau_2,$$

that is, $x_2(t-\tau_2) \le e^{-(r_2-a_2M_1)\tau_2}x_2(t)$. Hence, we have from (2.4),

$$D_+v(t) = \dot{x}_2(t) \ge x_2(t)(r_2 - a_2e^{-(r_2-a_2M_1)\tau_2}x_2(t)),$$

which implies that there exists a $T_2 > T_0$ such that $x_2(t) \ge m_1$ for $t > T_2$. Choose $T_{12} = \max\{T_1, T_2\}$. Then, for $t > T_{12}$,

$$D_+v(t) \ge m_1.$$

From (1.1), we have for $t > T_{12}$,

$$\dot{y}(t) \ge y(t)\Big(-d_1 + \frac{c_2m_1}{1+kM_1} - d_2y(t)\Big),$$

which implies that there exists a $T_3 > T_{12}$ such that $y(t) \geq m_2$ for $t > T_3$.

Let $T = T_3$; then we have for $t > T$,

$$m_1 \leq x_i(t) \leq M_1 (i = 1, 2); \qquad m_2 \leq y(t) \leq M_2.$$

Hence, system (1.1) is persistence. This completes the proof. □

3. Global stability of a positive equilibrium. Now we discuss the local and global stability of a positive equilibrium for system (1.1) by constructing a suitable Lyapunov function.

Let $E^*(x_1^*, x_2^*, y^*)$ be a positive equilibrium of system (1.1). Define $u_1(t) = x_1(t) - x_1^*$, $u_2(t) = x_2(t) - x_2^*$ and $u_3(t) = y(t) - y^*$. Then the linearized system of (1.1) at E^* is given by

$$\begin{aligned} \dot{u}_1(t) &= -b_1 x_1^* u_1(t) - b_2 x_1^* u_3(t - \tau_1) + \delta\left(u_2(t) - \frac{x_2^*}{x_1^*} u_1(t)\right), \\ \dot{u}_2(t) &= -a_2 x_2^* u_2(t - \tau_2) + \delta\left(u_1(t) - \frac{x_1^*}{x_2^*} u_2(t)\right), \\ \dot{u}_3(t) &= b_3 y^* u_1(t) - d_2 y^* u_3(t), \end{aligned} \tag{3.1}$$

where

$$b_1 = a_1 - \frac{k c_1 y^*}{(1 + k x_1^*)^2}, \qquad b_2 = \frac{c_1}{1 + k x_1^*}, \qquad b_3 = \frac{c_2}{(1 + k x_1^*)^2}.$$

Now we can rewrite the above equations as follows:

(3.2)

$$\begin{aligned} \frac{d}{dt}\left(\frac{1}{x_1^*} u_1(t) - b_2 \int_{t-\tau_1}^{t} u_3(s)\, ds\right) &= -\left(b_1 + \frac{\delta x_2^*}{x_1^{*2}}\right) u_1(t) + \frac{\delta}{x_1^*} u_2(t) - b_2 u_3(t), \\ \frac{d}{dt}\left(\frac{1}{x_2^*} u_2(t) - a_2 \int_{t-\tau_2}^{t} u_2(s)\, ds\right) &= \frac{\delta}{x_2^*} u_1(t) - \left(a_2 + \frac{\delta x_1^*}{x_2^{*2}}\right) u_2(t), \\ \frac{d}{dt}\left(\frac{1}{y^*} u_3(t)\right) &= b_3 u_1(t) - d_2 u_3(t). \end{aligned}$$

We denote that

$$\beta_1 = 2\left(b_1 + \frac{\delta x_2^*}{x_1^{*2}}\right) - \delta\left(\frac{1}{x_1^*} + \frac{1}{x_2^*}\right) - \tau_1 b_2\left(b_1 x_1^* + \frac{\delta x_2^*}{x_1^*}\right) - \tau_2 a_2 \delta$$
$$\beta_2 = 2\left(a_2 + \frac{\delta x_1^*}{x_2^{*2}}\right) - \delta\left(\frac{1}{x_1^*} + \frac{1}{x_2^*}\right) - \tau_1 b_2 \delta - \tau_2 a_2\left(2a_2 x_2^* + \delta + \frac{\delta x_1^*}{x_2^*}\right)$$
$$\beta_3 = 2d_2 - \tau_1 b_3\left((2b_2 + b_1)x_1^* + \delta + \frac{\delta x_2^*}{x_1^*}\right).$$

Then we can obtain the following result.

Theorem 3.1. *Assume that $E^*(x_1^*, x_2^*, y^*)$ is a positive equilibrium of system* (1.1). *If $b_1 > 0$, $\beta_1 > 0$, $\beta_2 > 0$ and $\beta_3 > 0$. Then E^* is locally asymptotically stable.*

Proof. The linearized system of (1.1) at E^* is given by (3.1). Let $z(t) = (u_1(t), u_2(t), u_3(t))$ be the solutions of (3.1) with initial condition (1.2). Define

$$\begin{aligned} V_1(z(t)) &= \left(\frac{1}{x_1^*}u_1(t) - b_2\int_{t-\tau_1}^{t} u_3(s)\,ds\right)^2 \\ &\quad + b_2\left(b_1 + b_2 + \frac{\delta}{x_1^*} + \frac{\delta x_2^*}{x_1^{*2}}\right)\int_{t-\tau_1}^{t}\int_{s}^{t} u_3^2(\theta)\,d\theta\,ds, \\ V_2(z(t)) &= \left(\frac{1}{x_2^*}u_2(t) - a_2\int_{t-\tau_2}^{t} u_2(s)\,ds\right)^2 \\ &\quad + a_2\left(a_2 + \frac{\delta}{x_2^*} + \frac{\delta x_1^*}{x_2^{*2}}\right)\int_{t-\tau_2}^{t}\int_{s}^{t} u_2^2(\theta)\,d\theta\,ds, \\ V_3(z(t)) &= \left(\frac{1}{y^*}u_3(t)\right)^2. \end{aligned}$$

Then $V_i(z(t)) > 0$, $i = 1, 2, 3$, for $z(t) \neq 0$. Along the positive solutions of system (3.1), we have from (3.2)

$$\begin{aligned}\dot{V}_1(z(t)) &= 2\bigg(\frac{1}{x_1^*}u_1(t) - b_2\int_{t-\tau_1}^t u_3(s)\,ds\bigg)\\ &\quad\times\bigg(-\bigg(b_1+\frac{\delta x_2^*}{x_1^{*2}}\bigg)u_1(t)+\frac{\delta}{x_1^*}u_2(t)-b_2u_3(t)\bigg)\\ &\le -\bigg(b_1+\frac{\delta x_2^*}{x_1^{*2}}\bigg)\bigg(\frac{2}{x_1^*}-\tau_1 b_2\bigg)u_1^2(t)+\frac{\tau_1 b_2\delta}{x_1^*}u_2^2(t)\\ &\quad+\tau_1 b_2\bigg(2b_2+b_1+\frac{\delta}{x_1^*}+\frac{\delta x_2^*}{x_1^{*2}}\bigg)u_3^2(t)\\ &\quad+\frac{2\delta}{x_1^{*2}}u_1(t)u_2(t)-\frac{2b_2}{x_1^*}u_1(t)u_3(t),\end{aligned}\tag{3.3}$$

$$\begin{aligned}\dot{V}_2(z(t)) &= 2\bigg(\frac{1}{x_2^*}u_2(t) - a_2\int_{t-\tau_2}^t u_2(s)\,ds\bigg)\\ &\quad\times\bigg(\frac{\delta}{x_2^*}u_1(t)-\bigg(a_2+\frac{\delta x_1^*}{x_2^{*2}}\bigg)u_2(t)\bigg)\\ &\le \frac{\tau_2 a_2\delta}{x_2^*}u_1^2(t)+\frac{2\delta}{x_2^{*2}}u_1(t)u_2(t)\\ &\quad-\bigg(\frac{2}{x_2^*}\bigg(a_2+\frac{\delta x_1^*}{x_2^{*2}}\bigg)-\tau_2 a_2\bigg(2a_2+\frac{\delta}{x_2^*}+\frac{2\delta x_1^*}{x_2^{*2}}\bigg)\bigg)u_2^2(t),\end{aligned}\tag{3.4}$$

$$\begin{aligned}\dot{V}_3(z(t)) &= \frac{2}{y^{*2}}u_3(t)\Big(b_3y^*u_1(t)-d_2y^*u_3(t)\Big)\\ &= \frac{2b_3}{y^*}u_1(t)u_3(t)-\frac{2d_2}{y^*}u_3^2(t).\end{aligned}\tag{3.5}$$

Now we define a Lyapunov function as follows

$$V(z(t)) = \frac{x_1^*}{b_2}V_1(z(t))+\frac{x_2^*}{b_2}V_2(z(t))+\frac{y^*}{b_3}V_3(z(t)).$$

Denote that $\beta=\min\{(\beta_1/b_2),(\beta_2/b_2),(\beta_3/b_3)\}>0$. Then we have by (3.3), (3.4) and (3.5),

$$\dot{V}(z(t)) \le -\frac{\beta_1}{b_2}u_1^2(t)-\frac{\beta_2}{b_2}u_2^2(t)-\frac{\beta_3}{b_3}u_3^2(t)\le -\beta(u_1^2(t)+u_2^2(t)+u_3^2(t)).$$

By the Barbalat lemma [**3**], we obtain

$$\lim_{t\to+\infty}(u_1^2(t)+u_2^2(t)+u_3^2(t))=0.$$

Hence, the zero solution of (3.1) or positive equilibrium E^* of system (1.1) is locally asymptotically stable. This completes the proof. □

Now we discuss the global stability of a positive equilibrium of (1.1). Define

$$\sigma_1=a_1-\frac{kc_1y^*}{1+kx_1^*},\quad \sigma_2=\frac{\delta}{x_2^*}(1+a_2M_1^*\tau_2),\quad \sigma_3=a_2(1+a_2M_1^*\tau_2),$$

and we suppose that

$$(W_4)\qquad \sigma_1>0,\qquad d_2\left(\sigma_1\sigma_3-\frac{\delta}{x_1^*}\sigma_2\right)>c_1c_2\sigma_3.$$

Theorem 3.2. *Let $E^*(x_1^*,x_2^*,y^*)$ be a positive equilibrium of system (1.1). If the inequalities (W_1), (W_3) and (W_4) hold, then E^* is globally asymptotically stable.*

Proof. Denote that $v_1(t)=\ln(x_1(t)/x_1^*)$, $v_2(t)=\ln(x_2(t)/x_2^*)$, $v_3(t)=\ln(y(t)/y^*)$, that is, $x_1(t)=x_1^*e^{v_1(t)}$, $x_2(t)=x_2^*e^{v_2(t)}$ and $y(t)=y^*e^{v_3(t)}$. Now we can rewrite system (1.1) as follows

$$\begin{aligned}\dot v_1(t)&=-\left(a_1-\frac{kc_1y^*}{(1+kx_1^*)(1+kx_1^*e^{v_1(t)})}\right)x_1^*(e^{v_1(t)}-1)\\&\qquad-\frac{c_1y^*}{1+kx_1^*e^{v_1(t)}}(e^{v_3(t-\tau_1)}-1)+\frac{\delta x_2^*}{x_1^*}(e^{v_2(t)-v_1(t)}-1),\\ \dot v_2(t)&=-a_2x_2^*(e^{v_2(t-\tau_2)}-1)+\frac{\delta x_1^*}{x_2^*}(e^{v_1(t)-v_2(t)}-1),\\ \dot v_3(t)&=\frac{c_2x_1^*}{(1+kx_1^*)(1+kx_1^*e^{v_1(t)})}(e^{v_1(t)}-1)-d_2y^*(e^{v_3(t)}-1).\end{aligned}\tag{3.6}$$

From (3.6), we have

$$
\begin{aligned}
\dot{v}_2(t) &= -a_2 x_2^*(e^{v_2(t)} - 1) + \frac{\delta x_1^*}{x_2^*}(e^{v_1(t)-v_2(t)} - 1) \\
&\quad + a_2 x_2^*(e^{v_2(t)} - e^{v_2(t-\tau_2)}) \\
&= -a_2 x_2^*(e^{v_2(t)} - 1) + \frac{\delta x_1^*}{x_2^*}(e^{v_1(t)-v_2(t)} - 1) \\
&\quad + a_2 x_2^* \int_{t-\tau_2}^{t} e^{v_2(s)} \frac{dv_2(s)}{ds}\, ds \\
&= -a_2 x_2^*(e^{v_2(t)} - 1) + \frac{\delta x_1^*}{x_2^*}(e^{v_1(t)-v_2(t)} - 1) \\
&\quad + a_2 \int_{t-\tau_2}^{t} x_2^* e^{v_2(s)} \Big(- a_2 x_2^*(e^{v_2(s-\tau_2)} - 1) \\
&\qquad\qquad + \frac{\delta x_1^*}{x_2^*}(e^{v_1(s)-v_2(s)} - 1)\Big)\, ds.
\end{aligned} \tag{3.7}
$$

By Theorem 2.1, there exists a $T > 0$ such that $m_1 \le x_i(t) \le M_1$, $i = 1, 2$, $m_2 \le y(t) \le M_2$ for $t > T$. Along the solutions of (3.6), we now calculative the Dini derivative of $|v_i(t)|$, $i = 1, 2, 3$. From (3.6) and (3.7), we have

$$
\begin{aligned}
D^+|v_1(t)| &\le -x_1^*\left(a_1 - \frac{kc_1 y^*}{1 + kx_1^*}\right)|e^{v_1(t)} - 1| \\
&\qquad + c_1 y^* |e^{v_3(t-\tau_1)} - 1| + \frac{\delta x_2^*}{x_1^*} Q_1(t), \\
D^+|v_2(t)| &\le -a_2 x_2^* |e^{v_2(t)} - 1| + \frac{\delta x_1^*}{x_2^*} Q_2(t) \\
&\qquad + a_2^2 M_1 x_2^* \int_{t-\tau_2}^{t} |e^{v_2(s-\tau_2)} - 1|\, ds \\
&\qquad + \frac{a_2 \delta M_1 x_1^*}{x_2^*} \int_{t-\tau_2}^{t} Q_2(s)\, ds, \\
D^+|v_3(t)| &\le c_2 x_1^* |e^{v_1(t)} - 1| - d_2 y^* |e^{v_3(t)} - 1|,
\end{aligned} \tag{3.8}
$$

where

$$Q_1(t) = \begin{cases} e^{v_2(t)-v_1(t)} - 1 & \text{for } v_1(t) \geq 0, \\ 1 - e^{v_2(t)-v_1(t)} & \text{for } v_1(t) < 0, \end{cases} \tag{3.9}$$

$$Q_2(t) = \begin{cases} e^{v_1(t)-v_2(t)} - 1 & \text{for } v_2(t) \geq 0, \\ 1 - e^{v_1(t)-v_2(t)} & \text{for } v_2(t) < 0. \end{cases} \tag{3.10}$$

From (3.9) and (3.10), we obtain

$$Q_1(t) \leq |e^{v_2(t)} - 1|, \qquad Q_2(t) \leq |e^{v_1(t)} - 1|. \tag{3.11}$$

Let $z(t) = (v_1(t), v_2(t), v_3(t))$ be the solution of (3.6) with initial condition (1.2). Define

$$\begin{aligned} V_1(t) &= |v_1(t)| + c_1 y^* \int_{t-\tau_1}^{t} |e^{v_3(s)} - 1|\, ds, \\ V_2(t) &= |v_2(t)| + \frac{a_2 M_1 \delta x_1^*}{x_2^*} \int_{t-\tau_2}^{t} \int_{s}^{t} |e^{v_1(\theta)} - 1|\, d\theta\, ds \\ &\quad + a_2^2 M_1^* x_2^* \bigg(\int_{t-\tau_2}^{t} \int_{s}^{t} |e^{v_2(\theta-\tau_2)} - 1|\, d\theta\, ds \\ &\qquad + \tau_2 \int_{t-\tau_2}^{t} |e^{v_2(s)} - 1|\, ds \bigg), \\ V_3(t) &= |v_3(t)|. \end{aligned} \tag{3.12}$$

Along the solutions of (3.6), we have by (3.8) and (3.11)

$$\begin{aligned} D^+ V_1(t) &\leq -\left(a_1 - \frac{k c_1 y^*}{1 + k x_1^*} \right) x_1^* |e^{v_1(t)} - 1| \\ &\quad - \left(-\frac{\delta}{x_1^*} \right) x_2^* |e^{v_2(t)} - 1| - (-c_1) y^* |e^{v_3(t)} - 1|, \\ D^+ V_2(t) &\leq -\left(-\frac{\delta}{x_2^*} (1 + \tau_2 a_2 M_1) \right) x_1^* |e^{v_1(t)} - 1| \\ &\quad - a_2 (1 - \tau_2 a_2 M_1) x_2^* |e^{v_2(t)} - 1|, \\ D^+ V_3(t) &\leq -(-c_2) x_1^* |e^{v_1(t)} - 1| - d_2 y^* |e^{v_3(t)} - 1|. \end{aligned} \tag{3.13}$$

From (3.13), we denote the following matrix

$$J = \begin{pmatrix} \sigma_1 & -\delta/(x_1^*) & -c_1 \\ -\sigma_2 & \sigma_3 & 0 \\ -c_2 & 0 & d_2 \end{pmatrix}.$$

By (W_4), we have $\sigma_1 > 0$, $\sigma_1\sigma_3 - (\delta/x_1^*)\sigma_2 > 0$, $\det(J) = d_2(\sigma_1\sigma_3 - (\delta/x_1^*)\sigma_2) - c_1c_2\sigma_3 > 0$. This shows that the matrix J is an M matrix. Hence, there exist three positive constants ρ_1, ρ_2 and ρ_3 such that $\mu_1 = \sigma_1\rho_1 - \sigma_2\rho_2 - c_2\rho_3 > 0$, $\mu_2 = -(\delta/x_1^*)\rho_1 + \sigma_3\rho_2 > 0$, $\mu_3 = -c_1\rho_1 + d_2\rho_3 > 0$. Now define a Lyapunov function as follows

$$V(z(t)) = \rho_1 V_1(z(t)) + \rho_2 V_2(z(t)) + \rho_3 V_3(z(t)).$$

Since $x_i(t) = x_i^* e^{v_i(t)} \geq m_1$, $i = 1, 2$, and $y(t) = y^* e^{v_3(t)} \geq m_2$ for $t > T$, we have $x_i^*|e^{v_i(t)} - 1| = x_i^* e^{\xi_i}|v_i(t)| \geq m_1|v_i(t)|$, $i = 1, 2$, and $y^*|e^{v_3(t)} - 1| = y^* e^{\xi_3}|v_3(t)| \geq m_2|v_3(t)|$, where $0 < \xi_i < v_i(t)$ or $v_i(t) < \xi_i < 0$, $i = 1, 2, 3$. Hence, along the solutions of (3.6), we have for $t > T$,

$$D^+V(z(t)) \leq -\mu_0(|V_1(z(t))| + |V_2(z(t))| + |V_3(z(t))|).$$

where $\mu_0 = \min\{\mu_1 m_1, \mu_2 m_2, \mu_3 m_3\}$. This shows that the zero solution of (3.6) or the positive equilibrium E^* of system (1.1) is globally asymptotically stable. This completes the proof. □

Example 1. Let $k = 1$, $\delta = 0.05$, $r_1 = 0.8$, $r_2 = 0.5$, $a_1 = 0.12$, $a_2 = 0.18$, $c_1 = 0.02$, $c_2 = 0.07$, $d_1 = 0.01$ and $d_2 = 0.22$. Then system (1.1) has only one positive equilibrium $E^*(6.444, 3.081, 0.230)$. Suppose that $\tau_1 = 0.2$ and $\tau_2 = 0.8$. Then, by simple calculation, we can choose $M_1 = 7$, $M_2 = 2.5$, $m_1 = 1.5$, $m_2 = 0.01$, and the inequalities (W_1) and (W_3) hold. From (W_4), we have

$$\sigma_1 = 0.119 > 0, \qquad \sigma_1\sigma_3 - \frac{\delta}{x_1^*}\sigma_2 = 0.042 > 0,$$
$$d_2\left(\sigma_1\sigma_3 - \frac{\delta}{x_1^*}\sigma_2\right) - c_1c_2\sigma_3 = 0.0087 > 0,$$

which implies that condition (W_4) holds. By Theorem 3.2, the equilibrium E^* is globally asymptotically stable (see Figure 1).

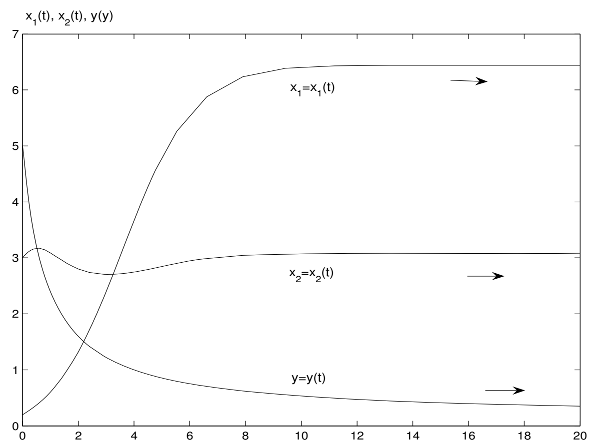

FIGURE 1.

4. Hopf bifurcation. In this section, we apply the Hopf bifurcation theorem to show the existence of a nontrivial periodic solution to system (1.1), and suppose that

$$\tau_1 = \tau_2 = \tau.$$

Then system (1.1) becomes

$$\begin{aligned} \dot{x_1}(t) &= x_1(t)\left(r_1 - a_1x_1(t) - \frac{c_1y(t-\tau)}{1+kx_1(t)}\right) + \delta(x_2(t) - x_1(t)), \\ \dot{x_2}(t) &= x_2(t)(r_2 - a_2x_2(t-\tau)) + \delta(x_1(t) - x_2(t)), \\ \dot{y}(t) &= y(t)\left(-d_1 + \frac{c_2x_1(t)}{1+kx_1(t)} - d_2y(t)\right). \end{aligned} \tag{4.1}$$

We use the delay as a parameter of bifurcation.

Let $E^*(x_1^*, x_2^*, y^*)$ be a positive equilibrium of system (4.1). Then the linearized system of (4.1) at E^* is given by

$$\begin{aligned} \dot{x}_1(t) &= -\left(b_1x_1^* + \frac{\delta x_2^*}{x_1^*}\right)x_1(t) + \delta x_2(t) - b_2x_1^*y(t-\tau), \\ \dot{x}_2(t) &= \delta x_1(t) - \frac{\delta x_1^*}{x_2^*}x_2(t) - a_2x_2^*x_2(t-\tau), \\ \dot{y}(t) &= b_3y^*x_1(t) - d_2y^*y(t). \end{aligned} \tag{4.2}$$

The characteristic equation of the linearized system (4.2) is as follows

$$\det\begin{pmatrix} -b_1x_1^* - (\delta x_2^*/x_1^*) - \lambda & \delta & -b_2x_1^*e^{-\lambda\tau} \\ \delta & -(\delta x_1^*/x_2^*) - a_2x_2^*e^{-\lambda\tau} - \lambda & 0 \\ b_3y^* & 0 & -d_2y^* - \lambda \end{pmatrix} = 0.$$

Thus, the following three degree exponential polynomial equation is obtained:

$$\lambda^3 + s_1\lambda^2 + s_2\lambda + s_3 + (s_4\lambda^2 + s_5\lambda + s_6)e^{-\lambda\tau} + s_7e^{-2\lambda\tau} = 0, \tag{4.3}$$

where

$$\begin{aligned}
s_1 &= d_2y^* + b_1x_1^* + \frac{\delta x_2^*}{x_1^*} + \frac{\delta x_1^*}{x_2^*},\\
s_2 &= d_2y^*\left(b_1x_1^* + \frac{\delta x_2^*}{x_1^*} + \frac{\delta x_1^*}{x_2^*}\right) + \frac{\delta b_1x_1^{*2}}{x_2^*},\\
s_3 &= \frac{\delta b_1d_2x_1^{*2}y^*}{x_2^*},\\
s_4 &= a_2x_2^*,\\
s_5 &= b_2b_3x_1^*y^* + a_2x_2^*\left(d_2y^* + b_1x_1^* + \frac{\delta x_2^*}{x_1^*}\right),\\
s_6 &= \frac{\delta b_2b_3x_1^{*2}y^*}{x_2^*} + d_2y^*(b_1x_1^* + \frac{\delta x_2^*}{x_1^*}),\\
s_7 &= a_2b_2b_3x_1^*x_2^*y^*.
\end{aligned}$$

Multiplying $e^{\lambda\tau}$ on both sides of (4.3), it is obvious to obtain

$$(\lambda^3 + s_1\lambda^2 + s_2\lambda + s_3)e^{\lambda\tau} + s_7e^{-\lambda\tau} = -s_4\lambda^2 - s_5\lambda - s_6. \tag{4.4}$$

Let $\lambda = i\omega$. Then (4.4) becomes

$$\begin{aligned}
(-i\omega^3 - s_1\omega^2 + is_2\omega + s_3)(\cos\omega\tau + i\sin\omega\tau) + s_7(\cos\omega\tau - i\sin\omega\tau)&\\
= s_4\omega^2 - s_6 - is_5\omega.&
\end{aligned}$$

Separating the real and imaginary parts, we have

$$\begin{aligned}
(-s_1\omega^2 + s_3 + s_7)\cos\omega\tau + (\omega^3 - s_2\omega)\sin\omega\tau &= s_4\omega^2 - s_6,\\
(-\omega^3 + s_2\omega)\cos\omega\tau + (-s_1\omega^2 + s_3 - s_7)\sin\omega\tau &= -s_5\omega.
\end{aligned}$$

By simple calculation, we obtain

$$(4.5)\qquad \begin{aligned} \cos\omega\tau &= \frac{n_4\omega^4+n_5\omega^2+n_6}{\omega^6+n_1\omega^4+n_2\omega^2+n_3},\\ \sin\omega\tau &= \frac{n_7\omega^5+n_8\omega^3+n_9\omega}{\omega^6+n_1\omega^4+n_2\omega^2+n_3}, \end{aligned}$$

where $n_1 = s_1^2 - 2s_2$, $n_2 = s_2^2 - 2s_1s_3$, $n_3 = s_3^2 - s_7^2$, $n_4 = -s_1s_4 + s_5$, $n_5 = s_4(s_3 - s_7) + s_1s_6 - s_2s_5$, $n_6 = -s_6(s_3 - s_7)$, $n_7 = s_4$, $n_8 = s_1s_5 - s_2s_4 - s_6$ and $n_9 = s_2s_6 - s_5(s_3 + s_7)$. Hence, we have by $\sin^2\omega\tau + \cos^2\omega\tau = 1$

$$(4.6)\qquad \omega^{12}+h_5\omega^{10}+h_4\omega^8+h_3\omega^6+h_2\omega^4+h_1\omega^2+h_0=0,$$

where $h_0 = n_3^2 - n_6^2$, $h_1 = 2n_2n_3 - 2n_5n_6 - n_9^2$, $h_2 = 2n_1n_3 + n_2^2 - 2n_4n_6 - n_5^2 - 2n_8n_9$, $h_3 = 2n_3 + 2n_1n_2 - 2n_4n_5 - 2n_7n_9 - n_8^2$, $h_4 = 2n_2 + n_1^2 - n_4^2 - 2n_7n_8$ and $h_5 = 2n_1 - n_7^2$. Define $z = \omega^2$; then (4.6) becomes

$$(4.7)\qquad z^6+h_5z^5+h_4z^4+h_3z^3+h_2z^2+h_1z+h_0=0.$$

Suppose

(W_5) Equation (4.7) has at least one positive real root.

Without loss of generality, assume that (4.7) has six positive real roots, defined by z_1, z_2, z_3, z_4, z_5, z_6, respectively. Then (4.6) has six positive roots

$$\omega_p = \sqrt{z_p}, \quad p = 1,2,3,4,5,6.$$

From (4.5), we have

$$\cos\omega_p\tau = \frac{n_4\omega_p^4+n_5\omega_p^2+n_6}{\omega_p^6+n_1\omega_p^4+n_2\omega_p^2+n_3}, \quad p=1,2,3,4,5,6.$$

Denote

$$(4.8)\qquad \tau_p^j = \frac{1}{\omega_p}\left(\arccos\frac{n_4\omega_p^4+n_5\omega_p^2+n_6}{\omega_p^6+n_1\omega_p^4+n_2\omega_p^2+n_3}+2j\pi\right),$$

where $p = 1, 2, 3, 4, 5, 6$; $j = 0, 1, 2, \ldots$. Then $\pm i\omega_p$ is a pair of purely imaginary roots of (4.3) with τ_p^j. Define

$$\tau_0 = \tau_{p_0}^0 = \min_{1 \leq p \leq 6} \{\tau_p^0\}, \quad \omega_0 = \omega_{p_0}. \tag{4.9}$$

In order to give the main results, we assume that

(W_6) $(s_1 + s_4)(s_2 + s_5) > s_3 + s_6 + s_7$.

(W_7) ω_0 and τ_0 satisfy the following inequality:

$$\begin{aligned} H_0 = & (-s_5\omega_0^2 + 2s_7\omega_0 \sin \omega_0\tau_0) \\ & \times (s_5 + (-3\omega_0^2 + s_2) \cos \omega_0\tau_0 - 2s_1\omega_0 \sin \omega_0\tau_0) \\ & + (-s_4\omega_0^3 + s_6\omega_0^2 + 2s_7\omega_0 \cos \omega_0\tau_0)(2s_4\omega_0 + 2s_1\omega_0 \cos \omega_0\tau_0 \\ & \qquad + (-3\omega_0^2 + s_2) \sin \omega_0\tau_0) \neq 0. \end{aligned}$$

Theorem 4.1. *Let $E^*(x_1^*, x_2^*, y^*)$ be a positive equilibrium of system* (4.1). *If* (W_5), (W_6) *and* (W_7) *hold, then the following conclusions are obtained*:

(i) *The zero solution of* (4.2) *or the positive equilibrium E^* of* (4.1) *is asymptotically stable for $\tau \in [0, \tau_0)$.*

(ii) *System* (4.1) *undergoes a Hopf bifurcation at E^* when $\tau = \tau_0$. That is, system* (4.1) *has a branch of periodic solutions bifurcating from E^* near $\tau = \tau_0$.*

Proof. By (W_5), we know that (4.8) and (4.9) hold. When $\tau = 0$, equation (4.3) becomes

$$\lambda^3 + (s_1 + s_4)\lambda^2 + (s_2 + s_5)\lambda + s_3 + s_6 + s_7 = 0. \tag{4.10}$$

Since $s_i > 0$, $i = 1, \ldots, 7$, we have from (W_6)

$$\begin{aligned} \triangle_1 &= s_1 + s_4 > 0, \\ \triangle_2 &= \det \begin{pmatrix} s_1 + s_4 & 1 \\ s_3 + s_6 + s_7 & s_2 + s_5 \end{pmatrix} \\ &= (s_1 + s_4)(s_2 + s_5) - (s_3 + s_6 + s_7) > 0, \\ \triangle_3 &= \det \begin{pmatrix} s_1 + s_4 & 1 & 0 \\ s_3 + s_6 + s_7 & s_2 + s_5 & s_1 + s_4 \\ 0 & 0 & s_3 + s_6 + s_7 \end{pmatrix} \\ &= (s_3 + s_6 + s_7)((s_1 + s_4)(s_2 + s_5) - (s_3 + s_6 + s_7)) > 0. \end{aligned}$$

By the Rowth-Hurwitz criteria, equation (4.10) has three roots with negative real part, so that the zero solution of (4.2) or the positive equilibrium E^* of (4.1) is asymptotically stable as $\tau = 0$.

Taking the derivative of λ with respect to τ in (4.4), we have

$$(3\lambda^2 + 2s_1\lambda + s_2)e^{\lambda\tau}\frac{d\lambda}{d\tau} + (\lambda^3 + s_1\lambda^2 + s_2\lambda + s_3)e^{\lambda\tau}\left(\lambda + \tau\frac{d\lambda}{d\tau}\right) + (2s_4\lambda + s_5)\frac{d\lambda}{d\tau} + s_7e^{-\lambda\tau}\left(-\lambda - \tau\frac{d\lambda}{d\tau}\right) = 0,$$

which implies that

$$\frac{d\lambda}{d\tau} = \frac{-\lambda \mathrm{L}(\lambda)}{(3\lambda^2 + 2s_1\lambda + s_2)e^{\lambda\tau} + 2s_4\lambda + s_5 + \tau L(\lambda)},$$

where

$$L(\lambda) = (\lambda^3 + s_1\lambda^2 + s_2\lambda + s_3)e^{\lambda\tau} - s_7e^{-\lambda\tau}. \tag{4.11}$$

For $\omega = \omega_0$ and $\tau = \tau_0$, we have by (4.4) and (4.11)

$$\begin{aligned}\left(\frac{d\lambda}{d\tau}\right)^{-1}\Bigg|_{\tau=\tau_0} &= \left(\frac{(3\lambda^2 + 2s_1\lambda + s_2)e^{\lambda\tau} + 2s_4\lambda + s_5}{-\lambda \mathrm{L}(\lambda)} - \frac{\tau}{\lambda}\right)\Bigg|_{\tau=\tau_0}\\ &= \left(\frac{(3\lambda^2 + 2s_1\lambda + s_2)e^{\lambda\tau} + 2s_4\lambda + s_5}{s_4\lambda^3 + s_5\lambda^2 + s_6\lambda + 2s_7\lambda e^{-\lambda\tau}} - \frac{\tau}{\lambda}\right)\Bigg|_{\tau=\tau_0}\\ &= \frac{(-3\omega_0^2 + i2s_1\omega_0 + s_2)(\cos\omega_0\tau_0 + i\sin\omega_0\tau_0) + i2s_4\omega_0 + s_5}{-is_4\omega_0^3 - s_5\omega_0^2 + is_6\omega_0 + i2s_7\omega_0(\cos\omega_0\tau_0 - i\sin\omega_0\tau_0)} - \frac{\tau_0}{i\omega_0}\\ &= \frac{H_{21} + iH_{22}}{H_{11} + iH_{12}} + i\frac{\tau_0}{\omega_0},\end{aligned}$$

where

$$\begin{aligned}H_{11} &= -s_5\omega_0^2 + 2s_7\omega_0\sin\omega_0\tau_0,\\ H_{12} &= -s_4\omega_0^3 + s_6\omega_0^2 + 2s_7\omega_0\cos\omega_0\tau_0,\\ H_{21} &= s_5 + (-3\omega_0^2 + s_2)\cos\omega_0\tau_0 - 2s_1\omega_0\sin\omega_0\tau_0,\\ H_{22} &= 2s_4\omega_0 + 2s_1\omega_0\cos\omega_0\tau_0 + (-3\omega_0^2 + s_2)\sin\omega_0\tau_0.\end{aligned}$$

Hence, we obtain from (W_7),

$$\mathrm{Re}\left(\frac{d\lambda}{d\tau}\right)^{-1}\Bigg|_{\tau=\tau_0} = \frac{H_{11}H_{21} + H_{12}H_{22}}{H_{11}^2 + H_{12}^2} = \frac{H_0}{H_{11}^2 + H_{12}^2} \neq 0.$$

Noticing that

$$\operatorname{sign}\left(\operatorname{Re}\left(\frac{d\lambda}{d\tau}\right)\bigg|_{\tau=\tau_0}\right)=\operatorname{sign}\left(\operatorname{Re}\left(\frac{d\lambda}{d\tau}\right)^{-1}\bigg|_{\tau=\tau_0}\right),$$

we know that system (4.1) undergoes a Hopf bifurcation at E^* as $\tau=\tau_0$. This completes the proof. $\square$

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Department of Applied Mathematics, Nanjing University of Finance and Economics, Nanjing, 210003 China
Email address: wyuquan@163.com