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STABILITY OF THE T-PERIODIC SOLUTION ON THE ES-S MODEL

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ABSTRACT. In this paper, by employing the powerful and effective coincidence degree method, we show the existence of T-periodic solutions of the extended simplified Schnakeberg (ES-S) model in $\mathcal{D}$, where $\mathcal{D}$ is a strictly positively invariant region of the ES-S model. Furthermore, Floquet theory is provided to show that the T-periodic solution $x_0(t)$ of the ES-S model is unique in $\mathcal{D}$ and locally uniformly asymptotically stable. This establishes a solid foundation for studying the patterns of the extended Schnakeberg (E-S) model. The novelty of the approach in this paper is to combine degree theory and Floquet theory together to study stability of the periodic solution in an ordinary differential equation system with continuous positive periodic coefficients. Actually, this provides a general method of qualitative analysis for the T-periodic solution in a nonautonomous system.

1. Introduction. In the last several decades, there have been a lot of models developed involving the study of cells reproduction pattern, such as the Brusselator model [**13, 14**], the Glycolysis model [**17**], the Gray-Scott model [**5, 6, 9, 10, 12**] and the Gierer-Meinhardt model [**3**]. Usually, the system of reaction-diffusion equations with zero-flux and periodic boundary conditions [**7**], which has stable steady-states, will induce the cell reproduction pattern by basic analysis due to the Turing principle [**11**, page 380]. For models with constant coefficients, their cell division patterns do not relate to the time factor. Time-related process of cell division is called aberrance of cell division, whose patterns have significant meaning to cancer pathology, especially, for the mechanism of splitting of the cancer cell. "Unlike normal cells, cancer cells do not carry on maturing once they have been made. In fact, the cells in a cancer can become even less mature over time. With all the reproducing, it is not surprising that more of the genetic information in

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the cell can become lost. So the cells become more and more primitive and tend to reproduce more quickly and even more haphazardly." [**18**] As a result, to explain cancer cell reproduction, one has to study time-related patterns. Therefore, we extend the Schnakeberg model with constant coefficients to the model with coefficients to be positive T-periodic functions (we call such a model an E-S model). The goal of this research is to study the pattern formations of the E-S model. This is quite a different job compared to the study of the Schnakeberg model by using Turing instability. In this case, we need to find a T-periodic solution for the E-S model in a certain patch; as a source of the T-periodic pattern, we have to study the stability of the T-periodic solution and its bifurcation along some parameters. From the mathematical point of view, the Turing instability method only works on an autonomous system, but what we need to focus on is a nonautonomous system. Therefore, study of the E-S model becomes much more complicated and challenging.

In this paper, by employing the powerful and effective coincidence degree method, we show existence of T-periodic solutions of the ES-S model in $\mathcal{D}$, where $\mathcal{D}$ is a strictly positively invariant region. Furthermore, Floquet theory is provided to show that the T-periodic solution $x_0(t)$ of the ES-S model is unique in $\mathcal{D}$ and locally uniformly asymptotically stable. This establishes a solid foundation for studying the patterns of the E-S model.

2. Invariant region of the ES-S model. The Schnakeberg model [**16**] is

$$\begin{cases} u_t(r,t) = d_1\Delta u(r,t) + a - u(r,t) + u^2(r,t)v(r,t) & r \in \Lambda, \\ v_t(r,t) = d_2\Delta v(r,t) + b - u^2(r,t)v(r,t) & r \in \Lambda \end{cases}$$

with boundary conditions

$$n(r)\cdot\nabla u(r,t) = n(r)\cdot\nabla v(r,t) = 0$$

for $r \in \partial\Lambda$, where $n(r)$ is the unit outward normal vector field along the boundary of $\Lambda = [0,l]\times[0,l]$, $l > 0$, and a, b, d_1, d_2 are positive constants. If we consider the case when reactants are well stirred, then the diffusion terms disappear. In this case, we get the simplified Schnakeberg (S-S) model [**11**, page 156]

$$\begin{cases} \dot{u} = a - u + u^2 v, \\ \dot{v} = b - u^2 v. \end{cases}$$

If, in the S-S model, we allow the coefficients a and b to be positive continuous T-periodic functions of t with period $T > 0$, then the corresponding model is called an ES-S model. Similarly, the Schnakeberg model will be called an E-S model if we replace constants a and b by positive continuous T-periodic functions.

Let

$$x(t) = \begin{cases} u(t) \\ v(t) \end{cases} \quad \text{and} \quad F(t, x(t)) = \begin{cases} a - u + u^2 v \\ b - u^2 v. \end{cases}$$

Then the ES-S model is defined by

$$\dot{x}(t) = F(t, x(t)) \tag{1}$$

with conditions

$$1.1 < a(t) < 1.6, \quad 0.04 < b(t) < 0.1. \tag{2}$$

Lemma 2.1. *There exists a strictly positively invariant region*

$$\mathcal{D} = \{(u, v) \in \mathbf{R}^2 : 1 \le u \le 2,\ 0.01 \le v \le 0.1\}$$

for the ES-S *model given by* (1) *with conditions* (2).

Proof. Clearly $\mathcal{D}$ is a closed convex subset of $\mathbf{R}^2$. We only need to check whether $n(u, v) \cdot F(t, (u, v)) < 0$ along the boundaries of $\mathcal{D}$, where $n(u, v)$ is the unit normal vector field along the boundary of $\mathcal{D}$ and $F(t, (u, v))$ is defined in (1). Notice that for, any $(u, v) \in \mathcal{D}$,

$$1 \le u \le 2, \quad 0.01 \le v \le 0.1. \tag{3}$$

Let $l_1 = \{(u, v) \in \mathbf{R}^2 : u = 1\}$, for any $(u, v) \in l_1 \cap \partial\mathcal{D}$, $n(u, v) = (-1, 0)$ and

$$F(t, (u, v)) = (a(t) - u(t) + u^2(t)v(t), b(t) - u^2(t)v(t)),$$

by (2) and (3),

$$n(u, v) \cdot F(t, (u, v)) = 1 - a(t) - v(t) < -v(t) < 0.$$

Let $l_2 = \{(u,v) \in \mathbf{R}^2 : u = 2\}$, for any $(u,v) \in l_2 \cap \partial\mathcal{D}$, $n(u,v) = (1,0)$. It follows from (2) and (3) that

$$n(u,v) \cdot F(t,(u,v)) = a(t) - 2 + 4v(t) < 0.$$

Let $l_3 = \{(u,v) \in \mathbf{R}^2 : v = 0.01\}$, for any $(u,v) \in l_3 \cap \partial\mathcal{D}$, $n(u,v) = (0,-1)$. Then using (2) and (3), we get

$$n(u,v) \cdot F(t,(u,v)) = -b(t) + 0.01u^2(t) < 0.$$

Let $l_4 = \{(u,v) \in \mathbf{R}^2 : v = 0.1\}$, for any $(u,v) \in l_4 \cap \partial\mathcal{D}$, $n(u,v) = (0,1)$. It follows from (2) and (3),

$$n(u,v) \cdot F(t,(u,v)) = b(t) - 0.1u^2(t) < 0.$$

Since $n(u,v) \cdot F(t,(u,v)) < 0$ for all $(u,v) \in \partial\mathcal{D}$, $\mathcal{D}$ is a strictly positively invariant region. $\square$

Linearize the system (1) with respect to its T-periodic solution $x(t) = (u(t), v(t))^T \in \mathcal{D}$ for any $t \in \mathbf{R}$ (if such a T-periodic solution exists). Then we get

$$\dot{W}(t) = A(t)W(t), \tag{4}$$

where

$$A(t) = F'_{x(t)} = \begin{pmatrix} -1 + 2u(t)v(t) & u^2(t) \\ -2u(t)v(t) & -u^2(t) \end{pmatrix}$$
$$W(t) = \begin{pmatrix} w_1(t) \\ w_2(t) \end{pmatrix}$$

is a variation vector field along the trajectory $x(t)$.

Proposition 2.1. *Linear system* (4) *satisfies* $\operatorname{tr}(A(t)) < 0$ *and* $\det(A(t)) > 0$ *for any* $t \in \mathbf{R}$.

Proof. By (2) and (3),

$$\operatorname{tr}(A(t)) = -1 + 2u(t)v(t) - u^2(t) < 0.$$

Furthermore,

$$\det(A(t)) = u^2(t) > 0. \qquad \square$$

3. Preliminaries.

Consider the nonlinear system

$$\dot{x}(t) = V(t, x(t)), \quad x(t_0) = x_0, \quad x(t) \in \mathbf{R}^2. \tag{5}$$

Lemma 3.1. *If $x^*(t)$ is an exponentially stable solution of* (5)*, then it is also a uniformly asymptotically stable solution of* (5)*.*

For the proof, see [**4**, pages 178–179].

Let $\mathcal{X} = \{x \in C([0,T]) \mid x(0) = x(T)\}$. Clearly $\mathcal{X}$ is a Banach space with the supremum norm. Define $Lx(t) = \dot{x}(t)$ with domain

$$\mathrm{Dom}\,(L) = \{x \in C^1([0,T]) \mid x(0) = x(T)\}.$$

It is easy to verify that $\mathrm{Dom}\,(L)$ is contained in $\mathcal{X}$, the range of L is $\mathrm{Im}\,(L) = \{z(t) \in \mathcal{X} \mid \int_0^T z(t)\,dt = 0\}$ and L is a Fredholm mapping of index 0. Let

$$\Theta = \{x \in \mathrm{Dom}\,(L) \mid x(t) \in \mathcal{D}, \text{ for all } t \in [0,T]\}. \tag{6}$$

Define $\mathcal{F}_1 : \Theta \to \mathcal{X}$ by $\mathcal{F}_1(x) = F(\cdot, x(\cdot))$ and $H_1(x)(t) = \mathcal{F}_1(x)(t) - Lx(t)$.

Now, construct a homotopy family

$$H_\lambda : (\mathrm{Dom}\,(L) \cap \Theta) \times [0,1] \longrightarrow \mathcal{X}$$

to be of the form

$$H_\lambda(x)(t) = \mathcal{F}_\lambda(x)(t) - Lx(t), \tag{7}$$

where $\mathcal{F}_\lambda : \Theta \times [0,1] \to \mathcal{X}$ with

$$\mathcal{F}_\lambda(x)(t) = \begin{pmatrix} \tilde{a}(t) - u(t) + u^2(t)v(t) \\ \tilde{b}(t) - u^2(t)v(t) \end{pmatrix}. \tag{8}$$

Here $\tilde{a}(t) = (1-\lambda)1.4 + \lambda a(t)$ and $\tilde{b}(t) = 0.05(1-\lambda) + \lambda b(t)$ with $\lambda \in [0,1]$. It is easy to verify that $\mathcal{F}_\lambda : \overline{\Theta} \times [0,1] \to \mathcal{X}$ is L-compact. For more details of degree theory, see [**8**, Chapters I–IV].

Lemma 3.2. *Given $\lambda \in [0,1]$, if $x_\lambda \in \overline{\Theta}$ is a T-periodic solution of the system*

$$\dot{x}(t) = \mathcal{F}_\lambda(x)(t), \tag{9}$$

then $x_\lambda \in \Theta$.

Proof. Clearly, $\tilde{a}$ and $\tilde{b}$ satisfy conditions (2). System (9) is an ES-S model. By Lemma 2.1, $\mathcal{D}$ is still a strictly positively invariant region of system (9). None of the T-periodic solutions of (9) in $\overline{\Theta}$ can touch the boundary of $\mathcal{D}$. □

Corollary 3.1. $0 \notin H_\lambda((\mathrm{Dom}\,(L) \cap \partial\mathcal{D}) \times [0,1])$.

Lemma 3.3. $D_L(H_0(x)(t), \Theta) = D_B(H_0(x)(t), \mathcal{D}) = 1$, *where D_L denotes Leray-Schauder degree and D_B denotes Brouwer degree.*

Proof. For the system $H_0(x)(t) = 0$, there is only one steady-state $p = (a+b, (b/a+b)) = (1.45, (1/29))$ in the strictly positively invariant region $\mathcal{D}$, which is a trivial T-periodic solution. Since $H_0(x)(t) = 0$ is an autonomous system, Proposition 2.1 and Bendixson's criteria [**15**, page 264] guarantee that p is only one T-periodic solution in $\mathcal{D}$.

For the system $H_0(x)(t) = 0$, the Leray-Schauder degree of H_0 in $\mathcal{D}$ is in fact reduced into the Brouwer degree. Therefore, by Proposition 2.1,

$$D_L(H_0, \Theta) = D_B(H_0, \mathcal{D}) = D_B(\mathcal{F}_0, \mathcal{D}) = \mathrm{sign}\,(\det A(t)) = 1,$$

where $A(t)$ is defined in (4). □

Lemma 3.4. *For system* (4) *with conditions* (2), *zero is the only T-periodic solution.*

Proof. Suppose (4) has a nontrivial T-periodic solution called $W_1(t)$. By Proposition 2.1 and Floquet theory [**2**, pages 93–105], its orbit Γ is orbitally asymptotically stable. For $s \in \mathbf{R}$, $sW_1(t)$ is also a T-periodic solution of (4). Then the orbit of $sW_1(t)$ cannot be attracted to Γ for any $s \in \mathbf{R}$. This leads to a contradiction to the orbital asymptotic stability of Γ. □

Remark 3.1. For the linear system (4), if $\operatorname{tr}(A(t))$ does not change sign in some simply connected region $E \subset \mathbf{R}^2$, then (4) has no nontrivial periodic solution in E; since system (4) is a linearization of a nonautonomous system, Bendixson's criteria cannot be used to prove Lemma 3.4.

Lemma 3.5. *Suppose $\mathcal{F} : \mathcal{X} \to \mathcal{X}$ is a completely continuous map of a Banach space such that $\mathcal{F}(0) = 0$ and $\mathcal{F}$ is Frechet differentiable at* 0 *with Frechet derivative $T \in K(\mathcal{X})$, where $K(\mathcal{X})$ is a set of all compact operators defined on $\mathcal{X}$. If $I - T \in L(\mathcal{X})$ is regular (invertible), then there exists an $\eta > 0$ such that, for $B = \{x \in \mathcal{X} : \|x\|_\infty < \eta\}$, we have*

$$D(\mathcal{F} - I, B) = D(T - I, B).$$

For the proof, see [**1**, Chapter 14].

Assume that system (4) is the linearization of system (1) with respect to $x_0(t)$; by Theorem 2.10 of [**2**, page 97], system (4) can be transformed into an autonomous system

$$\dot{Z}(t) = RZ(t), \tag{10}$$

where R is called a monodromy matrix of $A(t)$.

Lemma 3.6. *Let $A(t)$ and $W(t)$ be defined in* (4)*, $LW(t) = \dot{W}(t)$; $B(x_0(t), \varepsilon) \subset \Theta$ denotes a small neighborhood of $x_0(t)$, $B(0, \varepsilon) \subset \Theta \setminus \{x_0(t)\}$ denotes a small neighborhood of* 0*. Set*

$$Q(W(t), \lambda) = (\lambda R + (1 - \lambda)A(t))W(t) - LW(t).$$

Then

$$D_B(Q(\cdot, 1), B(0, \varepsilon) \cap \mathbf{R}^2) = 1.$$

Proof. Clearly, $\mathrm{tr}\,(R) = \rho_1 + \rho_2 = (1/T)\int_0^T \mathrm{tr}\,(A(s))\,ds\,(\mathrm{mod}(2\pi i/T)) < 0$, where ρ_1 and ρ_2 are eigenvalues of R. By Proposition 2.1, for any $\lambda \in [0,1]$,

$$\mathrm{tr}\,(\lambda R + (1-\lambda)A(t)) = \lambda \mathrm{tr}\,(R) + (1-\lambda)\mathrm{tr}\,(A(t)) < 0,$$

and Remark 3.1 implies that $Q(W(t),\lambda) = 0$ has only a trivial T-periodic solution in $B(0,\varepsilon)$. By degree invariance with respect to the homotopy family,

$$D_L(Q(\cdot,0),B(0,\varepsilon)) = D_L(Q(\cdot,1),B(0,\varepsilon)) = D_B(Q(\cdot,1),B(0,\varepsilon)\cap \mathbf{R}^2).$$

Consider the Taylor expansion of $H_1(x)(t)$ at $x_0(t) \in B(x_0(t),\varepsilon)$, where $H_1(x)(t)$ is defined in (7) as $\lambda = 1$. Then we have

$$H_1(x)(t) = H_1(x_0)(t) + M(t)(x(t)-x_0(t)) + h(t,x(t)-x_0(t)),$$

where $M = \mathcal{F}_1' - L$ is $(H_1)_x'$ and $h(t,x(t)-x_0(t))$ is a function of $o(\|x(t)-x_0(t)\|_\infty)$. Since $x_0(t)$ is the unique solution of $H_1(x)(t) = 0$ in $\mathcal{D}$, by excision property of the degree, $D_L(H_1(x)(t),\Theta) = D_L(H_1(x)(t),B(x_0(t),\varepsilon))$ and by Lemma 3.5,

$$D_L(H_1(x)(t),B(x_0(t),\varepsilon)) = D_L(M(t)(x(t)-x_0(t)),B(x_0(t),\varepsilon)).$$

Let $W(t) = x(t) - x_0(t)$. Then by Lemma 3.3,

$$\begin{aligned} D_L(H_0(x)(t),\Theta) &= D_L(H_1(x)(t),\Theta) \\ &= D_L(M(t)W(t),B(0,\varepsilon)) \\ &= D_L(Q(\cdot,0),B(0,\varepsilon)) \\ &= D_L(Q(\cdot,1),B(0,\varepsilon)) \\ &= D_B(Q(\cdot,1),B(0,\varepsilon)\cap\mathbf{R}^2) = 1. \qquad \square \end{aligned}$$

4. Main results.

Theorem 4.1. *For the* ES-S *model, there exists only one uniformly asymptotically stable T-periodic solution $x_0(t)$ in $\mathcal{D}$.*

Proof. (*Existence*). Combine Lemmas 3.2 and 3.3 and Corollary 3.1. By a general existence theorem of the Leray-Schauder type, we get

$$D_L(H_1(x)(t),\Theta) = D_L(H_0(x)(t),\Theta) = D_B(\mathcal{F}_0(x)(t),\mathcal{D}) = 1,$$

which implies that there at least exists one T-periodic solution $x_0(t) = (u_0(t), v_0(t))^T$ of the ES-S model in $\mathcal{D}$. If a and b are constants, it is easy to show that there is only one trivial T-periodic solution $x_0 \in \operatorname{int}(\mathcal{D})$; otherwise, we can easily verify that $x_0(t)$ is a nontrivial T-periodic solution of the ES-S model in $\mathcal{D}$ by substituting $x_0(t)$ into the ES-S model.

(*Uniqueness*). Define

$$C_T = \{x(t) \in \Theta | x(t) \text{ satisfies (1) with conditions (2)}\}.$$

Since $x_0(t) \in C_T$, C_T is not an empty set. If a and b are constants, there is only one constant solution in C_T.

If one of $a(t)$ and $b(t)$ is a nontrivial T-periodic function, then $x_0(t) \in C_T$ is a nontrivial T-periodic solution. Assume C_T is not a singleton; we pick

$$x_1(t) = \begin{pmatrix} u_1(t) \\ v_1(t) \end{pmatrix}, \quad x_2(t) = \begin{pmatrix} u_2(t) \\ v_2(t) \end{pmatrix}$$

in C_T and substitute them into (1) to get

$$\dot{x_i}(t) = F(t, x_i(t)), \quad i = 1, 2. \tag{11}$$

Define $z(t) = x_1(t) - x_2(t)$. By the mean value theorem, we get

$$\dot{z}(t) = z(t)\int_0^1 F_x'[t, x_2(t) + \theta(x_1(t) - x_2(t))]\, d\theta \tag{12}$$

and

$$\int_0^1 F_x'[t, x_2(t) + \theta(x_1(t) - x_2(t))]\, d\theta = \begin{pmatrix} -1 + 2n(t) & m(t) \\ -2n(t) & -m(t) \end{pmatrix},$$

where

$$m(t) = \int_0^1 [u_2(t) + \theta(u_1(t) - u_2(t))]^2 d\theta,$$
$$n(t) = \int_0^1 [u_2(t) + \theta(u_1(t) - u_2(t))][v_2(t) + \theta(v_1(t) - v_2(t))]\, d\theta.$$

Since

$$n(t) = \frac{1}{2}(v_2(t)u_1(t) + u_2(t)v_1(t)) + \frac{1}{3}(u_1(t) - u_2(t))(v_1(t) - v_2(t))$$

and

$$m(t) = \frac{1}{3}(u_1(t) - u_2(t))^2 + u_2(t)u_1(t),$$

$$\begin{aligned} \operatorname{tr}\left(\int_0^1 F_x'[t, x_2(t) + \theta(x_1(t) - x_2(t))]\, d\theta\right) \\ = -1 - \frac{1}{3}v_1(t)^2 - \frac{1}{3}v_2(t)^2 - \frac{1}{3}v_1(t)v_2(t) \\ + \frac{1}{3}v_2(t)u_1(t) + \frac{1}{3}v_1(t)u_2(t) \\ + \frac{2}{3}v_1(t)u_1(t) + \frac{2}{3}v_2(t)u_2(t). \end{aligned}$$

Notice the following facts:

$$\begin{aligned} \frac{1}{3}v_2(t)u_1(t) + \frac{2}{3}v_2(t)u_2(t) \le \frac{v_2(t)}{3}(u_1(t) + 2u_2(t)) \le 2v_2(t) \le 0.2, \\ \frac{1}{3}v_1(t)u_2(t) + \frac{2}{3}v_1(t)u_1(t) \le \frac{v_1(t)}{3}(u_2(t) + 2u_1(t)) \le 2v_1(t) \le 0.2. \end{aligned}$$

It follows that

$$\operatorname{tr}\left(\int_0^1 F_x'[t, x_2(t) + \theta(x_1(t) - x_2(t))]\, d\theta\right) < 0. \tag{13}$$

Equation (13) implies that the zero solution is the only T-periodic solution for (12) by Remark 3.1. Hence, $x_1(t) = x_2(t)$. C_T is a singleton.

(*Stability*). If $a(t)$ and $b(t)$ are constant functions, then $x_0 \in \mathcal{D}$ is a constant solution of system (1). By Proposition 2.1, x_0 is a locally uniformly asymptotically stable solution.

If one of $a(t)$ and $b(t)$ is a nontrivial T-periodic function, $x_0(t)$ is a nontrivial T-periodic solution of system (1). By Proposition 2.1, one Floquet exponent ρ_1 has negative real part. If $\operatorname{Real}(\rho_2) < 0$, $x_0(t)$ is locally uniformly asymptotically stable by [**2**, Theorem 2.13, page 101] and Lemma 3.1.

To show that Real $(\rho_2) < 0$ always holds by treating these two cases: (1) ρ_2 is a complex number. Notice that ρ_1 and ρ_2 are conjugate eigenvalues of R. Therefore, Real $(\rho_1) < 0$ implies that Real $(\rho_2) < 0$; (2) ρ_2 is a real number. Clearly ρ_1 is also a real number and $\rho_1 < 0$ implies that $\rho_2 \neq 0$ (otherwise, (4) must have one nontrivial T-periodic solution, which contradicts Lemma 3.4). If $\rho_2 > 0$, then $\det(R) < 0$, $\operatorname{sign}(\det(R)) = -1 = D_B(Q(\cdot,1), B(0,\varepsilon) \cap \mathbf{R}^2)$, which contradicts Lemma 3.6. □

5. Future works. In this paper, we proved the existence, uniqueness and stability of the periodic solution $x_0(t)$ of the ES-S model. This establishes a foundation for further study of patterns of the E-S model. Of course, the problem mentioned here is still open; the investigation of this question is currently underway.

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