ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 38, Number 5, 2008

# DYNAMICAL BEHAVIOR OF AN EPIDEMIC MODEL WITH COINFECTION OF TWO DISEASES

JIANQUAN LI, LADI WANG, HUIWEN ZHAO AND ZHIEN MA

ABSTRACT. We have formulated a simple epidemiological model with two diseases that can coinfect a single host. The first disease is assumed to be chronic, the second one acute. For infectives infected only by the first disease, we introduce the age of infection. For these two diseases, we obtain their reproduction numbers, respectively, and establish conditions for the existence and stability of the disease-free equilibrium, the boundary equilibrium, and the positive (coexistent) equilibrium. For infectious individuals infected only by the first disease, when some transfer rates depend on the age of infection and the corresponding model is governed by partial differential equations (PDEs), we give a sufficient condition for the existence of positive equilibrium, and its stability is determined by a transcendental equation; when all the associated rates are independent of the age of infection, the corresponding models are ordinary differential equations (ODEs). We obtain complete results on dynamics, find that the coexistent equilibrium of two diseases is globally stable if they exist, and find that the boundary equilibrium is globally stable if it is locally stable. Finally, we find that there is a difference between PDE and ODE models.

**1. Introduction.** It is well known that a carrier of human immunodeficiency virus (HIV) or a patient with tuberculosis (TB) commonly suffers with a slow progressing disease which lasts for a few years [**4, 5, 6, 9, 17, 20**], i.e., these diseases are chronic. Generally, infecting ability of a patient with the chronic disease may vary with the change of the age of infection (the time lapsed since infection) [**9, 13, 18**]. Thus, to formulate the spread of infection, it is necessary to incorporate the age of infection into an epidemic model with a chronic disease. On the other hand, a patient infected by a chronic disease may also be infected by other acute diseases [**1, 8**] such that coinfection of

*Keywords and phrases.* Epidemic model, age of infection, equilibrium, stability.
This work was partially supported by the Nature Science Foundation of China (10671209, 10531030) and the Foundation of Scientific Research of Air Force Engineering University.
Received by the editors on August 29, 2007, and in revised form on December 31, 2007.
DOI:10.1216/RMJ-2008-38-5-1457 

two or many diseases occurs in a single host. But, the introduction of the age of infection is not necessary for formulating the spread of an acute disease. Therefore, epidemic models with chronic and acute diseases generally include partial differential equations (PDEs) and ordinary differential equations (ODEs). Recently, on the immunological level and with mathematical analysis, coinfection has been studied by some researchers [**3, 7, 10, 13, 14, 16**].

In this paper, based on the classical SIR epidemic model with bilinear incidence, we establish and investigate an epidemic model with coinfection of two diseases in a single host. We assume that the first disease is chronic and the second one is acute. The age of infection is introduced for the first disease. In [**12, 15, 18**], it has been shown that the age of infection can cause qualitative changes in dynamics for some epidemic models with a single disease. Although the model discussed here is relatively simple, we can also find the difference between models with and without an age of infection. That is, it is possible that the boundary equilibria are bistable for a model with the age of infection, but the bistable case is impossible for a model without the age of infection.

This paper is organized as follows. In Section 2, an epidemic model with coinfection of two diseases is introduced. In Sections 3 and 4, we investigate the existence and stability of boundary and positive equilibria, respectively. In Section 5, we consider the case that all the mutation rate for the first disease is independent of the age of infection, and the global stability of the corresponding model is discussed. In Section 6, we discuss the difference between models with and without the age of infection, and an example is given to illustrate the main results.

**2. Model formulation.** Assume that two diseases are spreading in a population. The first disease is chronic, and the second one is acute. After a susceptible individual is infected by the first one, he (she) may also be infected by the second disease or recover. And the second one can also infect susceptible individuals directly. Thus, individuals in the population are divided into four classes: the class $S$ susceptible for two diseases, the first infected class $I_1$ infected only by the first disease, the second infected class $I_2$ infected by the second one and from either $S$ or $I_1$, the recover class $R$ which is from $I_1$ and $I_2$ and has permanent immunity for two diseases. Let $S(t)$, $I_1(t), I_2(t)$ and

$R(t)$ denote the numbers of individuals at time $t$ in classes $S$, $I_1, I_2$ and $R$, respectively. Since the first disease is a slowly progressing one, for class $I_1$ we introduce the age of infection, $\tau$, and the distribution of infectives infected only by the first disease at time $t$, $i_1(t,\tau)$. Clearly, $I_1(t) = \int_0^\infty i_1(t,\tau)\,d\tau$. A susceptible individual is infected by the first disease at a rate $\beta_1(\tau)$. The second disease is transmitted by the class $I_2$ to the susceptible class at a rate $\beta_2$. An individual already infected with the first disease can be coinfected by the second disease at a rate $\varepsilon(\tau)$. The individuals in class $I_1$ die of the first disease or recover at rates $\alpha_1(\tau)$ and $\gamma_1(\tau)$, respectively. The individuals in class $I_2$ die of two diseases or recover at rates $\alpha_2$ and $\gamma_2$, respectively. Let $bA$ be the input flow into the susceptible class, $b$ the natural death rate. Here, parameters $\alpha_2, \beta_2$ and $\gamma_2$ are nonnegative whereas $b$ and $A$ are positive. Functions $\beta_1(\tau)$, $\gamma_1(\tau)$, $\alpha_1(\tau)$ and $\varepsilon(\tau)$ are nonnegative, continuous and bounded.

Under the assumptions above, we have the following epidemic model:

(1)
$$\begin{aligned}
&\frac{dS(t)}{dt} = bA - bS(t) - S(t)\left[\int_0^\infty \beta_1(\tau) i_1(t,\tau)\,d\tau + \beta_2 I_2(t)\right],\\
&\frac{\partial i_1(t,\tau)}{\partial t} + \frac{\partial i_1(t,\tau)}{\partial \tau} = -[b+\alpha_1(\tau)+\gamma_1(\tau)]i_1(t,\tau) - \varepsilon(\tau) i_1(t,\tau) I_2(t),\\
&i_1(t,0) = S(t)\int_0^\infty \beta_1(\tau) i_1(t,\tau)\,d\tau,\\
&i_1(0,\tau) = \phi(\tau),\\
&\frac{dI_2(t)}{d\tau} = \beta_2 S(t) I_2(t) - (b+\alpha_2+\gamma_2) I_2(t) + I_2(t)\int_0^\infty \varepsilon(\tau) i_1(t,\tau)\,d\tau,\\
&\frac{dR(t)}{dt} = \int_0^\infty \gamma_1(\tau) i_1(t,\tau)\,d\tau + \gamma_2 I_2 - bR,
\end{aligned}$$

where the initial distribution $\phi(\tau)$ is assumed to be integrable and compactly supported in $[0,\infty)$.

Since the dynamics of $R(t)$ do not affect the evolution of $S$, $i_1$ and $I_2$, we omit the equation for $R(t)$ when studying the development of two diseases. Again, for simplification of notation, let $m(\tau) = b + \alpha_1(\tau) + \gamma_1(\tau)$ and $n = b + \alpha_2 + \gamma_2$; then, we will consider the following system:

(2)
$$\begin{aligned}
&\frac{dS(t)}{dt} = bA - bS(t) - S(t)\left[\int_0^\infty \beta_1(\tau)i_1(t,\tau)\,d\tau + \beta_2 I_2(t)\right],\\
&\frac{\partial i_1(t,\tau)}{\partial t} + \frac{\partial i_1(t,\tau)}{\partial \tau} = -m(\tau)i_1(t,\tau) - \varepsilon(\tau)i_1(t,\tau)I_2(t),\\
&i_1(t,0) = S(t)\int_0^\infty \beta_1(\tau)i_1(t,\tau)\,d\tau,\\
&i_1(0,\tau) = \phi(\tau),\\
&\frac{dI_2(t)}{dt} = \beta_2 S(t)I_2(t) - nI_2(t) + I_2(t)\int_0^\infty \varepsilon(\tau)i_1(t,\tau)\,d\tau.
\end{aligned}$$

Using $\lim_{\tau\to\infty} i_1(t,\tau) = 0$, which implies that there are no individuals with infinite age of infection, and integrating the second equation of (2) give

$$\begin{aligned}
\frac{dI_1(t)}{dt} = S(t)&\int_0^\infty \beta_1(\tau)i_1(t,\tau)\,d\tau\\
&- \int_0^\infty m(\tau)i_1(t,\tau)\,d\tau - I_2(t)\int_0^\infty \varepsilon(\tau)i_1(t,\tau)\,d\tau.
\end{aligned}$$

Let $N(t) = S(t) + I_1(t) + I_2(t)$, then

$$\begin{aligned}
\frac{dN(t)}{dt} &= bA - bS(t) - \int_0^\infty m(\tau)i_1(t,\tau)\,d\tau - nI_2(t)\\
&\le b[A - S(t) - I_1(t) - I_2(t)] = b[A - N(t)],
\end{aligned}$$

where $m(\tau) \ge b$ and $n \ge b$ are used. It follows that $\limsup_{t\to\infty} N(t) \le A$. Therefore, for $S(t)$, $I_1(t)$ and $I_2(t)$, we will consider (2) under the case $S(t) + I_1(t) + I_2(t) \le A$.

We note that, by standard methods, it is possible to prove the existence and uniqueness of solutions to the system (2) [**19**]. Moreover, it is easy to show that all the variables remain nonnegative and bounded for $t > 0$ for nonnegative initial data.

We introduce the following quantities which will be used throughout this paper:

$$\begin{aligned}
\pi_1(\tau) &= e^{-\int_0^\tau m(u)\,du} \le 1,\\
\pi_2(\tau) &= e^{-\int_0^\tau \varepsilon(u)\,du} \le 1,\\
R_1 &= A\int_0^\infty \beta_1(\tau)\pi_1(\tau)\,d\tau,\\
R_2 &= \frac{\beta_2 A}{n},\\
k &= \int_0^\infty \varepsilon(\tau)\pi_1(\tau)\,d\tau,\\
k' &= \frac{A}{R_2}\int_0^\infty \beta_1(\tau)\pi_1(\tau)[\pi_2(\tau)]^{bA(1-1/R_2)/n}\,d\tau,
\end{aligned}$$

where $R_1$ and $R_2$ are the reproduction numbers of two diseases, respectively.

**3. Existence and stability of boundary equilibrium.** With respect to the existence of boundary equilibrium, it is easy to get the following theorem:

**Theorem 1.** *System* (2) *always has the disease-free equilibrium* $E_0(A,0,0)$. *When* $R_1 > 1$, *system* (2) *has the boundary equilibrium* $E_1(A/R_1, bA\pi_1(\tau)(1-1/R_1), 0)$. *When* $R_2 > 1$, *system* (2) *has the boundary equilibrium* $E_2(A/R_2, 0, bA(1-1/R_2)/n)$.

With respect to the stability of boundary equilibrium, we have the following theorem:

**Theorem 2.** *For system* (2), *the disease-free equilibrium* $E_0$ *is asymptotically stable if* $\max\{R_1,R_2\} < 1$ *and unstable if* $\min\{R_1,R_2\} > 1$. *The boundary equilibrium* $E_1$ *is asymptotically stable if* $R_1 > 1$ *and* $k < (n(R_1-R_2)/bA(R_1-1))$, *and unstable if* $R_1 > 1$ *and* $k > (n(R_1-R_2)/bA(R_1-1))$. *The boundary equilibrium* $E_2$ *is asymptotically stable if* $R_2 > 1$ *and* $k' < 1$, *and unstable if* $R_2 > 1$ *and* $k' > 1$.

*Proof.* Linearizing system (2) about equilibria $E_0$, $E_1$ and $E_2$, we can obtain systems

$$
\begin{aligned}
&x'(t) = -bx(t) - A\left[\int_0^\infty \beta_1(\tau)y(t,\tau)\,d\tau + \beta_2 z(t)\right],\\
&\frac{\partial y(t,\tau)}{\partial t} + \frac{\partial y(t,\tau)}{\partial \tau} = -m(\tau)y(t,\tau),\\
&y(t,0) = A\int_0^\infty \beta_1(\tau)y(t,\tau)\,d\tau,\\
&z'(t) = n(R_2 - 1)z(t),
\end{aligned} \tag{3}
$$

(4)
$$
\begin{aligned}
&x'(t) = -bR_1x(t) - \frac{A}{R_1}\int_0^\infty \beta_1(\tau)y(t,\tau)\,d\tau - \frac{A}{R_1}\beta_2 z(t),\\
&\frac{\partial y(t,\tau)}{\partial t} + \frac{\partial y(t,\tau)}{\partial \tau} = -m(\tau)y(t,\tau) - bA\left(1 - \frac{1}{R_1}\right)\varepsilon(\tau)\pi_1(\tau)z(t),\\
&y(t,0) = b(R_1 - 1)x(t) + \frac{A}{R_1}\int_0^\infty \beta_1(\tau)y(t,\tau)\,d\tau,\\
&z'(t) = n\left(\frac{R_2}{R_1} - 1\right)z(t) + bAk\left(1 - \frac{1}{R_1}\right)z(t),
\end{aligned}
$$

and

(5)
$$
\begin{aligned}
&x'(t) = -bR_2x(t) - \frac{A}{R_2}\int_0^\infty \beta_1(\tau)y(t,\tau)\,d\tau - nz(t),\\
&\frac{\partial y(t,\tau)}{\partial t} + \frac{\partial y(t,\tau)}{\partial \tau} = -m(\tau)y(t,\tau) - \frac{bA}{n}\left(1 - \frac{1}{R_2}\right)\varepsilon(\tau)y(t,\tau),\\
&y(t,0) = \frac{A}{R_2}\int_0^\infty \beta_1(\tau)y(t,\tau)\,d\tau,\\
&z'(t) = b(R_2 - 1)x(t) + \frac{bA}{n}\left(1 - \frac{1}{R_2}\right)\int_0^\infty \varepsilon(\tau)y(t,\tau)\,d\tau,
\end{aligned}
$$

respectively.

For (3), (4) and (5), let $x(t) = x_0e^{\lambda t}$, $y(t,\tau) = p(\tau)e^{\lambda(t-\tau)}$, $z(t) = z_0e^{\lambda t}$. We have

$$(6)\qquad \begin{aligned} \lambda x_0 &= -bx_0 - A\left[\int_0^\infty \beta_1(\tau)p(\tau)aue^{-\lambda\tau}\,d\tau + \beta z_0\right],\\ p'(\tau) &= -m(\tau)p(\tau),\\ p(0) &= A\int_0^\infty \beta_1(\tau)p(\tau)e^{-\lambda\tau}\,d\tau,\\ \lambda z_0 &= n(R_2-1)z_0, \end{aligned}$$

and

$$(7)\qquad \begin{aligned} \lambda x_0 &= -bR_1x_0 - \frac{A}{R_1}\int_0^\infty \beta_1(\tau)p(\tau)e^{-\lambda\tau}\,d\tau - \frac{A}{R_1}\beta_2 z_0,\\ p'(\tau) &= -m(\tau)p(\tau) - bA\left(1-\frac{1}{R_1}\right)\varepsilon(\tau)\pi_1(\tau)e^{\lambda\tau}z_0,\\ p(0) &= b(R_1-1)x_0 + \frac{A}{R_1}\int_0^\infty \beta_1(\tau)p(\tau)e^{-\lambda\tau}\,d\tau,\\ \lambda z_0 &= n\left(\frac{R_2}{R_1}-1\right)z_0 + bAk\left(1-\frac{1}{R_1}\right)z_0, \end{aligned}$$

and

$$(8)\qquad \begin{aligned} \lambda x_0 &= -bR_2x_0 - \frac{A}{R_2}\int_0^\infty \beta_1(\tau)p(\tau)e^{-\lambda\tau}\,d\tau - nz_0,\\ p'(\tau) &= -m(\tau)p(\tau) - \frac{bA}{n}\left(1-\frac{1}{R_2}\right)\varepsilon(\tau)p(\tau),\\ p(0) &= \frac{A}{R_2}\int_0^\infty \beta_1(\tau)p(\tau)e^{-\lambda\tau}\,d\tau,\\ \lambda z_0 &= b(R_2-1)x_0 + \frac{bA}{n}\left(1-\frac{1}{R_2}\right)\int_0^\infty \varepsilon(\tau)p(\tau)e^{-\lambda\tau}\,d\tau, \end{aligned}$$

respectively.

(1) From the second equation of (6), we have

$$p(\tau) = p(0)\pi_1(\tau).$$

Substituting it into the third equation of (6) gives

$$p(0) = p(0)A\int_0^\infty \beta_1(\tau)\pi_1(\tau)e^{-\lambda\tau}\,d\tau.$$

It is easy to see that all the roots of equation $1 = A\int_0^\infty \beta_1(\tau)\pi_1(\tau)e^{-\lambda\tau}d\tau$ $=: \Delta_1(\lambda)$ have the negative real parts if $R_1 < 1$, and that equation $\Delta_1(\lambda) = 1$ has a positive root if $R_1 > 1$, since $\Delta_1(0) = R_1 > 1$ and $\lim_{\lambda\to\infty}\Delta_1(\lambda) = 0$. Again, (6) also has an eigenvalue, $\lambda = \beta_2 A - n = n(R_2 - 1)$, which is negative if $R_2 < 1$ and positive if $R_2 > 1$. Therefore, the disease-free equilibrium $E_0$ is asymptotically stable if $\max\{R_1, R_2\} < 1$ and unstable if $\min\{R_1, R_2\} > 1$.

(2) From the fourth equation of (7), we obtain an eigenvalue of (7), $\lambda = n(R_2/R_1 - 1) + kbA(1 - 1/R_1)$, which is negative if $R_1 > 1$ and $k < (n(R_1 - R_2)/bA(R_1 - 1))$, and positive if $R_1 > 1$ and $k > (n(R_1 - R_2)/bA(R_1 - 1))$.

The other eigenvalues of (7) are determined by the following equations

$$\begin{aligned}\lambda x_0 &= -bR_1x_0 - \frac{A}{R_1}\int_0^\infty \beta_1(\tau)p(\tau)e^{-\lambda\tau}\,d\tau,\\ p'(\tau) &= -m(\tau)p(\tau),\\ p(0) &= b(R_1 - 1)x_0 + \frac{A}{R_1}\int_0^\infty \beta_1(\tau)p(\tau)e^{-\lambda\tau}\,d\tau.\end{aligned}\tag{9}$$

From (9), we obtain the associated characteristic equation

$$\lambda + bR_1 = (\lambda + b)\frac{A}{R_1}\int_0^\infty \beta_1(\tau)\pi_1(\tau)e^{-\lambda\tau}\,d\tau,$$

which is equivalent to the following equation

$$\frac{\lambda + bR_1}{\lambda + b} = \frac{A}{R_1}\int_0^\infty \beta_1(\tau)\pi_1(\tau)e^{-\lambda\tau}\,d\tau.\tag{10}$$

We claim that all roots of equation (10) of $\lambda$ are with negative real parts. If not, we have

$$\left|\frac{\lambda + bR_1}{\lambda + b}\right| > 1, \quad \text{and} \quad \frac{A}{R_1}\left|\int_0^\infty \beta_1(\tau)\pi_1(\tau)e^{-\lambda\tau}\,d\tau\right| \le 1.$$

Thus, a contradiction occurs. Therefore, the boundary equilibrium $E_1$ is asymptotically stable if $R_1 > 1$ and $k < (n(R_1 - R_2)/bA(R_1 - 1))$ and unstable if $R_1 > 1$ and $k > (n(R_1 - R_2)/bA(R_1 - 1))$.

(3) From the second equation of (8), we have

$$p(\tau) = p(0)\pi_1(\tau)[\pi_2(\tau)]^{(bA/n)(1-(1/R_2))}.$$

Substituting it into the third equation yields the characteristic equation

$$(11)\quad 1 = \frac{A}{R_2}\int_0^\infty \beta_1(\tau)e^{-\lambda\tau}\pi_1(\tau)[\pi_2(\tau)]^{(bA/n)(1-(1/R_2))}\,d\tau := \Delta_2(\lambda).$$

It is easy to see that all the roots of $\Delta_2(\lambda) = 1$ have the negative real parts if $R_2 > 1$ and $k' < 1$ and that equation $\Delta_2(\lambda) = 1$ has a positive root if $R_2 > 1$ and $k' > 1$, since $\Delta_2(0) = k' > 1$ and $\lim_{\lambda\to\infty}\Delta_2(\lambda) = 0$.

The other eigenvalues of (8) are determined by the characteristic equation $\lambda^2 + bR_2\lambda + nb(R_2 - 1) = 0$. It is obvious that all roots of it are with the negative real parts for $R_2 > 1$.

Therefore, $E_2$ is asymptotically stable if $R_2 > 1$ and $k' < 1$, and unstable if $R_2 > 1$ and $k' > 1$.

The proof of Theorem 2 is complete. □

Regarding global stability of $E_0$ and $E_2$, we have

**Theorem 3.** *Assuming that $R_1 < 1$, the following results are true.*

(1) *If $R_2 < 1$, then $E_0$ is globally asymptotically stable.*

(2) *If $R_2 > 1$, then $E_2$ is globally asymptotically stable.*

*Proof.* Neglecting the term dependent on $I_2$ for the equation of $i_1(t,\tau)$, it follows from (2) that

$$\frac{\partial i_1(t,\tau)}{\partial t} + \frac{\partial i_1(t,\tau)}{\partial \tau} \le -m(\tau)i_1(t,\tau).$$

Integrating this inequality along characteristic lines we have

$$i_1(t,\tau) \le \begin{cases} i_1(t-\tau,0)\pi_1(\tau) & \tau < t, \\ \phi(\tau - t)(\pi_1(\tau)/\pi_1(\tau - t)) & \tau \ge t. \end{cases}.$$

Since $S \le A$, then, from the third equation of (2), we get

$$i_1(t,0) \le A\left[\int_0^t \beta_1(\tau)i_1(t-\tau,0)\pi_1(\tau)\,d\tau + \int_t^\infty \beta_1(\tau)\phi(\tau-t)\frac{\pi_1(\tau)}{\pi_1(\tau-t)}\,d\tau\right].$$

Moreover, we have

$$\lim_{t\to\infty} \sup i_1(t,0) \le A \int_0^\infty \beta_1(\tau)\pi_1(\tau)\,d\tau,$$
$$\lim_{t\to\infty} \sup i_1(t,0) = R_1 \lim_{t\to\infty} \sup i_1(t,0).$$

So $\lim_{t\to\infty} \sup i_1(t,0) = 0$ since $R_1 < 1$ and $\lim_{t\to\infty} \sup i_1(t,0) < \infty$. This implies that $\lim_{t\to\infty} i_1(t,\tau) = 0$ for every fixed $\tau$ and $\lim_{t\to\infty} \int_0^\infty i_1(t,\tau)\,d\tau = 0$. It follows that

$$\lim_{t\to\infty} \int_0^\infty \varepsilon(\tau) i_1(t,\tau)\,d\tau = 0 \text{ and } \lim_{t\to\infty} \int_0^\infty \beta_1(\tau) i_1(t,\tau)\,d\tau = 0.$$

Then, system (2) has the following asymptotically autonomous limiting system

$$\begin{aligned} \frac{dS(t)}{dt} &= bA - bS(t) - \beta_2 S(t) I_2(t), \\ \frac{dI_2(t)}{dt} &= \beta_2 S(t) I_2(t) - nI_2(t). \end{aligned} \tag{12}$$

For (12), it is easy to know that equilibrium $\overline{E}_0(A,0)$ is globally stable as $R_2 < 1$ and equilibrium $\overline{E}_2(n/\beta_2, bA(1-1/R_2)/n)$ is globally stable as $R_2 > 1$. Then the theory of asymptotically autonomous systems [**2**] implies that, for $R_1 < 1$, the asymptotic behavior of (2) is the same as that of (12). Therefore, Theorem 3 holds. □

**4. Positive equilibrium.** An equilibrium of system (2) $(S, i_1(\tau), I_2)$ satisfies the following equations

$$\begin{aligned} & bA - bS - S\left[\int_0^\infty \beta_1(\tau) i_1(\tau)\,d\tau + \beta_2 I_2\right] = 0, \\ & \frac{di_1(\tau)}{d\tau} = -m(\tau) i_1(\tau) - \varepsilon(\tau) i_1(\tau) I_2, \\ & i_1(0) = S\int_0^\infty \beta_1(\tau) i_1(\tau)\,d\tau, \\ & \beta_2 S I_2 - nI_2 + I_2 \int_0^\infty \varepsilon(\tau) i_1(\tau)\,d\tau = 0. \end{aligned} \tag{13}$$

To find the positive equilibrium, assume that $i_1(0) \neq 0$ and $I_2 \neq 0$ for (13); then, from the second equation of (13) we have

$$i_1(\tau) = i_1(0)\pi_1(\tau)[\pi_2(\tau)]^{I_2}.$$

Substituting it into the other equations of (13) gives

$$S = \frac{bA}{b + i_1(0)\int_0^\infty \beta_1(\tau)\pi_1(\tau)[\pi_2(\tau)]^{I_2}\,d\tau + \beta_2 I_2}, \tag{14}$$

$$S = \frac{1}{\int_0^\infty \beta_1(\tau)\pi_1(\tau)[\pi_2(\tau)]^{I_2}\,d\tau}, \tag{15}$$

and

$$S = \frac{1}{\beta_2}\left(n - i_1(0)\int_0^\infty \varepsilon(\tau)\pi_1(\tau)[\pi_2(\tau)]^{I_2}\,d\tau\right). \tag{16}$$

From (14) and (15) we have

$$\begin{aligned} & i_1(0)\int_0^\infty \varepsilon(\tau)\pi_1(\tau)[\pi_2(\tau)]^{I_2}\,d\tau \\ & \quad = \int_0^\infty \varepsilon(\tau)\pi_1(\tau)[\pi_2(\tau)]^{I_2}\,d\tau\left(bA - \frac{b+\beta_2 I_2}{\int_0^\infty \beta_1(\tau)\pi_1(\tau)[\pi_2(\tau)]^{I_2}\,d\tau}\right) \\ & \quad := f(I_2). \end{aligned}$$

From (15) and (16) we have

$$\begin{aligned} i_1(0)\int_0^\infty \varepsilon(\tau)\pi_1(\tau)[\pi_2(\tau)]^{I_2}\,d\tau &= n - \frac{\beta_2}{\int_0^\infty \beta_1(\tau)\pi_1(\tau)[\pi_2(\tau)]^{I_2}\,d\tau} \\ &:= g(I_2) \\ f(0) &= kbA\left(1 - \frac{1}{R_1}\right), \\ g(0) &= n\left(1 - \frac{R_2}{R_1}\right). \end{aligned}$$

Let $I_{2f}$ and $I_{2g}$ denote the zeros of functions $f(I_2)$ and $g(I_2)$, respectively.

Define function

$$f_1(I_2) := \frac{bA}{b+\beta_2 I_2} - \frac{1}{\int_0^\infty \beta_1(\tau)\pi_1(\tau)[\pi_2(\tau)]^{I_2}\,d\tau};$$

then

$$f(I_2) = (b+\beta_2 I_2)\int_0^\infty \varepsilon(\tau)\pi_1(\tau)[\pi_2(\tau)]^{I_2}\,d\tau f_1(I_2).$$

For the function $f_1(I_2)$, it is easy to show that $f_1'(I_2) < 0$, $f_1(0) = A(1-1/R_1)$ and $\lim_{I_2\to\infty} f_1(I_2) = -\infty$. So, $I_{2f}$ is a positive number when $R_1 > 1$. Further, when $R_1 > 1$, $f(I_2) > 0$ for $0 < I_2 < I_{2f}$; $f(I_2) < 0$ for $I_2 > I_{2f}$ and $f'(I_2) < 0$ for $0 < I_2 < I_{2f}$.

Since

$$g'(I_2) = \frac{\beta_2 \int_0^\infty \beta_1(\tau)\pi_1(\tau)[\pi_2(\tau)]^{I_2}\ln\pi_2(\tau)\,d\tau}{(\int_0^\infty \beta_1(\tau)\pi_1(\tau)[\pi_2(\tau)]^{I_2}\,d\tau)^2} < 0,$$

and $\lim_{I_2\to\infty} g(I_2) = -\infty$ then, when $R_1 > R_2$, $I_{2g}$ is a positive number. And, when $R_1 > R_2$, $g(I_2) > 0$ for $0 < I_2 < I_{2g}$ and $g(I_2) < 0$ for $I_2 > I_{2g}$.

From the inference above, for positive equilibrium $E^*(S^*, i_1(\tau), I_2^*)$, we know that $I_2^* < \min\{I_{2f}, I_{2g}\}$ when $R_1 > 1$ and $R_1 > R_2$.

Define function

$$g_1(I_2) := \frac{1}{\beta_2} g(I_2) = \frac{n}{\beta_2} - \frac{1}{\int_0^\infty \beta_1(\tau)\pi_1(\tau)[\pi_2(\tau)]^{I_2}\,d\tau}.$$

Notice that $I_{2f}$ and $I_{2g}$ are also the zeros of $f_1(I_2)$ and $g_1(I_2)$, respectively.

When $R_2 \le 1$, $bR_2/(b+\beta_2 I_2) < 1$, that is, $bA/(b+\beta_2 I_2) < n/\beta_2$. It follows that $g_1(I_2) > f_1(I_2)$, then $I_{2g} > I_{2f}$ as $R_1 > 1 \ge R_2$.

When $R_2 > 1$, $f_1(I_2) > g_1(I_2)$ for $I_2 < bA(1-1/R_2)/n =: \hat{I}_2$, and $f(I_2) < g(I_2)$ for $I_2 > \hat{I}_2$. Notice that

$$f_1(\hat{I}_2) = g_1(\hat{I}_2) = \frac{A}{R_2}\left(1 - \frac{1}{k'}\right);$$

then, when $k' < 1$, $f_1(\hat{I}_2) = g_1(\hat{I}_2) < 0$, it implies $I_{2g} < I_{2f} < \hat{I}_2$; when $k' > 1$, $f_1(\hat{I}_2) = g_1(\hat{I}_2) > 0$, it implies $I_{2g} > I_{2f} > \hat{I}_2$.

According to the monotonicity of $f(I_2)$ and $g(I_2)$, equation $f(I_2) = g(I_2)$ of $I_2$ has at least one root if $[f(0) - g(0)][I_{2f} - I_{2g}] < 0$.

On the other hand, when $R_1 > 1$, $f(0) < g(0)$ if and only if $k < (n/bA)(R_1 - R_2)/(R_1 - 1)$. Therefore, regarding the existence of positive equilibrium of (2), we have

**Theorem 4.** *System* (2) *has at least one positive equilibrium if one of the following conditions holds*:

(1) $R_1 > 1 \geq R_2$ *and* $k > (n/bA)(R_1 - R_2)/(R_1 - 1)$.

(2) $R_1 > R_2 > 1$, $k' < 1$, *and* $k < (n/bA)(R_1 - R_2)/(R_1 - 1)$.

(3) $R_1 > R_2 > 1$, $k' > 1$, $k > (n/bA)(R_1 - R_2)/(R_1 - 1)$.

*Remark* 5. Condition (2) in Theorem 4 also implies that the boundary equilibria, $E_1$ and $E_2$, both exist and are locally stable.

Linearizing system (2) about positive equilibrium $E^*(S^*, i_1(\tau), I_2^*)$, we can obtain system

(17)
$$\begin{aligned} x'(t) &= -\frac{bA}{S^*}x(t) - S^* \int_0^\infty \beta_1(\tau) y(t,\tau)\,d\tau - S^*\beta_2 z(t), \\ \frac{\partial y(t,\tau)}{\partial t} + \frac{\partial y(t,\tau)}{\partial \tau} &= \frac{1}{i_1(\tau)}\frac{di_1(\tau)}{d\tau} y(t,\tau) - \varepsilon(\tau) i_1(\tau) z(t), \\ y(t,0) &= S^* \int_0^\infty \beta_1(\tau) y(t,\tau)\,d\tau + \frac{i_1(0)}{S^*} x(t), \\ z'(t) &= \beta_2 I_2^* x(t) + I_2^* \int_0^\infty \varepsilon(\tau) y(t,\tau)\,d\tau, \end{aligned}$$

where $1/(i_1(\tau))(di_1(\tau)/d\tau) = -m(\tau) - \varepsilon(\tau) I_2^*$ and $(i_1(0)/S^*) = \int_0^\infty \beta_1(\tau) i_1(\tau)\,d\tau$ are used.

Let $x(t) = x_0 e^{\lambda t}, y(t,\tau) = p(\tau)e^{\lambda(t-\tau)}$, $z(t) = z_0 e^{\lambda t}$; it follows from (17) that

$$(18) \qquad \begin{aligned} \lambda x_0 &= -\frac{bA}{S^*}x_0 - S^* \int_0^\infty \beta_1(\tau) p(\tau) e^{-\lambda\tau}\,d\tau - S^*\beta_2 z_0, \\ \frac{dp(\tau)}{dt} &= \frac{1}{i_1(\tau)}\frac{di_1(\tau)}{d\tau} p(\tau) - \varepsilon(\tau) i_1(\tau) z_0 e^{\lambda\tau}, \end{aligned}$$

$$p(0) = S^* \int_0^\infty \beta_1(\tau)p(\tau)e^{-\lambda\tau}\, d\tau + \frac{i_1(0)}{S^*}x_0,$$
$$\lambda z_0 = \beta_2 I_2^* x_0 + I_2^* \int_0^\infty \varepsilon(\tau)p(\tau)e^{-\lambda\tau}\, d\tau.$$

From the second equation of (18), we have

$$p(\tau) = \frac{i_1(\tau)}{i_1(0)}p(0) - i_1(\tau)z_0 \int_0^\tau \varepsilon(v)e^{\lambda v}\, dv.$$

Substituting the third equation of (18) into the equation above, we have

$$p(\tau) = \frac{i_1(\tau)}{i_1(0)}\left[S^* \int_0^\infty \beta_1(\tau)p(\tau)e^{-\lambda\tau}\, d\tau + \frac{i_1(0)}{S^*}x_0\right] - i_1(\tau)z_0 \int_0^\tau \varepsilon(v)e^{\lambda v}\, dv. \tag{19}$$

We simplify these notations by defining the functions

$$M_1(\lambda) := \int_0^\infty \beta_1(\tau)p(\tau)e^{-\lambda\tau}\, d\tau$$
$$M_2(\tau) := \int_0^\infty \varepsilon(\tau)p(\tau)e^{-\lambda\tau}\, d\tau,$$
$$N_1(\lambda) := \int_0^\infty i_1(\tau)\beta_1(\tau)e^{-\lambda\tau}\, d\tau$$
$$N_2(\lambda) := \int_0^\infty i_1(\tau)\beta_1(\tau)e^{-\lambda\tau}\left(\int_0^\tau \varepsilon(v)e^{\lambda v}\, dv\right) d\tau,$$
$$N_3(\lambda) := \int_0^\infty i_1(\tau)\varepsilon(\tau)e^{-\lambda\tau}\, d\tau$$
$$N_4(\lambda) := \int_0^\infty i_1(\tau)\varepsilon(\tau)e^{-\lambda\tau}\left(\int_0^\tau \varepsilon(v)e^{\lambda v}\, dv\right) d\tau.$$

Multiplying $p(\tau)$ by $\beta_1(\tau)e^{-\lambda\tau}$ and $\varepsilon(\tau)e^{-\lambda\tau}$, respectively, and then integrating from 0 to $\infty$ yields

(20)
$$M_1(\lambda) = \frac{i_1(0)}{i_1(0) - S^* N_1(\lambda)} \left[ \frac{N_1(\lambda)}{S^*} x_0 - N_2(\lambda) z_0 \right],$$
(21)
$$\begin{aligned} M_2(\lambda) &= \frac{N_3(\lambda)}{i_1(0)} \left[ S^* M_1(\lambda) + \frac{i_1(0)}{S^*} x_0 \right] - N_4(\lambda) z_0 \\ &= \frac{i_1(0) N_3(\lambda)}{S^*[i_1(0) - S^* N_1(\lambda)]} x_0 - \left[ \frac{N_2(\lambda) N_3(\lambda) S^*}{i_1(0) - S^* N_1(\lambda)} + N_4(\lambda) \right] z_0. \end{aligned}$$

Substituting (20) and (21) into the first and fourth equations of (18), we obtain the characteristic equation

(22)
$$\begin{aligned} &\left[ \lambda + \frac{bA}{S^*} + \frac{i_1(0) N_1(\lambda)}{i_1(0) - S^* N_1(\lambda)} \right] \left[ \lambda + I_2^* \left( \frac{N_2(\lambda) N_3(\lambda) S^*}{i_1(0) - S^* N_1(\lambda)} + N_4(\lambda) \right) \right] \\ &\qquad = I_2^* \left[ \frac{i_1(0) N_2(\lambda)}{i_1(0) - S^* N_1(\lambda)} - \beta_2 \right] \left[ \frac{i_1(0) N_3(\lambda)}{i_1(0) - S^* N_1(\lambda)} + \beta_2 S^* \right] \end{aligned}$$

and have the following result.

**Theorem 6.** *The positive equilibrium of* (2) *is locally asymptotically stable if all roots,* $\lambda$*, of the characteristic equation* (22) *have negative real part.*

**5. Constant mutation rates.** Because equations (22) and $f(I_2) = g(I_2)$ (here, $f(I_2)$ and $g(I_2)$ are defined in Section 4) are both transcendental equations, it is difficult to solve roots of $f(I_2) = g(I_2)$ and to determine whether the positive equilibrium is stable. To gain insight into the transmission dynamics of the disease governed by system (2), we consider the special case where all the associated rates are independent of the age of infection. For (2) we define these constant rates as $\beta_1(\tau) := \beta_1$, $m(\tau) := m$ and $\varepsilon(\tau) := \varepsilon$.

Notice that $I_1(t) = \int_0^\infty i_1(t,\tau)\, d\tau$. Integrating the equation for $i_1(t,\tau)$ in (2) with respect to $\tau$ and using the initial condition $i_1(t,0)$

reduces the system of PDEs to the system of ODEs,

$$
\begin{aligned}
S' &= bA - bS - (\beta_1 I_1 + \beta_2 I_2)S,\\
I_1' &= \beta_1 S I_1 - m I_1 - \varepsilon I_1 I_2,\\
I_2' &= \beta_2 S I_2 - n I_2 + \varepsilon I_1 I_2.\\
&\quad (m \geq b, n \geq b)
\end{aligned} \tag{23}
$$

Adding the three equations in (23) gives

$$(S + I_1 + I_2)' = bA - bS - mI_1 - nI_2 \leq b[A - (S + I_1 + I_2)],$$

then the region $\Omega = \{(S, I_1, I_2) : S > 0, I_1 \geq 0, I_2 \geq 0, S + I_1 + I_2 \leq A\}$ is a positively invariant set of (23). It is easy to see that the reproduction numbers of two diseases for (23) are $R_1 = (\beta_1 A)/m$ and $R_2 = (\beta_2 A)/n$, respectively.

Correspondingly, for (23) $k$ and $k'$ become

$$k = \frac{\varepsilon}{m} \quad \text{and} \quad k' = \frac{R_1}{R_2} \frac{1}{1 + (bA\varepsilon/mn)(1 - (1/R_2))},$$

respectively.

**5.1. Boundary equilibrium of model (23).** According to Theorems 1 and 2 in Section 3, for (23) we have

**Theorem 7.** *Let*

$$\varepsilon_0 = \frac{mn}{bA}(R_1 - R_2), \qquad \varepsilon_1 = \frac{\varepsilon_0}{R_1 - 1}, \qquad \varepsilon_2 = \frac{\varepsilon_0}{R_2 - 1}.$$

*System* (23) *always has the disease-free equilibrium* $E_0(A, 0, 0)$, *which is asymptotically stable if* $\max\{R_1, R_2\} < 1$ *and unstable if* $\min\{R_1, R_2\} > 1$. *When* $R_1 > 1$, (23) *has the boundary equilibrium* $E_1(A/R_1, bA(1 - 1/R_1)/m, 0)$, *which is asymptotically stable if* $\varepsilon < \varepsilon_1$ *and unstable if* $\varepsilon > \varepsilon_1$. *When* $R_2 > 1$, (23) *has the boundary equilibrium* $E_2(A/R_2, 0, bA(1 - 1/R_2)/n)$, *which is asymptotically stable if* $\varepsilon > \varepsilon_2$ *and unstable if* $\varepsilon < \varepsilon_2$.

For Theorem 7, the existence of boundary equilibria can also be obtained by direct calculation; their stabilities are also proved by linearizing (23) about them. So the proof is omitted.

With respect to the global stability of boundary equilibrium we have the following results.

**Theorem 8.** (1) *When* $\max\{R_1, R_2\} < 1$, *the boundary equilibrium* $E_0$ *is globally asymptotically stable in* $\Omega$.

(2) *When* $R_1 > 1$ *and* $\varepsilon \le \varepsilon_1$, *the boundary equilibrium* $E_1$ *is globally asymptotically stable in* $\Omega$.

(3) *When* $R_2 > 1$ *and* $\varepsilon \ge \varepsilon_2$, *the boundary equilibrium* $E_2$ *is globally asymptotically stable in* $\Omega$.

*Proof.* (1) Define function $V_1 = I_1 + I_2$. Then differentiating $V_1$ along solutions of (23) gives

$$\begin{aligned} V_1'|_{(23)} &= I_1(\beta_1 S - m) + I_2(\beta_2 S - n) \\ &\le m(R_1 - 1)I_1 + n(R_2 - 1)I_2 \le \rho V_1, \end{aligned}$$

where $\rho = \max\{m(R_1 - 1), n(R_2 - 1)\}$. It follows that $V_1(t) \le V_1(0)e^{\rho t}$ where $V_1(0) = I_1(0) + I_2(0)$. Therefore, we have $\lim_{t\to\infty} V_1(t) = 0$ since $\rho < 0$ for $\max\{R_1, R_2\} < 1$. According to Theorem 7, the boundary equilibrium $E_0$ is globally asymptotically stable in $\Omega$ as $\max\{R_1, R_2\} < 1$.

(2) For boundary equilibrium $E_1(S^{(1)}, I_1^{(1)}, 0) = (A/R_1, bA(1 - 1/R_1)/m, 0)$, we consider the Lyapunov function

$$V_2 = \left(S - S^{(1)} - S^{(1)} \ln \frac{S}{S^{(1)}}\right) + \left(I_1 - I_1^{(1)} - I_1^{(1)} \ln \frac{I_1}{I_1^{(1)}}\right) + I_2.$$

Notice that

$$\begin{aligned} S' &= bA - bS - (\beta_1 I_1 + \beta_2 I_2)S \\ &= b(S^{(1)} - S) + \beta_1[I_1^{(1)}(S^{(1)} - S) + S(I_1^{(1)} - I_1)] - \beta_2 I_2 S, \\ I_1' &= I_1(\beta_1 S - m - \varepsilon I_2) = I_1[\beta_1(S - S^{(1)}) - \varepsilon I_2], \end{aligned}$$

and

$$\begin{aligned}I_2' &= I_2(\beta_2 S - n + \varepsilon I_1)\\ &= I_2[\beta_2(S - S^{(1)}) + \varepsilon(I_1 - I_1^{(1)}) + \beta_2 S^{(1)} - n + \varepsilon I_1^{(1)}]\\ &= I_2\left[\beta_2(S - S^{(1)}) + \varepsilon(I_1 - I_1^{(1)}) + n\left(\frac{R_2}{R_1} - 1\right) + \frac{\varepsilon bA}{m}\left(1 - \frac{1}{R_1}\right)\right]\\ &= I_2\left[\beta_2(S - S^{(1)}) + \varepsilon(I_1 - I_1^{(1)}) + \frac{bA}{m}\frac{R_1 - 1}{R_1}(\varepsilon - \varepsilon_1)\right].\end{aligned}$$

Then differentiating $V_2$ with respect to time yields

$$\begin{aligned}V_2'|_{(23)} &= \frac{S - S^{(1)}}{S}S' + \frac{I_1 - I_1^{(1)}}{I_1}I_1' + I_2'\\ &= -\frac{b + \beta_1 I_1^{(1)}}{S}(S - S^{(1)})^2 + \frac{bA}{m}\frac{R_1 - 1}{R_1}(\varepsilon - \varepsilon_1)I_2.\end{aligned}$$

Thus, when $R_1 > 1$ and $\varepsilon > \varepsilon_1$, $V_2'|_{(23)}$ is less than or equal to zero with equality only if $S = S^{(1)}$ and $I_2 = 0$; when $R_1 > 1$ and $\varepsilon = \varepsilon_1$, $V_2'|_{(23)}$ is less than or equal to zero with equality only if $S = S^{(1)}$. The invariant set of (23) on set $\{(S, I_1, I_2) \in \Omega : S = S^{(1)}, I_2 = 0\}$ or $\{(S, I_1, I_2) \in \Omega : S = S^{(1)}\}$ is the singleton $\{E_1\}$. It follows from LaSalle's invariance principle [**11**] that the boundary equilibrium $E_1$ is globally asymptotically stable in $\Omega$ as $R_1 > 1$ and $\varepsilon \le \varepsilon_1$.

(3) For boundary equilibrium $E_2(S^{(2)}, 0, I_2^{(2)}) = ((A/R_2), 0, bA\,(1 - (1/R_2)))/n$, consider the Lyapunov function

$$V_3 = \left(S - S^{(2)} - S^{(2)}\ln\frac{S}{S^{(2)}}\right) + I_1 + \left(I_2 - I_2^{(2)} - I_2^{(2)}\ln\frac{I_2}{I_2^{(2)}}\right).$$

Similar to the inference in (2), we can also obtain

$$V_3'|_{(23)} = -\frac{b + \beta_2 I_2^{(2)}}{S}(S - S^{(2)})^2 + \frac{bA}{n}\frac{R_2 - 1}{R_2}(\varepsilon_2 - \varepsilon)I_1,$$

and know that the boundary equilibrium $E_2$ is globally asymptotically stable in $\Omega$ when $R_2 > 1$ and $\varepsilon \ge \varepsilon_2$. □

**5.2. Positive equilibrium of model (23).** The positive equilibrium $E^*(S^*, I_1^*, I_2^*)$ of (23) satisfies equations

$$\begin{aligned} &bA - bS - (\beta_1 I_1 + \beta_2 I_2)S = 0, \\ &\beta_1 S - m - \varepsilon I_2 = 0, \\ &\beta_2 S - n + \varepsilon I_1 = 0. \end{aligned} \tag{24}$$

From the last two equations of (24) we obtain

$$\varepsilon(\beta_1 I_1^* + \beta_2 I_2^*) = \beta_1 n - \beta_2 m = \frac{mn}{A}(R_1 - R_2) > 0;$$

then the necessary condition for the existence of the positive equilibrium is $R_1 > R_2$.

Solving equations (24) gives

$$S^* = \frac{\varepsilon A}{\varepsilon + \varepsilon_0}, \qquad I_1^* = \frac{m}{\varepsilon}\left(\frac{R_1\varepsilon}{\varepsilon + \varepsilon_0} - 1\right), \qquad I_2^* = \frac{n}{\varepsilon}\left(1 - \frac{R_2\varepsilon}{\varepsilon + \varepsilon_0}\right).$$

Under the condition $R_1 > R_2$, $I_1^* > 0$ is equivalent to $\varepsilon(R_1 - 1) > \varepsilon_0$, i.e., $\varepsilon/\varepsilon_1 > 1$, and $I_2^* > 0$ is equivalent to $\varepsilon(R_2 - 1) < \varepsilon_0$, i.e., $\varepsilon/\varepsilon_2 < 1$. Notice that $\varepsilon_2 < 0$ for $R_2 < 1$ and that $I_1^* > 0$ implies that $R_1 > 1$; then, with respect to the existence of the positive equilibrium $E^*$, we have

**Theorem 9.** *System* (23) *has a unique positive equilibrium* $E^*$ *if one of the following cases holds.*

(1) $R_1 > R_2 \geq 1$ *and* $\varepsilon_1 < \varepsilon < \varepsilon_2$.

(2) $R_1 > 1 > R_2$ *and* $\varepsilon > \varepsilon_1$.

*Remark* 10. Conditions (1) and (2) in Theorem 9 correspond to conditions (1) and (3) in Theorem 4, respectively. But condition 2 in Theorem 4 is impossible for the case where mutation rates are constant. In fact, under this case, $k < (n/bA)(R_1 - R_2)/(R_1 - 1)$ is equivalent to $\varepsilon < \varepsilon_1$, $k' < 1$ is equivalent to $\varepsilon > \varepsilon_2$ and $\varepsilon_1 < \varepsilon_2$ for $R_1 > R_2 > 1$, so there is no value of $\varepsilon$ which satisfies $k < (n/bA)(R_1 - R_2)/(R_1 - 1)$ and $k' < 1$.

**Theorem 11.** *For system* (23), *the positive equilibrium* $E^*$, *if it exists, is globally asymptotically stable in* $\Omega$.

*Proof.* Define the Lyapunov function

$$V = \left(S - S^* - S^* \ln \frac{S}{S^*}\right) + \left(I_1 - I_1^* - I_1^* \ln \frac{I_1}{I_1^*}\right) + \left(I_2 - I_2^* - I_2^* \ln \frac{I_2}{I_2^*}\right).$$

From (23) we have

$$\begin{aligned} S' &= bA - bS - (\beta_1 I_1 + \beta_2 I_2)S \\ &= (S^* - S)(b + \beta_1 I_1^* + \beta_2 I_2^*) + \beta_1(I_1^* - I_1)S + \beta_2(I_2^* - I_2)S, \end{aligned}$$

$$\begin{aligned} I_1' &= I_1(\beta_1 S - m - \varepsilon I_2) \\ &= I_1[\beta_1(S - S^*) - \varepsilon(I_2 - I_2^*)], \end{aligned}$$

and

$$\begin{aligned} I_2' &= I_2(\beta_2 S - n + \varepsilon I_1) \\ &= I_2[\beta_2(S - S^*) + \varepsilon(I_1 - I_1^*)], \end{aligned}$$

then differentiating $V$ along solutions of (23) gives

$$\begin{aligned} V'|_{(23)} &= \frac{S - S^*}{S} S' + \frac{I_1 - I_1^*}{I_1} I_1' + I_2' \\ &= -\frac{(S - S^*)^2}{S}(b + \beta_1 I_1^* + I_2^*) \\ &= -\frac{(S - S^*)^2}{S}\frac{bA}{S^*}. \end{aligned}$$

Thus, $V'|_{(23)}$ is less than or equal to zero with equality only if $S = S^*$. The invariant set of (23) on set $\{(S, I_1, I_2) \in \Omega : S = S^*\}$ is the singleton $\{E^*\}$. It follows from LaSalle's invariance principle [**20**] that the positive equilibrium $E^*$ is globally asymptotically stable in $\Omega$ if it exists. □

The results obtained for (23) are summarized in Table 1.

TABLE 1.

| | | $E_0$ | $E_1$ | $E_2$ | $E^*$ |
|---|---|---|---|---|---|
| $R_1 < 1$, $R_2 < 1$ | | GAS | N | N | N |
| $R_1 < 1 < R_2$ | | US | N | GAS | N |
| $1 < R_1 \leq R_2$ | | US | US | GAS | N |
| $1 \leq R_2 < R_1$ | $\varepsilon \leq \varepsilon_1$ | US | GAS | US | N |
| | $\varepsilon_1 < \varepsilon < \varepsilon_2$ | US | US | US | GAS |
| | $\varepsilon \geq \varepsilon_2$ | US | US | GAS | N |
| $R_2 < 1 < R_1$ | $\varepsilon \leq \varepsilon_1$ | US | GAS | N | N |
| | $\varepsilon > \varepsilon_1$ | US | US | N | GAS |

In Table 1, N denotes *does not exist*, US denotes *unstable*, GAS denotes *globally stable*, where $\varepsilon_1$ and $\varepsilon_2$ are defined as in Theorem 7.

**6. Discussion.** In this paper, we have formulated a simple epidemiological model with two diseases that can coinfect a single host. The first disease was assumed to be chronic, the second one acute. For infectives infected only by the first disease, we introduce the age of infection. For these two diseases, we obtained their reproduction numbers, respectively, and established conditions for the existence and stability of the disease-free equilibrium, the boundary equilibrium and the positive (coexistent) equilibrium.

For the infectious individuals infected only by the first disease, when the incidence rates are $\beta_1(\tau)$ and $\varepsilon(\tau)$ and the disease-related death rate $\alpha_1(\tau)$ and the recovery rate $\gamma_1(\tau)$ depend on the age of infection, the corresponding models are governed by partial differential equations (PDEs), we gave a sufficient condition for the existence of positive equilibrium, its stability determined by a transcendental equation; when all the associated rates are independent of the age of infection, the corresponding models are ordinary differential equations (ODEs), we obtained complete results on the dynamics and found that the coexistent equilibrium of two diseases is globally stable if it exists, and that the boundary equilibrium is globally stable if it is locally stable.

Also, we find that there is a difference between PDE and ODE models. This difference is shown in Remarks 5 and 10. It is possible for a model with the age of infection that the boundary equilibria are bistable, but the bistable case is impossible for a model without the age of infection. In fact, for the PDE model, we can choose parameters as follows: $\beta_2 = 16.6$, $b = 2.0$, $m = 2.8$, $n = 2.6$, $\varepsilon = 0.6$, $A = 1$ and

$$\beta_1(\tau) = \begin{cases} 72.1(1.5-\tau)\tau & 0 \le \tau \le 1.5, \\ 0 & \text{elsewhere;} \end{cases}$$

then, $R_1 = 7.5311$, $R_2 = 6.3846$, $k' = 0.9889$, $k(bA(R_1-1)/n(R_1-R_2)) = 0.9249$, that is, Condition 2 in Theorem 4 is true. Further, besides the disease-free equilibrium $P_0$, the other equilibria are $E_1(0.1328, 1.7344e^{-2.8\tau}, 0)$, $E_2(0.1566, 0, 0.6487)$, $E^*(0.1419, 1.1982 \cdot e^{-2.932\tau}, 0.22)$, respectively, and the boundary equilibria $E_1$ and $E_2$ both are stable. But, for the ODE model, the case for two boundary equilibria to be bistable is impossible. Therefore, incorporating the age of infection into the epidemic model with chronic disease is necessary for modeling the disease spread.

## REFERENCES

**1.** L.J.S. Allen, M. Langlais and C.J. Phillips, *The dynamics of two viral infection in a single host population with applications to hantavirus*, Math. Biosci. **186** (2003), 191–217.

**2.** C. Castillo-Chavez and H.R. Thieme, *Asymptotically autonomous epidemic models*, in *Mathematical population dynamics*: *Analysis of heterogeneity*, Vol. 1, *Theory of epidemics*, O. Arino, D. Axelrod, M. Kimmel and M. Langlais, eds., Wuerz, Winnipeg, 1993.

**3.** F. Courchamp, C. Suppo, E. Fromont and C. Bouloux, *Dynamics of two feline retroviruses* (*FIV and FeLV*) *within one population of cats*, Proc. Roy. Soc. London **264** (1997), 785–794.

**4.** Z. Feng, C. Castillo-Chavez and A. Capurro, *A model for* TB *with exogenous reinfection*, Theoret. Pop. Biol. **57** (2000), 235–247.

**5.** Z. Feng, W. Huang and C. Castillo-Chavez, *On the role of variable latent periods in mathematical models for tuberculosis*, J. Dynam. Differ. Equations **13** (2001), 425–452.

**6.** Z. Feng, M. Iannelli and F.A. Milner, *A two-strain tuberculosis model with age of infection*, SIAM J. Appl. Math. **62** (2002), 1634–1656.

**7.** A.B. Gumel, S.M. Moghadas, Y. Yuan and P. Yu, *Bifurcation and stability analyses of a* 13-D SEIC *model using normal form reduction and numerical simulation*, Dynam. Continuous Discrete Impulsive Syst. Appl. Algorithms **10** (2003), 317–330.

**8.** C.C. Hung and S.C. Chang, *Impact of highly active antiretroviral therapy on incidence and management of human immunodeficiency virus-related opportunistic infections*, J. Antimicrobial Chemotherapy **54** (2004), 849–853.

**9.** J.M. Hyman, J. Li and E.A. Stanley, *The differential infectivity and staged progression models for the transmission of* HIV, Math. Biosci. **155** (1999), 77–109.

**10.** D. Kirschner, *Dynamics of co-infection with M. tuberculosis and* HIV-1, Theoret. Population Biol. **55** (1999), 94–109.

**11.** J.P. LaSalle, *The stability of dynamical systems*, Society Industrial Applied Mathematics, Philadelphia, 1976.

**12.** M. Martcheva and C. Castillo-Chavez, *Diseases with chronic stage in a population with varying size*, Math. Biosci. **182** (2003), 1–25.

**13.** M. Martcheva and S.S. Pilyugin, *The role of coinfection in multi-disease dynamics*, SIAM J. Appl. Math. **66** (2006), 843–872.

**14.** R. May and M. Nowak, *Coinfection and the evolution of parasite virulence*, Proc. Roy. Soc. London **261** (1995), 209–215.

**15.** F. Milner and A. Pugliese, *Periodic solutions: a robust numerical method for an* S-I-R *model of epidemics*, J. Math. Biol. **39** (1999), 471–492.

**16.** J. Mosquera and F. Adler, *Evolution of virulence: a united framework for coinfection and superinfection*, J. Theoret. Biol. **195** (1998), 293–313.

**17.** T.R. O'Brien, W.A. Blattner, D. Waters, et al., *Serum* HIV-1 RNA *levels and time to development of* AIDS *in the multicenter hemophilia cohort study*, J. American Medical Association **276** (1996), 105–110.

**18.** H.R. Thieme and C. Castillo-Chavez, *How may infection-age-dependent infectivity affect the dynamics of* HIV/AIDS?, SIAM J. Appl. Math. **53** (1993), 1447–1479.

**19.** G. Webb, *Theory of Nonlinear age-dependent population dynamics*, Marcel Dekker, New York, 1985.

**20.** M.T. Wong, M.J. Dolan, E. Kozlow, et al., *Patterns of virus burden and* T *cell phenotype are established early and are correlated with the rate of disease progression in* HIV *type* 1 *infected persons*, J. Infectious Disease **173** (1996), 877–887.

Department of Applied Mathematics and Physics, Air Force Engineering University, Xi'an 710051, China
**Email address: jianq_li@263.net**

Department of Applied Mathematics, Shanxi University of Finance and Economics, Taiyuan 030006, China
**Email address: wld58@126.com**

Department of Applied Mathematics and Physics, Air Force Engineering University, Xi'an 710051, China
**Email address: zhaoh@yahoo.com**

Department of Applied Mathematics, Xi'an Jiaotong University, Xi'an 710049, China
**Email address: zhma@mail.xjtu.edu.cn**