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THE EFFECT OF CONSTANT AND MIXED IMPULSIVE VACCINATION ON SIS EPIDEMIC MODELS INCORPORATING MEDIA COVERAGE

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ABSTRACT. An SIS epidemic model incorporating media coverage is presented in this paper. Impulsive vaccination to susceptible individuals is considered. In general, impulsive vaccination is a proportional vaccination, but when the number of susceptible individuals is very large, the number of people who need to be vaccinated is also proportionally increasing, considering limited vaccination ability of an area. So we first investigate constant impulsive vaccination. Using the discrete dynamical system determined by the stroboscopic map, we obtain the exact periodic infection-free solution and show that it is globally asymptotically stable if some conditions are satisfied. After constant impulsive vaccination to a large number of susceptible individuals, the number of susceptible individuals will gradually decrease; if the number of susceptible individuals decreases below the above constant, we will not use the above vaccination strategy. In that case we will use a common proportional impulsive vaccination. So, we consider mixed impulsive vaccination, and we also obtain the exact periodic infection-free solution and show that it is globally asymptotically stable.

1. Introduction. Vaccination is a commonly used method for controlling disease: the study of vaccines against infectious disease has been a boon to mankind. Many authors have investigated constant vaccination and impulsive vaccination to susceptible individuals, see [**5, 6, 9, 14, 15**], and the difference between constant vaccination and impulsive vaccination has also been studied in [**14**], where impulsive vaccination is a proportional vaccination. However, when the number of susceptible individuals is very large, the number of people who need to be vaccinated is also proportionally increasing, considering limited

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vaccination ability in an area. So, we consider constant impulsive vaccination when the number of susceptible individuals is very large. After constant impulsive vaccination to a large number of susceptible individuals, the number of susceptible individuals will gradually decrease; if the number of susceptible individuals decreases below the above constant, the above constant impulsive vaccination is not reasonable. At this time, we consider proportional impulsive vaccination if the number of susceptible individuals is smaller than the constant.

In real life, many infectious diseases transmit through both horizontal and vertical ways. These include such human diseases as rubella, herpes simplex, hepatitis B, and AIDS, etc. For human and animal disease, horizontal transmission typically occurs through direct or indirect physical contact with infectious hosts, or through disease vectors such as mosquitos, ticks, or other biting insects. Vertical transmission can be accomplished through transplacental transfer of disease agents; [**1**–**3, 7, 8**] considered this phenomenon. In our paper, we assume a fraction of the offsprings of infected hosts are infected at birth; hence, the infected birth flux will enter class I and vaccine treatment is only taken to a proportion of newborns who haven't been infected at birth.

In classical epidemic models, the incidence rate is assumed to be a mass action incidence rate with bilinear interactions given by βSI, where β is the probability of transmission per contact, and S and I represent the susceptible and infected populations, respectively. However, some factors such as media coverage, manner of life and density of population, may affect the incidence rate directly or indirectly, nonlinear incidence rate can be approximated by a variety of forms, such as $\beta I^p S^q$, $\beta(1 - cI)IS$ $(c > 0)$, $(kI^l S)/(1 + \alpha I^h)$ $(k, l, \alpha, h > 0)$ which were discussed in [**11, 12, 16**], respectively. In this paper, we suggest a general nonlinear incidence rate $(\beta_1 - \beta_2 I/(m + I))SI$ $(\beta_1 > \beta_2 > 0, m > 0)$ which reflects some characters of media coverage, where $\beta_1 = pc_1$, $\beta_2 = pc_2$ (p is the transmission probability under contacts in unit time), c_1 is the usual contact rate, c_2 is the maximum reduced contact rate through actual media coverage, that is, β_1 is the usual valid contact rate, β_2 is the maximum reduced valid contact rate through actual media coverage and m is the rate of reflection on the disease. Again, media coverage cannot totally interrupt disease transmission, so we have $\beta_1 > \beta_2$. We use $\beta_2 I/(m + I)$ to reflect the amount of contact rate reduced through media coverage. When infective individuals appear in a region, people reduce their contact with others to avoid being infected, and the more infective individuals being reported the less contact with others; hence, we take the above form. Few studies have appeared on this aspect [**4, 13**].

The paper is arranged as follows. In Section 2, an SIS model incorporating media coverage and vertical transmission is given. We consider constant impulsive vaccination of an SIS model in Section 3. Using the discrete dynamical system determined by the stroboscopic map, we obtain the exact periodic infection-free solution and show that it is globally asymptotically stable if some conditions are satisfied. In Section 4, we consider mixed impulsive vaccination. We also obtain the exact periodic infection-free solution and show that it is globally asymptotically stable if some conditions are satisfied.

2. An SIS model incorporating media coverage.

In this section, we give an SIS model incorporating media coverage and we study a population which is composed of two groups of individuals who are susceptible and infected with sizes denoted by $S(t)$, $I(t)$, respectively. The sum $S(t)+I(t)$ is the total population of $N(t)$. The natural birth and death rates are assumed to be identical and denoted by $b>0$; we assume that a fraction p $(0\le p\le 1)$ of the offspring from the infectious class are born into the infected I, the offspring from the susceptible and vaccinated classes are all susceptible individuals, the infective individuals recover and reenter into the susceptible class with rate λ, the incidence rate is $(\beta_1-\beta_2 I/(m+I))(SI/N)$ and the definitions of β_1, β_2 and m are the same as in Section 1. Influenza is among the diseases for which our model is an approximation. Based on the above assumptions, we have the following SIS epidemic model:

$$\tag{2.1} \begin{cases} \dot{S}=bN-pbI+\lambda I-(\beta_1-\beta_2 I/(m+I))(SI/N)-bS,\\ \dot{I}=pbI+(\beta_1-\beta_2 I/(m+I))(SI/N)-(b+\lambda)I. \end{cases}$$

In general ways, we should discuss constant vaccination and impulsive vaccination on the SIS model. The models are the following:

$$\tag{2.2} \begin{cases} \dot{S}=(1-\alpha)(bN-pbI)+\lambda I+\theta V\\ \qquad -(\beta_1-\beta_2 I/(m+I))SI-(b+q)S,\\ \dot{I}=pbI+(\beta_1-\beta_2 I/(m+I))SI-(b+\lambda)I,\\ \dot{V}=\alpha(bN-pbI)+qS-(b+\theta)V, \end{cases}$$

and

(2.3)
$$\begin{cases} \left.\begin{aligned} &\dot{S} = (1-\alpha)(bN - pbI) + \lambda I + \theta V - (\beta_1 - \beta_2 I/(m+I))(SI/N) - bS, \\ &\dot{I} = pbI + (\beta_1 - \beta_2 I/(m+I))(SI/N) - (b+\lambda)I, \\ &\dot{V} = \alpha(bN - pbI) - (b+\theta)V, \end{aligned}\right\} t \neq n\tau, \\ \left.\begin{aligned} &S(n\tau^+) = (1-q)S(n\tau^-), \\ &I(n\tau^+) = I(n\tau^-), \\ &V(n\tau^+) = V(n\tau^-) + qS(n\tau^-), \end{aligned}\right\} t = n\tau, \end{cases}$$

where a fraction $\alpha \in [0,1]$ of newborns are vaccinated at birth and the susceptible class is vaccinated with rate q. From models (2.2) and (2.3), we have obtained some interesting results.

3. Constant impulsive vaccination. In general, we should discuss proportional impulsive vaccination to susceptible individuals, but when the number of susceptible individuals is very large, the number of people who need to vaccinate is also proportionally increasing. In fact, the ability to vaccinate is limited in an area and the number of vaccinations can't always increase with the increasing number of susceptible individuals, but it can be controlled by the special vaccination ability of the area's medical personnel and other objective factors. According to this, we will modify the proportional impulsive vaccination to a constant impulsive vaccination, and this constant should not be more than the maximum vaccination number which vaccination personnel can accept.

In this section, we investigate the effect of constant impulsive vaccination to control epidemic disease. Based on (2.2), subject to the restriction $S(t) + I(t) + V(t) = N(t)$, without loss of generality, we let $N(t) = 1$. Then we only need to consider the following model:

$$(3.1) \qquad \begin{cases} \left.\begin{aligned} &\dot{I} = pbI + (\beta_1 - \beta_2 I/(m+I))(1 - I - V)I - (b+\lambda)I, \\ &\dot{V} = \alpha(b - pbI) - (b+\theta)V, \end{aligned}\right\} t \neq n\tau, \\ \left.\begin{aligned} &I(t^+) = I(t^-), \\ &V(t^+) = V(t^-) + h, \end{aligned}\right\} t = n\tau, \\ I(0^+) \geq 0, V(0^+) \geq 0, \end{cases}$$

where h is the amount of constant impulsive vaccination; obviously, we have $0 < h \leq 1$.

Lemma 3.1. *Suppose $x(t) = (I(t), V(t))$ is the solution of system (3.1) with initial values $I(0^+) \geq 0$ and $V(0^+) \geq 0$. Then $x(t) \geq 0$, that is, $I(t) \geq 0$ and $V(t) \geq 0$; and if $I(0^+) > 0$ and $V(0^+) > 0$, then $x(t) > 0$ holds for all $t \geq 0$.*

Proof. From (3.1), we have

$$\dot{I}|_{I=0} = 0, \qquad \dot{V}|_{I>0,V=0} = \alpha b(1 - pI) > 0,$$

and $V(t^+) > V(t)$, so the results are obviously true. The proof is complete. □

Theorem 3.1. *If $pb - \lambda + \theta > 0$, then the system (3.1) is uniformly ultimately bounded, that is, there exists a constant $M > 0$, such that for any solution $x(t) = (I(t), V(t))$, we have $I(t) < M$ and $V(t) < M$ when t large enough.*

Proof. Let $\beta = \alpha pb/(pb - \lambda + \theta)$ and $L(t) = \beta I(t) + V(t)$. We can calculate $D^+L(t)$ along the lines of system (3.1),

$$D^+L(t) \leq \beta\beta_1(1 - I)I - (b + \theta)L(t) + b\alpha,$$

when $I > 0$; since $\beta\beta_1(1 - I)I$ can obtain the maximum at $I = 1/2$, we have

$$\begin{cases} D^+L(t) \leq (1/4)\beta\beta_1 - (b + \theta)L(t) + b\alpha & t \neq n\tau, \\ L(t^+) \leq L(t) + h & t = n\tau. \end{cases}$$

From Theorem 1.4.1 of [**10**], we obtain

$$L(t) \leq \left(L(0^+) + \frac{(1/4)\beta\beta_1 + b\alpha}{b + \theta}\right)e^{-(b+\theta)t} + h\left(\frac{1 - e^{-(b+\theta)n\tau}}{1 - e^{-(b+\theta)\tau}}\right)e^{-(b+\theta)(t-n\tau)} + \frac{(1/4)\beta\beta_1 + b\alpha}{b + \theta},$$

for $t \in (n\tau, (n+1)\tau]$. Obviously, $L(t)$ is uniformly ultimately bounded, that is, there exists an $M > 0$, such that when t is large enough we have $L(t) < M$, i.e., system (3.1) is uniformly ultimately bounded. The proof is complete. □

In the following, we consider the subsystem of system (3.1),

$$\begin{cases} \dot{V} = b\alpha - (b+\theta)V & t \neq n\tau, \\ V(t^+) = V(t^-) + h & t = n\tau. \end{cases} \tag{3.2}$$

This subsystem is obtained by letting $I = 0$ in the system (3.1), that is, this subsystem represents infection-free system of system (3.1).

Regarding system (3.2), we have the following lemma.

Lemma 3.2. *System* (3.2) *has a globally attractive periodic solution* $\overline{V}(t) = b\alpha/(b+\theta) + he^{-(b+\theta)(t-n\tau)}/(1-e^{-(b+\theta)\tau})$, $t \in (n\tau, (n+1)\tau]$.

Proof. Suppose $V(t)$ is any solution of system (3.2) and $\widetilde{V}(t)$ is a periodic solution of system (3.2). From [**10**, Theorem 1.4.1], we have

$$V(t) = \frac{b\alpha}{b+\theta} + \left(V(0^+) - \frac{h}{1-e^{-(b+\theta)\tau}} - \frac{b\alpha}{b+\theta}\right) e^{-(b+\theta)t} + \frac{he^{-(b+\theta)(t-n\tau)}}{1-e^{-(b+\theta)\tau}}, \tag{3.3}$$

$t \in [n\tau, (n+1)\tau)$, where $V(0^+) = b\alpha/(b+\theta) + h/(1-e^{-(b+\theta)\tau})$. In the time interval $n\tau < t \leq (n+1)\tau$, system (3.2) has the following solution:

$$V(t) = \frac{b\alpha}{b+\theta} + \left(V(n\tau) - \frac{b\alpha}{b+\theta}\right) \exp(-(b+\theta)(t-n\tau)), \quad n\tau < t \leq (n+1)\tau. \tag{3.4}$$

Using the second equation of system (3.2), we deduce the stroboscopic map such that

$$V((n+1)\tau) = \frac{b\alpha}{b+\theta} + \left(V(n\tau) - \frac{b\alpha}{b+\theta}\right) \exp(-(b+\theta)\tau) \triangleq g(V(n\tau)).$$

The map g has a unique positive fixed point:

$$V^* = \frac{b\alpha}{b+\theta} + \frac{h}{1-\exp(-(b+\theta)\tau)}.$$

The fixed point V^* implies that there is a corresponding cycle of period τ in the vaccination population. So we obtain the complete expression for the periodic infection-free solution over the nth time interval $n\tau < t \le (n+1)\tau$:

$$\widetilde{V}(t) = \frac{b\alpha}{b+\theta} + \frac{he^{-(b+\theta)(t-n\tau)}}{1-e^{-(b+\theta)\tau}}, \quad t \in (n\tau, (n+1)\tau],$$

so we have

$$V(t) = \left(V(0^+) - \frac{h}{1-e^{-(b+\theta)\tau}} - \frac{b\alpha}{b+\theta}\right)e^{-(b+\theta)t} + \widetilde{V}(t)$$

and $\lim_{t\to\infty} |V(t) - \widetilde{V}(t)| = 0$. The proof is complete. □

Theorem 3.2. *The solution* $(0, \widetilde{V}(t))$ *of system* (3.1) *is an infection-free periodic solution and it is locally asymptotically stable if* $(pb - b - \lambda + \beta_1 - \beta_1 b\alpha/(b+\theta))\tau < \beta_1 h/(b+\theta)$.

Proof. We let $x(t) = I(t)$ and $y(t) = V(t) - \widetilde{V}(t)$. Then system (3.1) can be reduced to

$$\begin{cases} \left.\begin{aligned} \dot{x} &= (pb - b - \lambda)x + (\beta_1 - \beta_2 x/(m+x))(1 - x - y - \widetilde{V}(t))x, \\ \dot{y} &= -pb\alpha x - (b+\theta)y, \end{aligned}\right\} & t \ne n\tau, \\ \left.\begin{aligned} x(t^+) &= x(t^-), \\ y(t^+) &= y(t^-), \end{aligned}\right\} & t = n\tau, \end{cases}$$

its zero solution corresponding to the infection-free periodic solution $(0, \widetilde{V}(t))$ of system (3.1) and its linear system at zero is

$$\begin{cases} \left.\begin{aligned} \dot{x} &= (pb - b - \lambda + \beta_1 - \beta_1 \widetilde{V}(t))x, \\ \dot{y} &= -pb\alpha x - (b+\theta)y, \end{aligned}\right\} & t \ne n\tau, \\ \left.\begin{aligned} x(t^+) &= x(t^-), \\ y(t^+) &= y(t^-), \end{aligned}\right\} & t = n\tau. \end{cases}$$

Using Floquet theory, we can calculate its zero solution to be asymptotically stable if the condition of theorem is satisfied. The proof is complete. □

Theorem 3.3. *If* $(pb - b - \lambda + \beta_1)\tau < \beta_1 h/(b + \theta)$*, for any solution* $x(t) = (I(t), V(t))$ *of system* (3.1) *with positive initial values, we have* $\lim_{t\to\infty} I(t) = 0$ *and* $\lim_{t\to\infty} |V(t) - \widetilde{V}(t)| = 0$.

Proof. Suppose $x(t) = (I(t), V(t))$ is any solution of system (3.1) with initial values $I(0^+) > 0, V(0^+) > 0$. From system (3.1), we have $\dot{V} \geq -(b + \theta)V$. Then

$$\begin{cases} \dot{V} \geq -(b+\theta)V & t \neq n\tau, \\ V(t^+) = V(t^-) + h & t = n\tau, \end{cases}$$

and assume that $V(t)$ is any solution of the above system. We can choose a sufficiently small $\varepsilon > 0$ such that

$$\sigma = e^{(pb-b-\lambda+\beta_1+\beta_1\varepsilon)\tau - \beta_1 h/(b+\theta)} < 1.$$

We consider the following impulsive differential equation:

$$\begin{cases} \dot{u} = -(b+\theta)u & t \neq n\tau, \\ u(t^+) = u(t^-) + h & t = n\tau. \end{cases}$$

From Lemma 3.2 and the comparison theorem of the impulsive differential equation, we have $\lim_{t\to\infty} |u(t) - \widetilde{u}(t)| = 0$ and $V(t) \geq u(t)$, where $\widetilde{u}(t) = he^{-(b+\theta)(t-n\tau)}/(1 - e^{-(b+\theta)\tau})$, $t \in (n\tau, (n+1)\tau]$. Without loss of generality, we assume that $V(t) \geq u(t) > \widetilde{u}(t) - \varepsilon$ for any $t \geq 0$. From system (3.1), we obtain

$$\dot{I} \leq I(pb - b - \lambda + \beta_1 - \beta_1\widetilde{u}(t) + \beta_1\varepsilon)$$

and integrate it on $(n\tau, (n+1)\tau]$. Using the comparison theorem, we have

$$I((n+1)\tau) \leq I(n\tau)e^{(pb-b-\lambda+\beta_1+\beta_1\varepsilon)\tau - \beta_1\int_{n\tau}^{(n+1)\tau}\widetilde{u}(t)dt} = I(n\tau)\sigma,$$

and we have $I(n\tau) \leq I(0^+)\sigma^n$, so $\lim_{t\to\infty} I(n\tau) = 0$. On the other hand, when $t \neq n\tau$, we have $\dot{I} \leq (pb + \beta_1)I$, so

$$I(t) \leq I(n\tau)e^{(pb+\beta_1)(t-n\tau)} \leq I(n\tau)e^{(pb+\beta_1)\tau}, \quad t \in (n\tau, (n+1)\tau].$$

From the above discussion, we have $\lim_{t\to\infty} I(t) = 0$.

In the following we will prove $\lim_{t\to\infty}|V(t)-\widetilde{V}(t)|=0$. Since $\lim_{t\to\infty}I(t)=0$, we have for any sufficiently small $\varepsilon_1>0$ that there exists a $t_1>0$ such that $I(t)<\varepsilon_1$ for $t>t_1$. Then we have

$$b\alpha - pb\alpha\varepsilon_1 - (b+\theta)V(t) < \dot{V} < b\alpha - (b+\theta)V,$$

for $t\in(n\tau,(n+1)\tau]$ and $t>t_1$.

Consider the following two systems:

$$\begin{cases} \dot{V}_1 = b\alpha - (b+\theta)V_1 & t\neq n\tau,\\ V_1(t^+) = V_1(t^-)+h & t=n\tau,\\ V_1(0^+) = V_1(0^-), \end{cases} \tag{3.3}$$

and

$$\begin{cases} \dot{V}_2 = b\alpha - pb\alpha\varepsilon_1 - (b+\theta)V_2 & t\neq n\tau,\\ V_2(t^+) = V_2(t^-)+h & t=n\tau,\\ V_2(0^+) = V_2(0^-). \end{cases} \tag{3.4}$$

From Lemma 3.2, the solution $V_1(t)$ of system (3.3) satisfies $\lim_{t\to\infty}|V_1(t)-\widetilde{V}(t)|=0$ and the solution $V_2(t)$ of system (3.4) is

$$\begin{aligned} V_2(t) = {} & \left(V_2(0^+) - \frac{h}{1-e^{-(b+\theta)\tau}} - \frac{b\alpha - pb\alpha\varepsilon_1}{b+\theta}\right)e^{-(b+\theta)t} \\ & + \frac{b\alpha - pb\alpha\varepsilon_1}{b+\theta} + \frac{he^{-(b+\theta)(t-n\tau)}}{1-e^{-(b+\theta)\tau}}, \end{aligned}$$

so we have $\lim_{t\to\infty}|V_2(t)-\widetilde{V}(t)+pb\alpha\varepsilon_1/(b+\theta)|=0$.

From the comparison theorem of the impulsive differential equation, we have $V_2(t)<V(t)<V_1(t)$.

Now according to the above discussion, we obtain for any sufficiently small $\varepsilon_2>0$ that there exists a $t_2>t_1$ such that

$$\widetilde{V}(t) - \varepsilon_2 - \frac{pb\alpha\varepsilon_1}{b+\theta} \le V_2(t) \le V(t) \le V_1(t) \le \widetilde{V}(t) + \varepsilon_2 + \frac{pb\alpha\varepsilon_1}{b+\theta},$$

for all $t>t_2$. From the arbitrary properties of ε_1 and ε_2, we have $\lim_{t\to\infty}|V(t)-\widetilde{V}(t)|=0$. The proof is complete. $\square$

From Theorems 3.2 and 3.3, we have the following theorem.

Theorem 3.4. *If the condition of Theorem* 3.3 *is satisfied, then the solution* $(0, \widetilde{V}(t))$ *of system* (3.1) *is globally asymptotically stable.*

Remark 3.1. From the point of view of biological meaning, we assume $S(t) > h$, $t \geq 0$, and do not consider the case: $S(t^-) < h$, $t = n\tau$, since $S(t^+) < 0$ after the impulsive effect in this case. But this does not agree with the natural meaning.

4. Mixed impulsive vaccination. In this section we will modify constant vaccination. Constant vaccination at fixed time not only has a good natural background but also has been well controlled. So we consider using constant impulsive vaccination strategy when the number of susceptible individuals is very large, while we consider using proportional impulsive vaccination when the number of susceptible individuals is small. Thus, we can ensure positivity of the system, and this is a very good utility. Considering $S(t) + I(t) + V(t) = N(t)$ and letting $N(t) = 1$, then the modified model is the following

$$(4.1)\qquad \begin{cases} \left.\begin{aligned} \dot{S} &= (1-\alpha)(b - pbI) + \lambda I + \theta(1 - S - I) \\ &\qquad - (\beta_1 - \beta_2 I/(m+I))SI - bS, \\ \dot{I} &= pbI + (\beta_1 - \beta_2 I/(m+I))SI - (b+\lambda)I, \end{aligned}\right\} t \neq n\tau, \\ \left.\begin{aligned} &S(t^+) = \begin{cases} 0 & S(t^-) < h, \\ S(t^-) - h & S(t^-) \geq h, \end{cases} \\ &I(t^+) = I(t^-), \end{aligned}\right\} t = n\tau, \end{cases}$$

where h is the amount of impulsive vaccination, initial value $S(0^+) \geq 0$ and $I(0^+) \geq 0$; when the number of susceptible individuals is less than this constant h, we take the proportional impulsive vaccination $\Delta S(t) = -S(t^-)$, that is, all susceptible individuals are vaccinated. This is because medical personnel in this area have the ability to vaccinate all susceptible individuals, but when the number of susceptible individuals is more than this constant h, considering the maximum vaccination ability is h, we take constant impulsive vaccination $\Delta S(t) = -h$.

Lemma 4.1. *Suppose $x(t) = (S(t), I(t))$ is the solution of system* (4.1) *with initial values $S(0^+) \geq 0$ and $I(0^+) \geq 0$. Then $x(t) \geq 0$, i.e., $S(t) \geq 0$ and $I(t) \geq 0$; and if $S(0^+) > 0$ and $I(0^+) > 0$, then $x(t) > 0$ for all $t \geq 0$.*

The proof of Lemma 4.1 is similar to Lemma 3.1, so we omit it.

Obviously, system (4.1) has an invariant set $\{(S, I) | S \geq 0, I > 0, S + I \leq 1\}$.

Consider the two following impulsive systems:

$$\begin{cases} \dot{S} = (1-\alpha)b + \theta - (b+\theta)S & t \neq n\tau, \\ S(t^+) = 0 & t = n\tau, \end{cases} \tag{4.2}$$

and

$$\begin{cases} \dot{S} = (1-\alpha)b + \theta - (b+\theta)S & t \neq n\tau, \\ S(t^+) = S(t^-) - h & t = n\tau. \end{cases} \tag{4.3}$$

Regarding these two systems, we have the following lemmas.

Lemma 4.2. *System* (4.2) *has a globally asymptotically stable τ-periodic solution*:

$$\widetilde{S}_r(t) = \frac{(1-\alpha)b+\theta}{b+\theta} + \left(S_r^0 - \frac{(1-\alpha)b+\theta}{b+\theta}\right)e^{-(b+\theta)(t-n\tau)}, \quad t \in (n\tau, (n+1)\tau],$$

where $S_r^0 = 0$, that is,

$$\widetilde{S}_r(t) = \frac{(1-\alpha)b+\theta}{b+\theta}\left(1 - e^{-(b+\theta)(t-n\tau)}\right), \quad t \in (n\tau, (n+1)\tau].$$

Lemma 4.3. *System* (4.3) *has a globally asymptotically stable τ-periodic solution*:

$$\widetilde{S}_c(t) = \frac{(1-\alpha)b+\theta}{b+\theta} + \left(S_c^0 - \frac{(1-\alpha)b+\theta}{b+\theta}\right)e^{-(b+\theta)(t-n\tau)}, \quad t \in (n\tau, (n+1)\tau],$$

where $S_c^0 = ((1-\alpha)b+\theta/(b+\theta) - h/(1-e^{-(b+\theta)\tau})$, that is,

$$\widetilde{S}_c(t) = \frac{(1-\alpha)b+\theta}{b+\theta} - \frac{he^{-(b+\theta)(t-n\tau)}}{1-e^{-(b+\theta)\tau}}, \quad t \in (n\tau, (n+1)\tau].$$

Specifically, $\widetilde{S}_c(t)$ will be a nonnegative periodic solution if $(((1-\alpha)b + \theta)/(b+\theta))(1-e^{-(b+\theta)\tau}) \geq h$.

The proof of Lemmas 4.2 and 4.3 is similar to Lemma 3.2. Here we omit it.

If $I = 0$ in the system (4.1), we have the infection-free subsystem:

$$\text{(4.4)} \qquad \left\{ \begin{array}{l} \dot{S} = (1-\alpha)b + \theta - (b+\theta)S, \quad t \neq n\tau, \\ S(t^+) = \begin{cases} 0 & S(t^-) < h, \\ S(t^-) - h & S(t^-) \geq h, \end{cases} \end{array} \right\} t = n\tau.$$

We have the following lemma about this system.

Lemma 4.4. *System* (4.4) *has a periodic solution*:

$$\text{(4.5)} \qquad \widetilde{S}(t) = \begin{cases} \widetilde{S}_r(t) & (((1-\alpha)b+\theta)/(b+\theta))(1-e^{-(b+\theta)\tau}) < h, \\ \widetilde{S}_c(t) & ((1-\alpha)b+\theta/b+\theta)(1-e^{-(b+\theta)\tau}) \geq h, \end{cases}$$

which is asymptotically stable.

Proof. We will show the following two cases for convenience.

Case I. $(((1-\alpha)b+\theta)/(b+\theta))(1-e^{-(b+\theta)\tau}) < h$. In this case $\widetilde{S}_r((n+1)\tau^-) < h$. If there is a $k \in Z$ such that $S(t^+) = 0$ $(t = k\tau)$, then we claim that $S(t^+) = 0, S(t) = \widetilde{S}_r(t)$, $t \geq k\tau$ for all $t = n\tau, n > k$. Otherwise, if there does not exist a k such that $S(t^+) = 0$ $(t = k\tau)$, then $S(t^+) = S(t^-) - h$ for all $t = n\tau$, and from the condition $(((1-\alpha)b+\theta)/(b+\theta))(1-e^{-(b+\theta)\tau}) < h$, we have $((1-\alpha)b+\theta/b+\theta) - (h/1-e^{-(b+\theta)\tau}) < 0$, that is, $S_c^0 < 0$. From Lemma 4.3 we know that $S(t)$ will be negative at some time, which is a contradiction to the positivity of the system, so all solutions will tend to $\widetilde{S}_r(t)$ ultimately, and the stability of $\widetilde{S}_r(t)$ can be proved by the same method as the above discussion.

Case II. $(((1-\alpha)b+\theta)/(b+\theta))(1-e^{-(b+\theta)\tau}) \geq h$. In this case $\widetilde{S}_r((n+1)\tau^-) \geq h$ and $S(t^+) = 0$ cannot hold when $t = n\tau (n > 1)$, that is, we have $S(t^+) = S(t^-) - h$ for all $n\tau (n > 1)$. The conclusion is true from Lemma 4.3.

Define

$$R_1 = \frac{\beta_1}{b(1-p)+\lambda}\frac{1}{\tau}\int_0^\tau \widetilde{S}_c(t)\,dt,$$
$$R_2 = \frac{\beta_1}{b(1-p)+\lambda}\frac{1}{\tau}\int_0^\tau \widetilde{S}_r(t)\,dt.$$

Theorem 4.1. *Suppose* $((1-\alpha)b+\theta)/(b+\theta) \geq h/(1-e^{-(b+\theta)\tau})$. *Then the periodic solution* $(\widetilde{S}(t), 0)$ *is locally asymptotically stable if* $R_1 < 1$. *Suppose* $((1-\alpha)b+\theta)/(b+\theta) < h/(1-e^{-(b+\theta)\tau})$. *Then the periodic solution* $(\widetilde{S}(t), 0)$ *is locally asymptotically stable if* $R_2 < 1$.

Using the comparison theorem in impulsive differential equations, we can prove Theorem 4.1, because $\widetilde{S}_r(t)$ and $\widetilde{S}_c(t)$ cannot exist at the same time.

Theorem 4.2. *Under the conditions of Theorem* 4.1, *if* $\lambda - \theta - (1-\alpha)pb < 0$, *then the solution* $(S(t), I(t))$ *of system* (4.1) *from region* $\{(S,I)|S \geq 0, I \geq 0, S+I \leq 1\}$ *must have* $\lim_{t\to\infty}|S(t) - \widetilde{S}(t)| = 0$ *and* $\lim_{t\to\infty} I(t) = 0$.

Proof. Suppose $(S(t), I(t))$ is the solution of system (4.1), where $0 < S(t) \leq 1$ and $0 < I(t) \leq 1$. We will show the following three cases.

Case I. $(((1-\alpha)b+\theta)/(b+\theta))(1-e^{-(b+\theta)\tau}) < h$. From $R_2 < 1$, we can choose a sufficiently small $\varepsilon_1 > 0$ such that

$$\sigma_1 = \exp\left((pb - b - \lambda + \beta_1\varepsilon_1)\tau + \frac{\beta_1((1-\alpha)b+\theta)}{b+\theta}\left(\tau - \frac{1-e^{-(b+\theta)\tau}}{b+\theta}\right)\right) < 1.$$

From the first equation of system (4.1), we have

$$\dot{S} \leq (1-\alpha)b + \theta - (\theta+b)S.$$

From Lemma 4.4 and the comparison theorem, for sufficiently small $\varepsilon_1 > 0$, there exists a $t_1 > 0$, such that when $t > t_1$ we have

$$S(t) \leq \widetilde{S}_r(t) + \varepsilon_1. \tag{4.6}$$

Without loss of generality, we can assume that the above inequality holds for all $t > 0$, then from the second equation of (4.1) we have

$$\dot{I} \leq (pb - b - \lambda + \beta_1(\widetilde{S}_r(t) + \varepsilon_1))I \triangleq \varphi(t)I. \tag{4.7}$$

For $\varphi(t)$ continuous and bounded on $(n\tau, (n+1)\tau]$, there exists a constant $M' > 0$ such that $|\varphi(t)| < M'$. Integrating (4.7) on $(0, t)$, where $t \in (n\tau, (n+1)\tau]$, yields

$$\begin{aligned}
I(t) &\leq I(0)\exp\left(\int_0^t \varphi(t)\,dt\right) \\
&= I(0)\exp\left(\left(\int_0^\tau + \int_\tau^{2\tau} + \cdots + \int_{(n-1)\tau}^{n\tau} + \int_{n\tau}^{t}\right)\varphi(t)\,dt\right) \\
&\leq I(0)\exp\left(n\left(\int_0^\tau \varphi(t)\,dt\right) + \int_{n\tau}^{(n+1)\tau} |\varphi(t)|\,dt\right) \\
&\leq I(0)\exp\left(n\left(\int_0^\tau \varphi(t)\,dt\right) + M'\tau\right) \\
&= I(0)\sigma_1^n \exp(M'\tau) \longrightarrow 0,
\end{aligned}$$

as $n \to \infty$. Therefore, $I(t) \to 0$, as $t \to \infty$. So, for sufficiently small $\varepsilon_2 > 0$, there exists a $t_2 > t_1$ such that $I(t) < \varepsilon_2$. Then from the first equation of system (4.1), we have

$$\dot{S} \geq (1-\alpha)b + \theta - (b+\theta)S - (b+\theta+\beta_1)\varepsilon_2,$$

for $t > t_2$, where we can choose ε_2 sufficiently small such that $(((1-\alpha)b+\theta)/(b+\theta) - (b+\theta+\beta_1)\varepsilon_2/(b+\theta))(1 - e^{-(b+\theta)\tau}) < h$. Similar to Lemma 4.4, the comparison system

$$\begin{cases}
\dot{S} = (1-\alpha)b + \theta - (b+\theta)S - (b+\theta+\beta_1)\varepsilon_2 & t \neq n\tau, \\
S(t^+) = \left.\begin{cases} 0 & S(t^-) < h, \\ S(t^-) - h & S(t^-) \geq h, \end{cases}\right\} & t = n\tau,
\end{cases}$$

has a globally stable periodic solution: $(((1-\alpha)b+\theta-(b+\theta+\beta_1)\varepsilon_2)/(b+\theta))(1-e^{-(b+\theta)(t-n\tau)}) = \widetilde{S}_r(t)-((b+\theta+\beta_1)\varepsilon_2/(b+\theta))(1-e^{-(b+\theta)(t-n\tau)})$, $t\in(n\tau,(n+1)\tau]$, and we have

$$\begin{aligned}\widetilde{S}_r(t) - \frac{b+\theta+\beta_1}{b+\theta}\varepsilon_2(1-e^{-(b+\theta)(t-n\tau)}) \\ &\geq \widetilde{S}_r(t) - \frac{b+\theta+\beta_1}{b+\theta}\varepsilon_2(1-e^{-(b+\theta)\tau}) \\ &\triangleq \widetilde{S}_r(t) - \varepsilon_3,\end{aligned}$$

where $\varepsilon_3 = ((b+\theta+\beta_1)/(b+\theta))\varepsilon_2(1-e^{-(b+\theta)\tau})$. From the comparison theorem, we have

$$S(t) \geq \widetilde{S}_r(t) - \varepsilon_3. \tag{4.8}$$

From (4.6) and (4.8) we have $\lim_{t\to\infty}|S(t)-\widetilde{S}(t)| = 0$.

Case II. $(((1-\alpha)b+\theta)/(b+\theta))(1-e^{-(b+\theta)\tau}) > h$. From $R_1 < 1$, we can choose a sufficiently small $\varepsilon_1 > 0$ such that

$$\sigma_2 = \exp\left((pb-b-\lambda+\beta_1\varepsilon)\tau + \beta_1\left(\frac{(1-\alpha)b+\theta}{b+\theta}\tau - \frac{h}{b+\theta}\right)\right) < 1.$$

From the first equation of system (4.1), we have

$$\dot{S} \leq (1-\alpha)b+\theta-(\theta+b)S.$$

From Lemma 4.4 and the comparison theorem, for $\varepsilon_1 > 0$, there exists a $t_1 > 0$, such that when $t > t_1$ we also have

$$S(t) \leq \widetilde{S}_c(t) + \varepsilon_1. \tag{4.9}$$

Without loss of generality, we can assume that the above inequality holds for all $t > 0$. Then from the second equation of (4.1) we have

$$\dot{I} \leq (pb-b-\lambda+\beta_1(\widetilde{S}_c(t)+\varepsilon_1))I \triangleq \psi(t)I. \tag{4.10}$$

For $\psi(t)$ continuous and bounded on $(n\tau,(n+1)\tau]$, there exists a constant $M'' > 0$ such that $|\psi(t)| < M''$. Integrating (4.10) on $(0,t)$, where $t \in (n\tau, (n+1)\tau]$, yields

$$\begin{aligned} I(t) &\le I(0)\exp\bigg(\int_0^t \psi(t)\,dt\bigg) \\ &= I(0)\exp\bigg(\bigg(\int_0^{\tau} + \int_{\tau}^{2\tau} + \cdots + \int_{(n-1)\tau}^{n\tau} + \int_{n\tau}^{t}\bigg)\psi(t)\,dt\bigg) \\ &\le I(0)\exp\bigg(n\bigg(\int_0^{\tau} \psi(t)\,dt\bigg) + \int_{n\tau}^{(n+1)\tau} |\psi(t)|\,dt\bigg) \\ &\le I(0)\exp\bigg(n\bigg(\int_0^{\tau} \psi(t)\,dt\bigg) + M''\tau\bigg) \\ &= I(0)\sigma_2^n \exp(M''\tau) \longrightarrow 0, \end{aligned}$$

as $n \to \infty$. Therefore $I(t) \to 0$ as $t \to \infty$. So, for sufficiently small $\varepsilon_2 > 0$, there exists a $t_2 > t_1$ such that

$$\dot{S} \ge (1-\alpha)b + \theta - (b+\theta)S - (b+\theta+\beta_1)\varepsilon_2,$$

for $t > t_2$, where we can choose $\varepsilon_2 > 0$ sufficiently small such that $(((1-\alpha)b+\theta)/(b+\theta) - (b+\theta+\beta_1)\varepsilon_2/(b+\theta))(1 - e^{-(b+\theta)\tau}) > h$. Similar to Lemma 4.4, the comparison system

$$\begin{cases} \dot{S} = (1-\alpha)b + \theta - (b+\theta)S - (b+\theta+\beta_1)\varepsilon_2 & t \ne n\tau, \\ S(t^+) = \left.\begin{cases} 0 & S(t^-) < h, \\ S(t^-) - h & S(t^-) \ge h, \end{cases}\right\} & t = n\tau, \end{cases}$$

has a globally stable periodic solution:

$$\widetilde{S}_c(t) - \frac{b+\theta+\beta_1}{b+\theta}\varepsilon_2.$$

Thus, we obtain for $t > t_2$,

$$S(t) \ge \widetilde{S}_c(t) - \frac{b+\theta+\beta_1}{b+\theta}\varepsilon_2. \tag{4.11}$$

From (4.9) and (4.11) we have $\lim_{t\to\infty} |S(t) - \widetilde{S}(t)| = 0$.

Case III. $(((1-\alpha)b+\theta)/(b+\theta))(1-e^{-(b+\theta)\tau}) = h$. Similar to Case II, we can prove $I(t) \to 0$. So for sufficiently small $\varepsilon_2 > 0$, there exists a $t_2 > t_1$ such that

$$\dot{S} \geq (1-\alpha)b + \theta - (b+\theta)S - (b+\theta+\beta_1)\varepsilon_2.$$

For $t > t_2$, we have

$$\left(\frac{(1-\alpha)b+\theta}{b+\theta} - \frac{b+\theta+\beta_1}{b+\theta}\varepsilon_2\right)(1-e^{-(b+\theta)\tau}) < h$$

for

$$\frac{(1-\alpha)b+\theta}{b+\theta}(1-e^{-(b+\theta)\tau}) = h.$$

From Lemma 4.4, we obtain

$$\begin{aligned} S(t) &\geq \widetilde{S}_r(t) - \frac{b+\theta+\beta_1}{b+\theta}\varepsilon_2(1-e^{-(b+\theta)(t-n\tau)}) \\ &\geq \widetilde{S}_r(t) - \frac{b+\theta+\beta_1}{b+\theta}\varepsilon_2(1-e^{-(b+\theta)\tau}) \\ &= \widetilde{S}_r(t) - \varepsilon_3, \end{aligned}$$

and at this time we have $\widetilde{S}_r(t) = \widetilde{S}_c(t)$, so we obtain $S(t) \geq \widetilde{S}_c(t) - \varepsilon_3$, and we have $\lim_{t\to\infty} |S(t) - \widetilde{S}(t)| = 0$. The proof is complete. □

From Theorems 4.1 and 4.2, we can get the following theorem.

Theorem 4.3. *Suppose* $\lambda - \theta - (1-\alpha)pb < 0$. *Then the periodic solution* $(\widetilde{S}(t), 0)$ *of system* (4.1) *is globally asymptotically stable in the invariant set* $\{(S,I)|S \geq 0, I > 0, S+I \leq 1\}$ *if* $((1-\alpha)b+\theta)/(b+\theta) \geq h/(1-e^{-(b+\theta)\tau})$ *and* $R_1 < 1$ *hold or* $((1-\alpha)b+\theta)/(b+\theta) < h/(1-e^{-(b+\theta)\tau})$ *and* $R_2 < 1$ *hold.*

5. Conclusion. In this paper, we considered a vaccinated SIS model with vertical transmission and media coverage. In general ways, we discussed constant vaccination and impulsive vaccination to susceptible individuals in another paper, that is, models (2.2) and (2.3), and compared the effectiveness of this two vaccination policy.

We obtained the result that impulsive vaccination is more effective. But, when the number of susceptible individuals is very large, the number of people who need to be vaccinated is also proportionally increasing; in addition, vaccination ability in an area is limited, so, in Section 3, we consider constant impulsive vaccination when the number of susceptible individuals is very large, that is, we consider system (3.1). From Theorem 3.1 we know that system (3.1) is uniformly ultimately bounded if $pb - \lambda + \theta > 0$. Using the discrete dynamical system determined by the stroboscopic map, we also obtain the periodic infection-free solution of system (3.1), which is globally asymptotically stable if $(pb - b - \lambda + \beta_1)\tau < \beta_1 h/(b + \theta)$, see Theorem 3.4.

After constant impulsive vaccination to a large number of susceptible individuals, the number of susceptible individuals will gradually decrease. When this constant impulsive vaccination is not reasonable, we will consider using proportional impulsive vaccination. So, in Section 4, we consider using a constant impulsive vaccination strategy when the amount of susceptible individuals is larger than the constant h, while we consider using proportional impulsive vaccination when the number of susceptible individuals is smaller than the constant h, that is, system (4.1). Using the discrete dynamical system determined by the stroboscopic map, we also obtain the exact periodic infection-free solution and show that it is globally asymptotically stable under some conditions, see Theorem 4.3.

In reality, the aim of using a vaccination policy is to reduce the numbers of infected people, so we only consider the behavior of the periodic infection-free solution of our model. But we do not consider the existence and behavior dynamics of the positive periodic solution of our model. We leave this to future work.

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