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MODELING AND ANALYSIS OF A DELAYED COMPETITIVE SYSTEM WITH IMPULSIVE PERTURBATIONS

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ABSTRACT. In this paper, a periodic delayed competitive system with impulsive perturbations is proposed. By using the property of globally asymptotic stability of a periodic single-species growth population model with impulse, sufficient conditions for the permanence of the above impulsive system without delays are derived. Later, the existence of positive periodic solutions of the above impulsive system with delays is discussed. As an application, an example and its numerical simulations are presented to illustrate the feasibility of the main results. Biological interpretations on our main results are also given.

1. Introduction.

In [**6**], Golpasamy introduced the following autonomous two-species competitive system with a single constant delay

$$\begin{aligned} y_1'(t) &= y_1(t)\left[r_1 - a_1 y_1(t-\tau) - \frac{c_2 y_2(t-\tau)}{1+y_2(t-\tau)}\right], \\ y_2'(t) &= y_2(t)\left[r_2 - a_2 y_1(t-\tau) - \frac{c_1 y_1(t-\tau)}{1+y_1(t-\tau)}\right], \end{aligned} \tag{1.1}$$

where r_i, a_i and c_i, $i = 1, 2$, are all positive constants, and τ is a nonnegative constant. From system (1.1), it is easy to see that one species is governed by the following well-known Wright equation, see [**7**], when the other is absent

$$y'(t) = y(t)[r - ay(t-\tau)]. \tag{1.2}$$

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In general, in system (1.1) the intra-species and interaction terms may have different delays, see [**6, 16, 19**], and any biological and environmental parameters are naturally subject to fluctuation in time. So it is realistic to consider a periodic competitive system involving multiple delays modeled by

$$\begin{aligned} y_1'(t) &= y_1(t)\left[r_1(t) - a_1(t)y_1(t-\tau_1(t)) - \frac{c_2(t)y_2(t-\sigma_2(t))}{1+y_2(t-\sigma_2(t))}\right], \\ y_2'(t) &= y_2(t)\left[r_2(t) - a_2(t)y_2(t-\tau_2(t)) - \frac{c_1(t)y_1(t-\sigma_1(t))}{1+y_1(t-\sigma_1(t))}\right], \end{aligned} \tag{1.3}$$

where $r_i(t)$, $a_i(t)$, $c_i(t) \in C(R,(0,\infty))$, $\tau_i(t)$, $\sigma_i \in C(R,[0,\infty))$, $i = 1,2$, are ω-periodic functions.

On the other hand, an ecological system can often be affected by human activities. In many practical situations the population of the system may increase by stocking or decrease by harvesting. If two species of system (1.3) will be harvested, a corresponding delayed periodic competitive system with harvesting can be written as

$$\begin{aligned} y_1'(t) &= y_1(t)\left[r_1(t) - a_1(t)y_1(t-\tau_1(t)) - \frac{c_2(t)y_2(t-\sigma_2(t))}{1+y_2(t-\sigma_2(t))}\right] - E_1(y_1), \\ y_2'(t) &= y_2(t)\left[r_2(t) - a_2(t)y_2(t-\tau_2(t)) - \frac{c_1(t)y_1(t-\sigma_1(t))}{1+y_1(t-\sigma_1(t))}\right] - E_2(y_2). \end{aligned} \tag{1.4}$$

Here the functions $E_1(y_1)$ and $E_2(y_2)$ are nonnegative and represent the effects of harvesting on y_1 and y_2, respectively. If $E_1(y_1) = \overline{E}_1$, $E_2(y_2) = \overline{E}_2$ are positive constants, the terms represent constant time rates at which y_1 and y_2, respectively, are harvested from system (1.4). If $E_1(y_1) \equiv 0$, $E_2(y_2) \equiv 0$, the terms represent no harvesting. If $E_1(y_1) = \overline{E}_1 y_1$, $E_2(y_2) = \overline{E}_2 y_2$, the terms represent that harvesting is proportional to the current density of y_1 and y_2, respectively.

Model (1.4) has invariably assumed that human activities occur continuously, whereas it is often the case that the harvesting of two species is seasonal or occurs in regular pulses. Based on the above facts, the continuous activities of human are then removed from the model and replaced with impulsive harvesting. Such a revised version is subject to short term perturbations which are often assumed to be in the form of impulses in the modeling process. Consequently, the impulsive differential equation provides a natural description of such a system. Equations of this kind are found in almost every domain of applied sciences. Numerous examples are given in [**1**]. Some impulsive equations have been recently introduced in population dynamics in relation to: chemotherapeutic treatment of disease [**8**], vaccination [**11, 21**], population ecology [**6**–**8**] and integrated pest management [**17, 18**], etc.

In this paper, with the idea of impulsive perturbations, we consider the following delayed periodic competitive impulsive system

$$(1.5)\quad \begin{cases} \left.\begin{aligned} y_1'(t) &= y_1(t)\left[r_1(t) - a_1(t)y_1(t-\tau_1(t)) - \frac{c_2(t)y_2(t-\sigma_2(t))}{1+y_2(t-\sigma_2(t))}\right] \\ y_2'(t) &= y_2(t)\left[r_2(t) - a_2(t)y_2(t-\tau_2(t)) - \frac{c_1(t)y_1(t-\sigma_1(t))}{1+y_1(t-\sigma_1(t))}\right] \end{aligned}\right\}, \ t \neq t_k,\ k \in \mathcal{N}, \\ \left.\begin{aligned} y_1(t_k^+) &= (1+b_{1k})y_1(t_k) \\ y_2(t_k^+) &= (1+b_{2k})y_2(t_k) \end{aligned}\right\}, \ t = t_k,\ k \in \mathcal{N} \end{cases}$$

with

$$(1.6)\qquad \begin{aligned} y_i(t) = \phi_i(t), \ \text{for } -r \le t \le 0, \qquad &\phi_i \in L([-r,0],[0,\infty)), \\ &\phi_i(0) > 0, \quad i = 1,2, \end{aligned}$$

where $L([-r,0],[0,\infty))$ denotes the set of Lebesgue measurable functions on $[-r,0]$, $r = \max_{1\le i\le 2}\max_{t\in[0,\omega]}\{\tau_i(t),\sigma_i(t)\}$, and $\mathcal{N}$ is the set of positive integers. When $b_{ik} > 0$, the perturbations stand for stocking while $b_{ik} < 0$ means harvesting.

Now, we consider the following four hypotheses and two definitions.

(H1) $0 < t_1 < t_2 < \dots$ are fixed impulsive points with $\lim_{k\to\infty} t_k = \infty, k \in \mathcal{N}$;

(H2) $\{b_{ik}\}$ is a real sequence and $b_{ik} > -1$, $i = 1,2$, $k \in \mathcal{N}$;

(H3) $\prod_{0<t_k<t}(1+b_{ik})$ is a periodic function of period ω;

(H4) $r_i(t)$, $a_i(t)$ and $c_i(t)$ are positive ω-periodic functions, $\tau_i(t)$, $\sigma_i(t)$, $k = 1,2 \in ([t_0,\infty),[0,\infty))$, are Lebesgue measurable periodic functions of period ω and $t-\tau_i(t) \to \infty$, $t-\sigma_i(t) \to \infty$ as $t \to \infty$.

Here, in the sequel we assume that a product equals a unit if the number of factors is equal to zero.

Definition 1.1. A function $y(t) = (y_1(t), y_2(t)) \in ([-r, \infty), [0, \infty))$ is said to be a solution of equation (1.5) on $[-r, \infty)$ if

(I) $y(t)$ is absolutely continuous on each interval $[0, t_1]$ and $(t_k, t_{k+1}]$, $k \in \mathcal{N}$;

(II) For any t_k, $k \in \mathcal{N}$, $y(t_k^+)$ and $y(t_k^-)$ exist and $y(t_k^-) = y(t_k)$, $i = 1, 2$;

(III) $y(t)$ satisfies (1.5) for almost everywhere in $[0, \infty)\backslash\{t_k\}$ and satisfies $y_i(t_k^+) - y_i(t_k) = b_{ik} y_i(t_k)$ for $t = t_k$, $i = 1, 2$, $k \in \mathcal{N}$.

Definition 1.2. A function $y(t)$ is an ω-periodic solution of equation (1.5) if it is a solution of equation (1.5) and satisfies $y(t + \omega) = y(t)$.

2. Permanence. In this section, we will discuss the permanence of system (1.5). To do this we need the following Lemmas 2.1 and 2.2.

Lemma 2.1 [**10**, Lemma 4.1]. *Consider the following a single species impulsive system*

$$\begin{cases} y'(t) = y(t)[g(t) - h(t)y(t)] & t \neq t_k,\ k \in \mathcal{N}, \\ y(t_k^+) = (1 + b_k)y(t_k), \end{cases} \tag{2.1}$$

where $g(t)$ and $h(t)$ are all ω-periodic functions and $h(t) > 0$ for all $t \geq 0$. $b_k > -1$ is a real constant and there exists a positive integer q such that $t_{k+q} = t_k + \omega$, $b_{k+q} = b_k$. If

$$\prod_{k=1}^{q} (1 + b_k) e^{\int_0^\omega g(t)dt} > 1. \tag{2.2}$$

Then system (2.1) has a unique positive ω-periodic solution $y^*(t, y_0^*)$ for which $y^*(0, y_0^*) = y_0^*$ and $y^*(t, y_0^*) > 0$, $t \in [0, +\infty)$, and $y^*(t, y_0^*)$ is global asymptotically stable in the sense that $\lim_{t\to\infty} |y(t, y_0) - y^*(t, y_0^*)| = 0$, where $y(t, y_0)$ is any solution of system (2.1) with positive initial value $y(0, y_0) = y_0 > 0$.

Remark 2.1. The above result is motivated by the work of Liu and Chen [**12**]. In [**12**, Theorem 2.1], $g(t) > 0$ is also assumed; recalling the whole proof carefully, the conclusion holds if we only require $h(t) > 0$

(for detailed studies, we refer to [**12**] and references cited therein). From the biological viewpoints, the net birth $g(t)$ is not necessarily positive, since the environment fluctuates randomly; in some conditions $g(t)$ may be negative.

From Theorem 1.4.3 of [**9**], the following comparison theorem for the impulsive equation is obvious. See also [**3**, Lemma 2.2].

Lemma 2.2 [**3**, Lemma 2.2]. *Suppose that $y(t)$ is the solution of system* (2.1) *with initial value $y_0 > 0$ and $s(t)$ satisfies the following inequalities*

$$\begin{cases} s'(t) \le s(t)[g(t) - h(t)s(t)] & t \ne t_k,\ k \in \mathcal{N}, \\ s(t_k^+) = (1 + b_k)s(t_k) & t = t_k, \\ s(0) \le y_0. \end{cases} \tag{2.3}$$

The variable $m(t)$ satisfies the reversed inequalities in system (2.3). *Then*

$$s(t) \le y(t) \le m(t).$$

Now, we will establish sufficient conditions for the permanence of system (1.5) when all delays are zero, that is,

$$\begin{cases} \left.\begin{aligned} y_1'(t) &= y_1(t)\left[r_1(t) - a_1(t)y_1(t) - \frac{c_2(t)y_2(t)}{1 + y_2(t)}\right] \\ y_2'(t) &= y_2(t)\left[r_2(t) - a_2(t)y_2(t) - \frac{c_1(t)y_1(t)}{1 + y_1(t)}\right] \end{aligned}\right\} t \ne t_k,\ k \in \mathcal{N}, \\ \left.\begin{aligned} y_1(t_k^+) &= (1 + b_{1k})y_1(t_k) \\ y_2(t_k^+) &= (1 + b_{2k})y_2(t_k) \end{aligned}\right\},\ t = t_k. \end{cases} \tag{2.4}$$

From Theorem 2.1 of [**4**], we can know that assumption (H3) implies that there exists a $q \in \mathcal{N}$ such that $t_{k+q} = t_k + \omega$, $b_{i(k+q)} = b_{ik}$, $k \in \mathcal{N}$. From Lemma 2.1, it is easy to see that if

$$\prod_{k=1}^{q}(1 + b_{1k})e^{\int_0^\omega r_1(t)dt} > 1 \quad \left(\text{or } \prod_{k=1}^{q}(1 + b_{2k})e^{\int_0^\omega r_2(t)\,dt} > 1\right), \tag{2.5}$$

then system (2.4) has a semi-trivial ω-periodic solution $(y_1^*(t), 0)$ (or $(0, y_2^*(t))$, where $y_1^*(t)$ and $y_2^*(t)$ are respectively the unique positive ω-periodic solution of the following systems (2.6) and (2.7).

$$\begin{cases} y_1'(t) = y_1(t)[r_1(t) - a_1(t)y_1(t)] & t \neq t_k,\ k \in \mathcal{N}, \\ y_1(t_k^+) = (1 + b_{1k})y_1(t_k) & t = t_k. \end{cases} \tag{2.6}$$

$$\begin{cases} y_2'(t) = y_2(t)[r_2(t) - a_2(t)y_2(t)] & t \neq t_k,\ k \in \mathcal{N}, \\ y_2(t_k^+) = (1 + b_{2k})y_2(t_k) & t = t_k. \end{cases} \tag{2.7}$$

Theorem 2.1. *If there exists a positive integer q such that $t_{k+q} = t_k + \omega$, $b_{i(k+q)} = b_{ik}$ and*

$$\begin{aligned} &\prod_{k=1}^{q} (1 + b_{1k}) e^{\int_0^\omega (r_1(t) - c_2(t)y_2^*(t)/(1+y_2^*(t)))\,dt} > 1, \\ &\prod_{k=1}^{q} (1 + b_{2k}) e^{\int_0^\omega (r_2(t) - c_1(t)y_1^*(t)/(1+y_1^*(t)))\,dt} > 1, \end{aligned} \tag{2.8}$$

then system (2.4) is permanent, that is, there exist constants $\triangle \geq \delta > 0$ such that for each solution $(y_1(t), y_2(t))$ of system (2.4), there exists a constant $T > 0$ such that $\delta \leq y_1(t),\ y_2(t) \leq \triangle$ for $t \geq T$. Here $y_1^*(t)$ and $y_2^*(t)$ respectively satisfy systems (2.6) and (2.7), $k \in \mathcal{N}$.

Proof. From system (2.4), we have

$$\begin{cases} y_1(t) = y_1(t_{k-1}^+) \exp\left\{\int_0^t \left[r_1(s) - a_1(s)y_1(s) - \dfrac{c_2(s)y_2(s)}{1 + y_2(s)}\right]\right\}, \\ \hfill t \in (t_{k-1}, t_k], \\ y_2(t) = y_2(t_{k-1}^+) \exp\left\{\int_0^t \left[r_2(s) - a_2(s)y_2(s) - \dfrac{c_1(s)y_1(s)}{1 + y_1(s)}\right]\right\}, \\ \hfill t \in (t_{k-1}, t_k], \\ y_1(t_k^+) = (1 + b_{1k})y_1(t_k), \\ y_2(t_k^+) = (1 + b_{2k})y_2(t_k). \end{cases} \tag{2.9}$$

It is easy to see that when $t \geq 0$, if $y_1(0^+) \geq 0$ and $y_2(0^+) \geq 0$, then we have $y_1(t) \geq 0$, $y_2(t) \geq 0$, and further $y_1(0^+) > 0$, $y_2(0^+) > 0$, then

$y_1(t) > 0, y_2(t) > 0$. Suppose that $(y_1(t), y_2(t))$ is a positive solution of system (2.4) with positive initial value $(y_1(0), y_2(0))$; we consider the following impulsive system

$$\text{(2.10)} \quad \begin{cases} m'(t) = m(t)[r_1(t) - a_1(t)m(t)] & t \neq t_k,\ k \in \mathcal{N}, \\ m(t_k^+) = (1 + b_{1k})m(t_k) & m(0) = y_1(0) > 0,\ t = t_k, \end{cases}$$

and denote its solution by $m(t)$. According to Lemma 2.1, we have $\lim_{t\to\infty}(m(t) - y_1^*(t)) = 0$, which implies that there exists a $T_1 > 0$ such that $m(t) \le y_1^*(t) + \alpha_1$ for $t \ge T_1$, where the constant $\alpha_1 > 0$ is sufficient small. By Lemma 2.2, we have $y_1(t) \le m(t) \le y_1^*(t) + \alpha_1$ for $t \ge T_1$.

Similarly, for $y_2(t)$, we can get that there exists a $T_2 > 0$ such that $y_2(t) \le y_2^*(t) + \alpha_2$ for $t \ge T_2$, where the constant $\alpha_2 > 0$ is also sufficient small. Thus, we obtain that there exists a $T_3 = \max\{T_1, T_2\}$ such that

$$\text{(2.11)} \qquad y_1(t) \le y_1^*(t) + \alpha_1, \quad y_2(t) \le y_2^*(t) + \alpha_2, \text{ for } t \ge T_3.$$

Now, we consider the impulsive equation

$$\text{(2.12)} \qquad \begin{cases} s'(t) = s(t)\left[r_1(t) - a_1(t)s(t) - \dfrac{c_2(t)(y_2^*(t) + \alpha_2)}{1 + y_2^*(t) + \alpha_2}\right], \\ \hfill t \neq t_k,\ k \in \mathcal{N}, \\ s(t_k^+) = (1 + b_{1k})s(t_k), \hfill t = t_k \end{cases}$$

Since the constant $\alpha_2 > 0$ is sufficient small, and $(y_2^*(t)/1 + y_2^*(t))$ is a monotone increasing function with respect to $y_2^*(t)$, from the first inequality of (2.8) we can choose the constant α_2 such that

$$\text{(2.13)} \qquad \prod_{k=1}^{q}(1 + b_{1k})e^{\int_0^\omega (r_1(t) - c_2(t)(y_2^*(t)+\alpha_2)/(1+y_2^*(t)+\alpha_2))\,dt} > 1.$$

Consequently, it follows from Lemma 2.1 that system (2.12) has a unique positive ω-periodic solution, and we denote the solution by $y_{1*}(t)$. We denote the solution of (2.12) satisfying $s(T_3) = y_1(T_3)$ by $s(t)$ and continue to choose a positive constant $\beta_1 > 0$ such that $\beta_1 < \min_{t\in[0,\omega]} y_{1*}(t)$. By Lemmas 2.1 and 2.2, and the asymptotic property of $y_{1*}(t)$, there exists a $T_4 > T_3$ such that

$$\text{(2.14)} \qquad y_{1*}(t) - \beta_1 \le s(t) \le y_1(t), \quad t > T_4.$$

Similarly, for $y_2(t)$, it follows from the second inequality of (2.8) that we can choose the constant $\alpha_1 > 0$ to be sufficient small such that

$$\prod_{k=1}^{q}(1+b_{2k})e^{\int_0^\omega (r_2(t)-c_1(t)(y_1^*(t)+\alpha_1)/(1+y_1^*(t)+\alpha_1))\,dt} > 1. \tag{2.15}$$

Hence, there exist $y_{2*}(t)$, β_2 and T_5 which correspond to $y_{1*}(t)$, β_1 and T_4, respectively, such that

$$0 < y_{2*}(t) - \beta_2 \le y_2(t), \quad t > T_5. \tag{2.16}$$

Let

$$\delta = \min_{t\in[0,\omega]}\{y_{1*}(t)-\beta_1, y_{2*}(t)-\beta_2\}, \quad \triangle = \max_{t\in[0,\omega]}\{y_1^*(t)+\alpha_1, y_2^*(t)+\alpha_2\}.$$

So we get $\delta \le y_1(t),\ y_2(t) \le \triangle$ for $t > \max\{T_4, T_5\}$. The proof is complete. □

3. Positive periodic solution. In this section, we will discuss the existence of positive periodic solutions of system (1.5). To do so, we first establish Lemma 3.1 which is motivated by Yan et al. [**20**]. This lemma enables us to reduce the existence of solution of equation (1.5) to the corresponding problem for a delay differential equation without impulses (see (3.1)).

Now, let us consider the nonimpulsive competitive system with delays

$$\begin{aligned} u_1'(t) &= u_1(t)\left[r_1(t) - A_1(t)u_1(t-\tau_1(t)) - \frac{C_2(t)u_2(t-\sigma_2(t))}{1+B_2(t)u_2(t-\sigma_2 t))}\right],\\ u_2'(t) &= u_2(t)\left[r_2(t) - A_2(t)u_2(t-\tau_2(t)) - \frac{C_1(t)u_1(t-\sigma_1(t))}{1+B_1(t)u_1(t-\sigma_1(t))}\right] \end{aligned} \tag{3.1}$$

with the initial condition which is similar to (1.6), where

$$\begin{aligned} A_1(t) &= \prod_{0<t_k<t-\tau_1(t)}(1+b_{1k})a_1(t), & A_2(t) &= \prod_{0<t_k<t-\tau_2(t)}(1+b_{2k})a_2(t),\\ B_1(t) &= \prod_{0<t_k<t-\sigma_1(t)}(1+b_{1k}), & B_2(t) &= \prod_{0<t_k<t-\sigma_2(t)}(1+b_{2k}),\\ C_1(t) &= B_1(t)c_1(t), & C_2(t) &= B_2(t)c_2(t). \end{aligned} \tag{3.2}$$

Lemma 3.1. *Assume that* (H1)–(H4) *hold. Then*

(1) *If* $u(t) = (u_1(t), u_2(t))$ *is a solution of equation* (3.1) *on* $[-r, \infty]$, *then*

$$y(t) = \Big(\prod_{0<t_k<t} (1 + b_{1k}) u_1(t), \ \prod_{0<t_k<t} (1 + b_{2k}) u_2(t) \Big)$$

is a solution of equation (1.5) *on* $[-r, \infty]$;

(2) *If* $y(t) = (y_1(t), y_2(t))$ *is a solution of equation* (1.5) *on* $[-r, \infty]$, *then*

$$u(t) = \Big(\prod_{0<t_k<t} (1 + b_{1k})^{-1} y_1(t), \ \prod_{0<t_k<t} (1 + b_{2k})^{-1} y_2(t) \Big)$$

is a solution of equation (3.1) *on* $[-r, \infty]$.

Proof. Firstly, we prove (1). It is easy to see that

$$y(t) = \Big(\prod_{0<t_k<t} (1 + b_{1k}) u_1(t), \ \prod_{0<t_k<t} (1 + b_{2k}) u_2(t) \Big)$$

is absolutely continuous on the interval $(t_k, t_{k+1}]$, and for any $t \neq t_k$, $k \in \mathcal{N}$, we have

$$\begin{aligned}
&y_1'(t) - y_1(t)\Big[r_1(t) - a_1(t) y_1(t - \tau_1(t)) - \frac{c_2(t) y_2(t - \sigma_2(t))}{1 + y_2(t - \sigma_2(t))} \Big] \\
&= \prod_{0<t_k<t} (1 + b_{1k}) \Big[u_1'(t) - u_1(t) \\
&\quad \times \Big[r_1(t) - a_1(t) \prod_{0<t_k<t-\tau_1(t)} (1 + b_{1k}) u_1(t - \tau_1(t)) \\
&\qquad - \frac{c_2(t) \prod_{0<t_k<t-\sigma_2(t)} (1 + b_{2k}) u_2(t - \sigma_2(t))}{1 + \prod_{0<t_k<t-\sigma_2(t)} (1 + b_{2k}) u_2(t - \sigma_2(t))} \Big] \Big] \\
&= \prod_{0<t_k<t} (1 + b_{1k}) \Big[u_1'(t) - u_1(t) \\
&\qquad \Big[r_1(t) - A_1(t) u_1(t - \tau_1(t)) - \frac{C_2(t) u_2(t - \sigma_2(t))}{1 + B_2(t) u_2(t - \sigma_2(t))} \Big] \Big] \\
&= 0,
\end{aligned} \tag{3.3}$$

and

$$
\begin{aligned}
(3.4)\quad & y_2'(t) - y_2(t)\left[r_2(t) - a_2(t)y_2(t-\tau_2(t)) - \frac{c_1(t)y_1(t-\sigma_1(t))}{1+y_1(t-\sigma_1(t))}\right] \\
&= \prod_{0<t_k<t}(1+b_{2k})\Bigg[u_2'(t) - u_2(t) \\
&\quad \times \Bigg[r_2(t) - a_2(t)\prod_{0<t_k<t-\tau_2(t)}(1+b_{2k})u_2(t-\tau_2(t)) \\
&\qquad - \frac{c_1(t)\prod_{0<t_k<t-\sigma_1(t)}(1+b_{1k})u_1(t-\sigma_1(t))}{1+\prod_{0<t_k<t-\sigma_1(t)}(1+b_{1k})u_1(t-\sigma_1(t))}\Bigg]\Bigg] \\
&= \prod_{0<t_k<t}(1+b_{2k})\Bigg[u_2'(t) - u_2(t) \\
&\quad \times \left[r_2(t) - A_2(t)u_1(t-\tau_2(t)) - \frac{C_1(t)u_1(t-\sigma_1(t))}{1+B_1(t)u_1(t-\sigma_1(t))}\right]\Bigg] \\
&= 0.
\end{aligned}
$$

On the other hand, for every $t_k \in \{t_k\}$, we have

$$
\begin{aligned}
y_1(t_k^+) &= \lim_{t\to t_k^+}\prod_{0<t_j<t}(1+b_{1j})u_1(t) = \prod_{0<t_j\le t_k}(1+b_{1j})u_1(t_k), \\
y_2(t_k^+) &= \lim_{t\to t_k^+}\prod_{0<t_j<t}(1+b_{2j})u_2(t) = \prod_{0<t_j\le t_k}(1+b_{2j})u_2(t_k),
\end{aligned}
$$

and

$$
y_1(t_k) = \prod_{0<t_j<t_k}(1+b_{1j})u_1(t_k), \quad y_2(t_k) = \prod_{0<t_j<t_k}(1+b_{2j})u_2(t_k).
$$

Thus, for every $k = 1, 2, \dots$, we have

$$
(3.5)\qquad y_1(t_k^+) = (1+b_{1k})y_1(t_k), \quad y_2(t_k^+) = (1+b_{2k})y_2(t_k).
$$

It follows from (3.3), (3.4) and (3.5) that $y(t)$ is a solution of equation (1.5).

By a similar procedure as above, (2) can be proven, and hence we omit its details. The proof is complete. □

In the following, based on Mawhin's continuation theorem, we shall consider the existence of positive periodic solutions of system (1.5). We will make some preparations.

Let X and Z be normed vector spaces, $L : \operatorname{Dom} L \subset X \to Z$ a linear mapping, and let $N : X \to Z$ be a continuous mapping. The mapping L will be called a Fredholm mapping of index zero if $\dim \operatorname{Ker} L = \operatorname{codim} \operatorname{Im} L < +\infty$ and $\operatorname{Im} L$ is closed in Z. If L is a Fredholm mapping of index zero, then there exist continuous projectors $P : X \to X$ and $Q : Z \to Z$ such that $\operatorname{Im} P = \operatorname{Ker} L$, $\operatorname{Ker} Q = \operatorname{Im} L = \operatorname{Im}(I - Q)$. It follows that $L|\operatorname{Dom} L \cap \operatorname{Ker} P : (I - P)X \to \operatorname{Im} L$ is invertible. We denote the inverse of that map by K_p. If Ω is an open bounded subset of X, the mapping N will be called L-compact on $\overline{\Omega}$ if $QN(\overline{\Omega})$ is bounded and $K_p(I - Q)N : \overline{\Omega} \to X$ is compact. Since $\operatorname{Im} Q$ is isomorphic to $\operatorname{Ker} L$, there exists an isomorphism $J : \operatorname{Im} Q \to \operatorname{Ker} L$.

Lemma 3.2 [**4**]. *Let L be a Fredholm mapping of index zero, and let N be L-compact on $\overline{\Omega}$. Suppose*

(a) *for each $\lambda \in (0, 1)$, every solution x of $Lx = \lambda Nx$ is such that $x \notin \partial\Omega$;*

(b) *$QNx \neq 0$ for each $x \in \partial\Omega \cap \operatorname{Ker} L$ and*

$$\deg \Big\{ JQN, \Omega \cap \operatorname{Ker} L, 0 \Big\} \neq 0.$$

Then the equation $Lx = Nx$ has at least one solution lying in $\operatorname{Dom} L \cap \overline{\Omega}$.

Now we state the result on the existence of positive periodic solutions of system (1.5).

Theorem 3.1. *Assume that* (H1)–(H4) *hold, and further assume the following inequalities*

$$\bar{r}_1 > \bar{c}_2, \quad \bar{r}_2 > \bar{c}_1 \tag{3.6}$$

hold. Then system (1.5) *has at least one positive ω-periodic solution, where $\bar{r}_i = 1/\omega \int_0^\omega r_i(t)\, dt$, $\bar{c}_i = 1/\omega \int_0^\omega c_i(t)\, dt$, $i = 1, 2$.*

Proof. Let $x_1(t) = \ln u_1(t)$ and $x_2(t) = \ln u_2(t)$. Corresponding to system (3.1), we have

$$\begin{aligned} x_1'(t) &= r_1(t) - A_1(t)e^{x_1(t-\tau_1(t))} - \frac{C_2(t)e^{x_2(t-\sigma_2(t))}}{1 + B_2(t)e^{x_2(t-\sigma_2(t))}}, \\ x_2'(t) &= r_2(t) - A_2(t)e^{x_2(t-\tau_2(t))} - \frac{C_1(t)e^{x_1(t-\sigma_1(t))}}{1 + B_1(t)e^{x_1(t-\sigma_1(t))}}. \end{aligned} \tag{3.7}$$

In order to use Lemma 3.2, we take

$$X = Z = \Big\{x(t) = (x_1(t), x_2(t)) \in C(R, R^2), x(t+\omega) = x(t)\Big\}.$$

Then X and Z are Banach spaces when they are endowed with the norms $\|x\| = \max_{t\in[0,\omega]} |x_1(t)| + \max_{t\in[0,\omega]} |x_2(t)|$ for any $x \in X$ (or Z). Set

$$N\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} r_1(t) - A_1(t)e^{x_1(t-\tau_1(t))} - \dfrac{C_2(t)e^{x_2(t-\sigma_2(t))}}{1 + B_2(t)e^{x_2(t-\sigma_2 t)}} \\ r_2(t) - A_2(t)e^{x_2(t-\tau_2(t))} - \dfrac{C_1(t)e^{x_1(t-\sigma_1(t))}}{1 + B_1(t)e^{x_1(t-\sigma_1(t))}} \end{bmatrix}$$

and

$$Lx = x',\ Px = \frac{1}{\omega}\int_0^\omega x(t)\,dt,\ x \in X; \quad Qz = \frac{1}{\omega}\int_0^\omega z(t)\,dt,\ z \in Z.$$

Obviously,

$$\operatorname{Ker} L = \{x \mid x \in X, x = h, h \in R^2\},\ \operatorname{Im} L = \Big\{z \mid z \in Z, \int_0^\omega z(t)\,dt = 0\Big\},$$

and $\dim \operatorname{Ker} L = \operatorname{codim} \operatorname{Im} L = 2$. Since $\operatorname{Im} L$ is closed in Z, L is a Fredholm mapping of index zero. It is easy to see that P and Q are continuous projectors such that $\operatorname{Im} P = \operatorname{Ker} L$, $\operatorname{Ker} Q = \operatorname{Im} L = \operatorname{Im}(I - Q)$. Moreover, the generalized inverse (to L) $K_p : \operatorname{Im} L \to \operatorname{Ker} P \cap \operatorname{Dom} L$ is given by $K_p(z) = \int_0^t z(s)\,ds - 1/\omega \int_0^\omega \int_0^t z(s)\,ds\,dt$. Thus,

$$QNx = \frac{1}{\omega}\int_0^\omega Nx(t)\,dt,$$

$$K_p(I-Q)Nx = \int_0^t Nx(s)\,ds - \frac{1}{\omega}\int_0^\omega \int_0^t Nx(s)\,ds\,dt - \left(\frac{t}{\omega} - \frac{1}{2}\right)\int_0^\omega Nx(s)\,ds.$$

It is obvious that QN and $K_p(I-Q)N$ are continuous. Moreover, $QN(\overline{\Omega})$ is bounded and $K_p(I-Q)N(\overline{\Omega})$ are relatively compact for any open bounded set $\Omega \subset X$. Hence, N is L-compact on $\overline{\Omega}$; here Ω is any open bounded set in X.

Corresponding to the equation $Lx = \lambda Nx$, $\lambda \in (0,1)$, we have

$$\begin{aligned} x_1'(t) &= \lambda\left[r_1(t) - A_1(t)e^{x_1(t-\tau_1(t))} - \frac{C_2(t)e^{x_2(t-\sigma_2(t))}}{1+B_2(t)e^{x_2(t-\sigma_2(t))}}\right],\\ x_2'(t) &= \lambda\left[r_2(t) - A_2(t)e^{x_2(t-\tau_2(t))} - \frac{C_1(t)e^{x_1(t-\sigma_1(t))}}{1+B_1(t)e^{x_1(t-\sigma_1(t))}}\right]. \end{aligned} \tag{3.8}$$

Suppose that $(x_1(t), x_2(t)) \in X$ is a solution of system (3.8). By integrating (3.8) over the interval $[0,\omega]$, we obtain

$$\begin{aligned} \int_0^\omega A_1(t)e^{x_1(t-\tau_1(t))}\,dt + \int_0^\omega \frac{C_2(t)e^{x_2(t-\sigma_2(t))}}{1+B_2(t)e^{x_2(t-\sigma_2(t))}}\,dt &= \bar{r}_1\omega,\\ \int_0^\omega A_2(t)e^{x_2(t-\tau_2(t))}\,dt + \int_0^\omega \frac{C_1(t)e^{x_1(t-\sigma_1(t))}}{1+B_1(t)e^{x_1(t-\sigma_1(t))}}\,dt &= \bar{r}_2\omega. \end{aligned} \tag{3.9}$$

It follows from (3.8) and (3.9) that we have

$$\int_0^\omega |x_1'(t)|\,dt < 2\bar{r}_1\omega, \qquad \int_0^\omega |x_2'(t)|\,dt < 2\bar{r}_2\omega. \tag{3.10}$$

Besides, from (3.9), we have

$$\int_0^\omega A_1(t)e^{x_1(t-\tau_1(t))}\,dt < \bar{r}_1\omega, \qquad \int_0^\omega A_2(t)e^{x_2(t-\tau_2(t))}\,dt < \bar{r}_2\omega. \tag{3.11}$$

Since $x(t) \in X$, it follows that there exists a $\xi_i \in [0,\omega]$ such that $x_i(\xi_i) = \min_{t\in[0,\omega]} x_i(t)$, $i = 1,2$. Then, from (3.11) we have

$$e^{x_i(\xi_i)} < \frac{\bar{r}_i\omega}{\int_0^\omega A_i(t)\,dt} \Longrightarrow x_i(\xi_i) < \ln\frac{\bar{r}_i\omega}{\int_0^\omega A_i(t)\,dt}, \quad i = 1,2, \tag{3.12}$$

which, together with (3.10), lead to

(3.13)

$$x_i(t) \le x_i(\xi_i) + \int_0^\omega |x_i'(t)|\,dt < \ln \frac{\bar{r}_i\omega}{\int_0^\omega A_i(t)\,dt} + 2\bar{r}_i\omega \overset{\text{def}}{:=} M_i, \quad i = 1, 2.$$

On the other hand, there exists an $\eta_i \in [0,\omega]$ such that $x_i(\eta_i) = \max_{t\in[0,\omega]} x_i(t)$, $i = 1, 2$. Notice that $C_j(t) = B_j(t)c_j(t)$, $j = 1, 2$, so from (3.9) we obtain

(3.14)

$$\begin{aligned} e^{x_i(\eta_i)} \int_0^\omega A_i(t)\,dt &\ge \bar{r}_i\omega - \int_0^\omega \frac{C_j(t)e^{x_j(t-\sigma_j(t))}}{1 + B_j(t)e^{x_j(t-\sigma_j(t))}}\,dt \\ &> \bar{r}_i\omega - \int_0^\omega \frac{C_j(t)e^{x_j(t-\sigma_j(t))}}{B_j(t)e^{x_j(t-\sigma_j(t))}}\,dt = (\bar{r}_i - \bar{c}_j)\omega, \end{aligned}$$

$i, j = 1, 2$ and $i \ne j$. By this and assumption (3.6) of Theorem 3.1, we have

$$x_i(\eta_i) > \ln \frac{(\bar{r}_i - \bar{c}_j)\omega}{\int_0^\omega A_i(t)\,dt}, \quad i, j = 1, 2 \text{ and } i \ne j. \tag{3.15}$$

As a consequence, by (3.10) and (3.15) we have

$$\begin{aligned} x_i(t) \ge x_i(\eta_i) - \int_0^\omega |x_i'(t)|\,dt > \ln \frac{(\bar{r}_i - \bar{c}_j)\omega}{\int_0^\omega A_i(t)\,dt} - 2\bar{r}_i\omega \overset{\text{def}}{:=} \widehat{M}_i, \\ i, j = 1, 2 \text{ and } i \ne j. \end{aligned} \tag{3.16}$$

Equations (3.13) and (3.16) imply that

$$\max_{t\in[0,\omega]} |x_i(t)| \le \{|M_i|, |\hat{M}|_i\} \overset{\text{def}}{:=} F_i, \quad i = 1, 2. \tag{3.17}$$

Obviously, M_i, $\widehat{M}_i$ and F_i are independent of λ, respectively. Denote $F = F_1 + F_2 + f_0$, where $f_0 > 0$ is taken sufficiently large such that each solution (v_1^*, v_2^*) of the equation

$$\begin{aligned} \bar{r}_1\omega - \int_0^\omega A_1(t)e^{v_1}\,dt - \int_0^\omega \frac{C_2(t)e^{v_2}}{1 + B_2(t)e^{v_2}}\,dt = 0, \\ \bar{r}_2\omega - \int_0^\omega A_2(t)e^{v_2}\,dt - \int_0^\omega \frac{C_1(t)e^{v_1}}{1 + B_1(t)e^{v_1}}\,dt = 0 \end{aligned} \tag{3.18}$$

satisfies $|v_1^*| + |v_2^*| < f_0$. Now we take $\Omega = \{(x_1, x_2) \in X : \|x\| < F\}$. This satisfies condition (a) of Lemma 3.2. When $(x_1, x_2) \in \partial\Omega \cap \operatorname{Ker} L = \partial\Omega \cap R^2$, (x_1, x_2) is a constant vector in R^2 with $|x_1|+|x_2| = F$. If equation (3.18) has one or many solutions, then $QN(x_1, x_2) \neq 0$. If equation (3.18) has no solution, then it is natural that $QN(x_1, x_2) \neq 0$, which implies that condition (b) of Lemma 3.2 is also satisfied.

In the following, we define $\Phi : \operatorname{Dom} L \times [0,1] \to X$ by

$$\Phi(x_1, x_2, \mu) = \begin{bmatrix} \bar{r}_1\omega - \int_0^\omega A_1(t)e^{x_1}\,dt \\ \bar{r}_2\omega - \int_0^\omega A_2(t)e^{x_2}\,dt \end{bmatrix} + \mu \begin{bmatrix} -\int_0^\omega \frac{C_2(t)e^{x_2}}{1+B_2(t)e^{x_2}}\,dt \\ -\int_0^\omega \frac{C_1(t)e^{x_1}}{1+B_1(t)e^{x_1}}\,dt \end{bmatrix},$$

where $\mu \in [0,1]$ is a parameter. By a similar analysis as above, we can show that when $(x_1, x_2) \in \partial\Omega \cap \operatorname{Ker} L = \partial\Omega \cap R^2$, $\Phi(x_1, x_2, \mu) \neq 0$. Taking $J = I : \operatorname{Im} Q \to \operatorname{Ker} L$, $(x_1, x_2) \to (x_1, x_2)$ and using the property of topological degree, we have

$$\begin{aligned} &\deg\Big\{JQN(x_1, x_2), \Omega \cap \operatorname{Ker} L, (0,0)\Big\} \\ &\qquad = \deg\Big\{\Phi(x_1, x_2, 1), \Omega \cap \operatorname{Ker} L, (0,0)\Big\} \\ &\qquad = \deg\Big\{\Phi(x_1, x_2, 0), \Omega \cap \operatorname{Ker} L, (0,0)\Big\} \\ &\qquad = \deg\Big\{\bar{r}_1\omega - \int_0^\omega A_1(t)\,dt\,e^{x_1}, \\ &\qquad\qquad \bar{r}_2\omega - \int_0^\omega A_2(t)\,dt\,e^{x_2}, \Omega \cap \operatorname{Ker} L, (0,0)\Big\}. \end{aligned}$$

Obviously, the following system

$$\bar{r}_1\omega - \int_0^\omega A_1(t)\,dt\,e^{\mu_1} = 0, \qquad \bar{r}_2\omega - \int_0^\omega A_2(t)\,dt\,e^{\mu_2} = 0$$

has a unique solution (μ_1^*, μ_2^*). Hence,

$$\begin{aligned} &\deg\{JQN(x_1, x_2), \Omega \cap \operatorname{Ker} L, (0,0)\} \\ &\qquad = \operatorname{sign}\left\{\int_0^\omega A_1(t)\,dt \int_0^\omega A_2(t)\,dt\,e^{(\mu_1^*+\mu_2^*)}\right\} \\ &\qquad = 1 \neq 0. \end{aligned}$$

By now all assumptions required in Lemma 3.2 hold. It follows by Lemma 3.2 that system (3.7) has an ω-periodic solution. By the change of $x_i(t) = \ln u_i(t)$, $i = 1, 2$, and Lemma 3.1, we obtain that system (1.5) has a positive ω-periodic solution. □

4. An example and its numerical simulations. As an application of our main results, we consider the following delayed competitive system with impulsive perturbations

$$\text{(4.1)}\qquad \begin{cases} \left.\begin{aligned} y_1'(t) &= y_1(t)\Big[2+\sin\pi t-(1+0.5\cos\pi t)y_1(t-1) \\ &\qquad\qquad -\frac{(1.5+\sin\pi t)y_2(t-0.5)}{1+y_2(t-0.5)}\Big] \\ y_2'(t) &= y_2(t)\Big[2.5+\sin\pi t-(2+0.5\cos\pi t)y_2(t-2) \\ &\qquad\qquad -\frac{(2+\sin\pi t)y_1(t-1)}{1+y_1(t-1)}\Big] \end{aligned}\right\}, & t\neq t_k, \\ \left.\begin{aligned} y_1(t_k^+) &= (1+a_k)y_1(t_k) \\ y_2(t_k^+) &= (1+b_k)y_2(t_k) \end{aligned}\right\}, & t=t_k,\ k\in\mathcal{N}. \end{cases}$$

From Remark 1.1 of [**20**], we can know that assumption (H3) implies $\prod_{t\le t_k<t+\omega}(1+b_{ik}) = 1$, and hence we can choose $a_{2k} = -0.5$, $a_{2k-1} = 1$, $b_{2k} = 1$, $b_{2k-1} = -0.5$, and initial value $(0.05, 0.09)$. It is easy to verify that the assumptions (H1)–(H4) are satisfied, and $\bar{r}_1 = 2 > \bar{c}_2 = 1.5$, $\bar{r}_2 = 2.5 > \bar{c}_1 = 2$. Then system (4.1) has a 2-periodic solution. Numerical simulations show the feasibility of this result. From Figure 1, we may observe that the two species tend to a periodic coexistence, and there exists a positive 2-periodic solution of system (4.1).

5. Discussion. In this paper, we establish a nonautonomous delayed two-species competitive model with impulsive perturbations. By applying the property of globally asymptotic stability of an impulsive single-species growth population model, sufficient conditions are derived that guarantee the permanence of the above impulsive system without delays. Later, we discuss the existence of a positive periodic

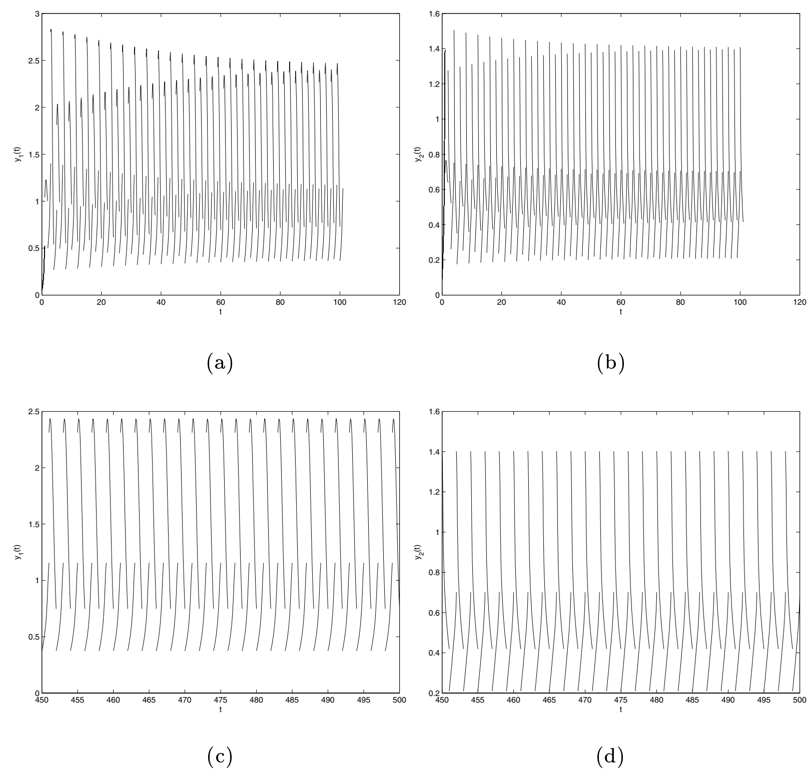

FIGURE 1. Two species tend to periodic oscillation. (a) Time-series of the species y_1 with t over [0, 100]. (b) Time-series of the species y_2 with t over [0, 100]. (c) Time-series of the species y_1 with t over [450, 500]. (d) Time-series of the species y_2 with t over [450, 500].

solution of the above impulsive system with delays by using both analysis technique and coincidence degree theory. Numerical simulations are also presented to illustrate the feasibility of our main results. We can see that the two species tend to a periodic coexistence, and there exists a 2-periodic solution, see Figure 1.

Now, we give biological interpretations of our main results. From the assumptions (2.8) of Theorem 2.1, we can see that if the impulsive perturbations and the intrinsic growth rates are suitably large while the inter-specific competitive rates are suitably small, then system (2.4) is permanent. The assumptions (3.6) of Theorem 3.1 may be interpreted by saying that the intra-specific intrinsic growth rates are larger than the inter-specific competitive rates. From Theorems 2.1 and 3.1, we also can see that the intra-specific competitive rates have no affect on the permanence of system (2.4) and the existence of positive periodic solutions of system (1.5). Besides, Theorem 3.1 shows that system (1.5) has at least one positive periodic solution irrespective of the sizes of the delays, that is, the delays have no affect on the existence of positive periodic solutions.

We expect to further discuss the permanence of system (1.5) when $\tau_i(t) \neq 0$ and $\sigma_i(t) \neq 0$. Of course, the uniqueness and global stability of positive periodic solutions of system (1.5) are also more important. We leave them for future work.

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REFERENCES

1. D. Bainov and P. Simeonov, *Impulsive differential equations: Periodic solutions and applications*, Longman Scientific and Technical, New York, 1993.

2. G. Ballinger and X. Liu, *Permanence of population growth models with impulsive effects*, Math. Comput. Modelling **26** (1997), 59–72.

3. A. d'Onofrio, *On pulse vaccination strategy in the* SIR *epidemic model with vertical transmission*, Appl. Math. Letters **18** (2005), 729–732.

4. J.W. Dou, L.S. Chen and K.T. Li, *A monotone-interative method for finding periodic solutions of an impulsive competition system on tumor-normal cell interaction*, Discrete Continuous Dynamical Systems **4** (2004), 555–562.

5. R.E. Gaines and J.L. Mawhin, *Coincidence degree and nonlinear differential equations*, Springer-Verlag, Berlin, 1977.

6. K. Gopalsamy, *Stability and oscillation in delay differential equations of population dynamics*, in *Mathematics and its applications*, Vol.74, Kluwer Academic Press, Dordrecht, 1992.

7. Y. Kuang, *Delay differential equations: With applications in population dynamics*, Academic Press, Boston, 1993.

8. A. Lakmeche and O. Arino, *Bifurcation of non-trivial periodic solutions of impulsive differential equations arising chemotherapeutic treatment*, Dynamic Continuous Discrete Impulsive Systems **7** (2000), 265–287.

9. V. Lakshmikantham, D.D. Bainov and P.S. Simeonov, *Theory of impulsive differential equations*, World Scientific, Singapore, 1989.

10. Z.J. Liu, *The study of effect of delay and impulse on population dynamical system*, Doctoral dissertation, Dalian University of Technology, 2006.

11. X.N. Liu and L.S. Chen, *Complex dynamics of Holling type* II *Lotka-Volterra predator-prey system with impulsive perturbations on the predator*, Chaos Solitons Fractals **16** (2003), 311–320.

12. ———, *Global dynamics of the periodic logistic system with periodic impulsive perturbations*, J. Math. Anal. Appl. **289** (2004), 279–291.

13. ———, *Periodic solution of a two-species competitive system with toxicant and birth pulse*, Chaos Solitons Fractals **32** (2007), 1703–1712.

14. X.N. Liu and Y. Takeuchi, *Periodic and global dynamics of an impulsive delay Lasota-Wazewska model*, J. Math. Anal. Appl. **327** (2007), 326–341.

15. Z.J. Liu and R.H. Tan, *Impulsive harvesting and stocking in a Monod-Haldane functional response predator-prey system*, Chaos Solitons Fractals **34** (2007), 454–464.

16. Z.J. Liu, R.H. Tan and L.S. Chen, *Global stability in a periodic delayed predator-prey system*, Appl. Math. Comput. **186** (2007), 389–403.

17. B. Liu, Z.D. Teng and L.S. Chen, *Analysis of a predator-prey model with Holling* II *functional response concerning impulsive control strategy*, Comput. Appl. Math. **193** (2006), 347–362.

18. S.Y. Tang and L.S. Chen, *Modelling and analysis of integrated pest management strategy*, Discrete Continuous Dynamical Systems **4** (2004), 759–768.

19. W.D. Wang and Z.E. Ma, *Harmless delays for uniform persistence*, J. Math. Anal. Appl. **158** (1991), 256–268.

20. J.R. Yan, A.M. Zhao and W.P. Yan, *Existence and global attractivity of periodic solution for an impulsive delay differential equation with Allee effect*, J. Math. Anal. Appl. **309** (2005), 489–504.

21. G.Z. Zeng, L.S. Chen and L.H. Sun, *Complexity of an* SIR *epidemic dynamics model with impulsive vaccination control*, Chaos Solitons Fractals **26** (2005), 495–505.

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