

THE ANALYSIS OF TWO EPIDEMIC MODELS WITH CONSTANT IMMIGRATION AND QUARANTINE

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ABSTRACT. Combining models with classifying immigration and with quarantine, this paper constructs an *SIQS* model and an *SIQR* model which incorporate constant immigration and quarantine for the special case of simple mass action incidence rate. The decline of the disease-related death rate and the increase of the individuals' recovery rate after individuals are quarantined are considered in the paper. Then, the unique endemic equilibria of the two models are attained, and local and global stability of the endemic equilibria is also proved.

1. Introduction. As many people know, quarantine is the separation and/or restriction of movement of persons who, because of recent exposure to a communicable disease, risk acquiring that disease and subsequently exposing others. It is apparent, as presented in the hadith, that principals of quarantine should be defined. This prevents people from entering a plague area, as well as preventing others from leaving. The quarantine is a new concept that has recently been discovered for human beings is still being applied today. It is intuitive why a healthy person is banned from going to the epidemic area. Unless one has a great knowledge of modern medical science, it is hard to understand why leaving an area of epidemic is important, especially to those who are healthy. It makes sense when the healthy person, who lives in an epidemic area, runs to another safer area in order to avoid the infection. Chen Jun-jie [2] and Herbert Hethcote et al. [5] have considered models of *SIQS* and *SIQR* with different reaction incidence rates, which are the simple mass reaction incidence rate, the standard reaction incidence rate and the quarantine-adjusted incidence rate. Immigration is an essential problem in the epidemic models [1, 3, 4, 7,

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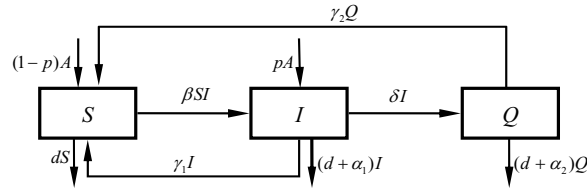
8]. Nowadays, only susceptible immigration is not considered enough in modeling the epidemic; Li Jianquan et al. [7] classified immigration into three classes and obtained global stability of the models.

Differential equation models have been used to study the dynamics of many diseases in human and wild animal populations. In this paper, the traditional K-M compartment model [6] is adopted to analyze the problem. Assume the total population N is divided into four classes: the susceptible class S , the normal infectious class I in which individuals are not quarantined, the quarantined infectious class Q , in which the individuals have been removed and isolated either voluntarily or coercively from the infectious class, and the removed class R .

2. The $SIQS$ model with classifying immigration. Consider the following $SIQS$ model, where infection does not confer immunity. In the model, some of the susceptible individuals become infected and then some infected individuals remain in the I class for their entire infectious period before they return to the susceptible class, while other infected individuals are transferred into a quarantined class Q . The individuals in the infectious class including the normal infectious class I and the class Q of quarantined individuals remain there until they are no longer infectious, at which time they return to the susceptible individuals. To formulate the model, the assumptions are:

- i) There is a constant flow of A new members into the population in unit time, of which a fraction p ($0 \leq p \leq 1$) is infective;
- ii) There is a natural death rate constant $d > 0$ in each class;
- iii) Each infective makes βN contact sufficient to transmit in unit time;
- iv) A fraction $\delta \geq 0$ of infectious individuals isolate from the infectious class;
- v) In unit time, the infectious individuals and the quarantined individuals recover into class S with the fractions γ_1 and γ_2 , respectively, and die from the infection with the different fraction α_1 and α_2 , evidently, $\alpha_1 > \alpha_2 > 0$, $\gamma_2 > \gamma_1 > 0$.

Based on these assumptions, the fraction made by an infective that is with a susceptible and thus can produce a new infection is S/N and the rate of new infection is $\beta N \cdot S/N \cdot I = \beta SI$, which is called the

FIGURE 1. The general transfer diagram for the $SIQS$ model.

simple mass action incidence rate. The transfer diagram is as shown in Figure 1.

From the above assumptions, the model for transmission of infectious diseases can be described by

$$(1) \quad \begin{cases} S' = (1-p)A + \gamma_1 I + \gamma_2 Q - dS - \beta SI, \\ I' = \beta SI + pA - (\gamma_1 + d + \alpha_1 + \delta)I, \\ Q' = \delta I - (d + \alpha_2 + \gamma_2)Q. \end{cases}$$

Due to $N = S + I + Q$, from (1) we obtain the equation

$$(2) \quad N'(t) = A - dN - \alpha_1 I - \alpha_2 Q.$$

In the absence of disease, the population size N approaches the carrying capacity A/d . The differential equation (2) implies that solution of (1) starting in $R_+^3 = \{(S, I, Q) \in R^3 \mid S \geq 0, I \geq 0, Q \geq 0\}$ either approaches, enters or remains in the subset of R_+^3 defined by $D = \{(S, I, Q) \in R^3 \mid 0 < S + I + Q \leq A/d, S \geq 0, I \geq 0, Q \geq 0\}$. Thus, it suffices to consider solutions in the region D . Solutions of the initial value problem starting in D and defined by (1) exist and are unique on a maximal interval. Since solutions remain bounded in the positively invariant region D , the maximal interval is $[0, +\infty)$.

If $p = 0$, analysis of the stability of the $SIQS$ system and the following $SIQR$ system is very similar to the research which Hethcote did in 2002 [5]. The two models have their basic reproduction numbers R_0 and $R_0 = 1$ is the threshold, that is, there is the unique disease-free equilibrium which is globally asymptotically stable, if $R_0 > 1$, the disease-free equilibrium, is unstable and the epidemic equilibrium is globally asymptotically stable. In the paper, analysis of the case $p = 0$ is omitted, and the case of $0 < p \leq 1$ is considered.

Lemma 1. *When $0 < p \leq 1$, system (1) has a unique positive equilibrium (I^*, Q^*, N^*) which is locally asymptotically stable.*

Proof. Substituting $S = N - I - Q$ into (1), then from the last two equations of (1) and (2), we have

$$(3) \quad \begin{cases} I' = \beta(N - I - Q)I + pA - (\gamma_1 + d + \alpha_1 + \delta)I, \\ Q' = \delta I - (d + \alpha_2 + \gamma_2)Q, \\ N' = A - dN - \alpha_1 I - \alpha_2 Q. \end{cases}$$

So the equilibrium (I^*, Q^*, N^*) of system (1) satisfies the following equations

$$(4) \quad \begin{cases} \beta(N - I - Q)I + pA - (\gamma_1 + d + \alpha_1 + \delta)I = 0, \\ \delta I - (d + \alpha_2 + \gamma_2)Q = 0, \\ A - dN - \alpha_1 I - \alpha_2 Q = 0. \end{cases}$$

From the last two equations of (4), we obtain

$$I = \frac{(A - dN)(d + \alpha_2 + \gamma_2)}{\alpha_1(d + \alpha_2 + \gamma_2) + \alpha_2\delta}, \quad Q = \frac{\delta(A - dN)}{\alpha_1(d + \alpha_2 + \gamma_2) + \alpha_2\delta}.$$

Substituting them into the first equation of (4), we have

$$\begin{aligned} & \beta \{ (d + \alpha_2 + \gamma_2) [(\alpha_1 + d)N - A] + \delta [(\alpha_2 + d)N - A] \} \\ & \quad = [\alpha_1(d + \alpha_2 + \gamma_2) + \alpha_2\delta] \\ & \quad \quad \left\{ (d + \alpha_1 + \gamma_1 + \delta) - \frac{pA[\alpha_1(d + \alpha_2 + \gamma_2) + \alpha_2\delta]}{(A - dN)(d + \alpha_2 + \gamma_2)} \right\}. \end{aligned}$$

Denote

$$\begin{aligned} h(N) &= \beta \{ (d + \alpha_2 + \gamma_2) [(\alpha_1 + d)N - A] + \delta [(\alpha_2 + d)N - A] \}, \\ g(N) &= [\alpha_1(d + \alpha_2 + \gamma_2) + \alpha_2\delta] \\ & \quad \left\{ (d + \alpha_1 + \gamma_1 + \delta) - \frac{pA[\alpha_1(d + \alpha_2 + \gamma_2) + \alpha_2\delta]}{(A - dN)(d + \alpha_2 + \gamma_2)} \right\}. \end{aligned}$$

It is easy to see that $h(N)$ is a nondecreasing function of N and $h(N) < 0$ for sufficiently small N . At the same time, $g'(N) < 0$,

$g(0) > 0$, $\lim_{N \rightarrow A/d^-} g(N) = -\infty$, so in the interval $(0, A/d)$, $h(N)$ and $g(N)$ intersect at a unique point. Therefore, in D equation (4) has a unique root denoted with $P^*(I^*, Q^*, N^*)$, which is also the unique positive and epidemic equilibrium of (3). To prove the local asymptotic stability of $P^*(I^*, Q^*, N^*)$, the Jacobian matrix of system (3) at P^* is obtained as follows

$$J^* = \begin{bmatrix} -\beta I^* - (pA)/I^* & -\beta I^* & \beta I^* \\ \delta & -(d + \alpha_2 + \gamma_2) & 0 \\ -\alpha_1 & -\alpha_2 & -d \end{bmatrix}.$$

The characteristic equation of J^* is

$$f(\lambda) = a_0\lambda^3 + a_1\lambda^2 + a_2\lambda + a_3 = 0,$$

where

$$a_0 = 1, a_1 = \beta I^* + \frac{pA}{I^*} + 2d + \alpha_2 + \gamma_2,$$

$$a_2 = \left(\beta I^* + \frac{pA}{I^*} \right) (2d + \alpha_2 + \gamma_2) + (\alpha_1 + \delta) \beta I^* + d(d + \alpha_2 + \gamma_2),$$

$$a_3 = d(d + \alpha_2 + \gamma_2) \left(\beta I^* + \frac{pA}{I^*} \right) + \beta(\delta + \alpha_1)(\alpha_2 + d)I^* + \alpha_1\beta\gamma_2I^*.$$

Then

$$\Delta_1 = a_1 = \beta I^* + \frac{pA}{I^*} + 2d + \alpha_2 + \gamma_2 > 0,$$

$$\Delta_2 = \begin{vmatrix} a_1 & a_0 \\ a_3 & a_2 \end{vmatrix} = a_1 a_2 - a_3$$

$$= \left(\beta I^* + \frac{pA}{I^*} + 2d + \alpha_2 + \gamma_2 \right) \left[\left(\beta I^* + \frac{pA}{I^*} \right) (2d + \alpha_2 + \gamma_2) + (\alpha_1 + \delta) \beta I^* + d(d + \alpha_2 + \gamma_2) \right]$$

$$- d(d + \alpha_2 + \gamma_2) \left(\beta I^* + \frac{pA}{I^*} \right)$$

$$- \beta(\delta + \alpha_1)(\alpha_2 + d)I^* - \alpha_1\beta\gamma_2I^*$$

$$\begin{aligned}
&= \left(\beta I^* + \frac{pA}{I^*} \right)^2 (2d + \alpha_2 + \gamma_2) + \left(\beta I^* + \frac{pA}{I^*} \right) \\
&\quad \times [d^2 + (2d + \alpha_2 + \gamma_2)(d + \alpha_2 + \gamma_2)] \\
&\quad + (\alpha_1 + \delta) \beta I^* \left(\beta I^* + \frac{pA}{I^*} + d \right) + \beta I^* \delta \gamma_2 \\
&\quad + d(d + \alpha_2 + \gamma_2) \left(\beta I^* + \frac{pA}{I^*} + 2d + \alpha_2 + \gamma_2 \right) \\
&> 0.
\end{aligned}$$

Hence, the Routh-Hurwitz conditions are satisfied. Thus, it follows that the endemic equilibrium P^* of (3) is always locally asymptotically stable. \square

Theorem 1. *When $0 < p \leq 1$, the endemic equilibrium P^* of system (1) is always globally asymptotically stable.*

Proof. From the first equation of (4), we can easily obtain

$$\beta(N^* - I^* - Q^*) + \frac{pA}{I^*} - (\gamma + d + \alpha + \delta) = 0.$$

In order to prove the global stability of P^* , we rewrite (3) as follows

$$(5) \quad \begin{cases} I' = I \left[-\frac{pA(I-I^*)}{I \cdot I^*} + \beta(N - N^*) - \beta(I - I^*) - \beta(Q - Q^*) \right], \\ Q' = \delta(I - I^*) - (d + \alpha_2 + \gamma_2)(Q - Q^*), \\ N' = -d(N - N^*) - \alpha_1(I - I^*) - \alpha_2(Q - Q^*). \end{cases}$$

Define the Liapunov function

$$\begin{aligned}
V(I, Q, N) &= \frac{\alpha_2(\delta + \alpha_1) + \alpha_1(\gamma_2 + 2d)}{\alpha_2\beta} \left(I - I^* - I^* \ln \frac{I}{I^*} \right) \\
&\quad + \frac{1}{2} \left\{ [(Q - Q^*) - (N - N^*)]^2 + \frac{2d + \gamma_2}{\alpha_2} (N - N^*)^2 \right. \\
&\quad \left. + \frac{\alpha_1(2d + \gamma)}{\alpha_2\delta} (Q - Q^*)^2 \right\} \geq 0,
\end{aligned}$$

which is a positive function in the region R_+^3 . Then the time derivative of $V(I, Q, N)$ along the solution (5) is given by

$$\begin{aligned}
\left. \frac{dV}{dt} \right|_{(5)} &= \frac{\alpha_2(\delta + \alpha_1) + \alpha_1(\gamma_2 + 2d)}{\alpha_2\beta} \\
&\cdot \left[-\frac{pA(I - I^*)^2}{I \cdot I^*} + \beta(I - I^*)(N - N^*) \right. \\
&\quad \left. - \beta(I - I^*)^2 - \beta(I - I^*)(Q - Q^*) \right] \\
&+ [(Q - Q^*) - (N - N^*)] \\
&\cdot [\delta(I - I^*) - (d + \alpha_2 + \gamma_2)(Q - Q^*) + d(N - N^*)] \\
&+ \alpha_1(I - I^*) + \alpha_2(Q - Q^*) + \frac{2d + \gamma_2}{\alpha_2}(N - N^*) \\
&\cdot [-d(N - N^*) - \alpha_1(I - I^*) - \alpha_2(Q - Q^*)] \\
&+ \frac{\alpha_1(2d + \gamma)}{\alpha_2\delta}(Q - Q^*) \cdot [\delta(I - I^*) - (d + \alpha_2 + \gamma_2)(Q - Q^*)] \\
&= -\frac{\alpha_2(\delta + \alpha_1) + \alpha_1(\gamma_2 + 2d)}{\alpha_2\beta} \left[\frac{pA(I - I^*)^2}{I \cdot I^*} + \beta(I - I^*)^2 \right] \\
&\quad - \left[(d + \gamma_2) + \frac{\alpha_1(2d + \gamma_2)(d + \alpha_2 + \gamma_2)}{\alpha_2\delta} \right] (Q - Q^*)^2 \\
&\quad - \frac{d(\alpha_2 + 2d + \gamma_2)}{\alpha_2}(N - N^*)^2 \\
&\leq 0.
\end{aligned}$$

Note that $V' = 0$ on the set D holds if and only if $(I, Q, N) = (I^*, Q^*, N^*)$. Because the largest positively invariant subset is the equilibrium (I^*, Q^*, N^*) , the Liapunov-Lasalle theorem implies that $P^*(I^*, Q^*, N^*)$ is globally attractive in R_+^3 . Combining with the local stability of (I^*, Q^*, N^*) , we can conclude that $P^*(I^*, Q^*, N^*)$ is globally asymptotically stable in the region R_+^3 . So the endemic equilibrium P^* is globally asymptotically stable in the region D . \square

Figures 2 and 3 using Matlab reflect how system (1) develops from different initial values and in two different situations. We choose $(1000, 0, 0)$, $(700, 300, 0)$ and $(400, 300, 300)$, which can represent three different cases engendering initially, as original values. They represent

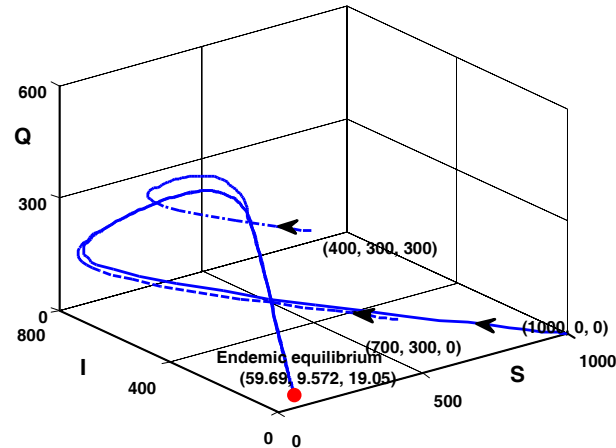


FIGURE 2. The trajectories of system (1) from initial points $(1000, 0, 0)$, $(700, 300, 0)$ and $(400, 300, 300)$, where $A = 2$, $p = 0.5$, $\alpha_1 = 0.1$, $d = 0.001$, $\gamma_1 = 0.1$, $\beta = 0.01$, $\alpha_2 = 0.05$, $\delta = 0.5$ and $\gamma_2 = 0.2$.

that initially there are no infectious people in the region, there are infectious individuals and none of them quarantined in the region, and there are infectious individuals and some of them quarantined. In the two figures, what we find is that the trajectories from different initial points go to one point respectively, which is the endemic equilibrium. We can prove the global asymptotic stability of the point.

From the figures, we can divide the period when the epidemic affects the change of population in the region into the following phases:

i) During the origin of the infection, because of the huge base of susceptible population, the infectious population increases dramatically. For the lag effect of the isolation, although the quarantined population increases too, it can't match the leap of the infectious population.

ii) Along with the epidemic developing gradually, the susceptible population declines slightly, and the number of normal infectious individuals decline too; but, the quarantined individuals increase more rapidly than in phase 1 in numbers for the lag effect of the isolation also.

iii) In this section, the susceptible population and the normal infectious population go to stable gradually, and the quarantined population begins to decrease.

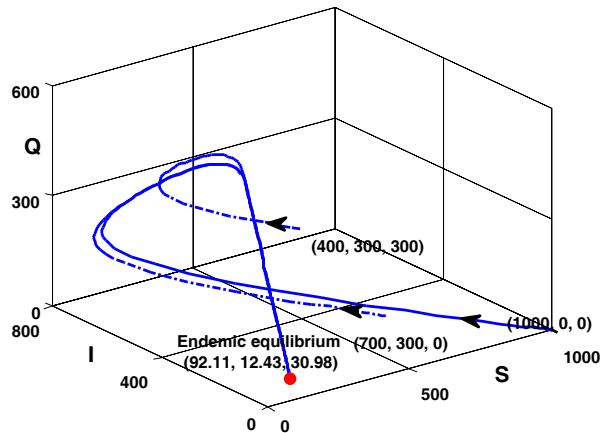


FIGURE 3. The trajectories of system (1) from initial points $(1000, 0, 0)$, $(700, 300, 0)$ and $(400, 300, 300)$, where $A = 2$, $p = 0.5$, $\alpha_1 = 0.1$, $d = 0.001$, $\gamma_1 = 0.1$, $\beta = 0.01$, $\alpha_2 = 0.02$, $\delta = 0.8$ and $\gamma_2 = 0.3$.

iv) The population of all four classes in the region keeps stable, and the epidemic exists in the region persistently.

In the two figures, the constants A , p , α_1 , d , γ_1 and β are fixed, and the constants α_2 , δ and γ_2 which can reflect the condition of isolation are changed. Note that the quarantined therapy in Figure 2 is better than Figure 3, and the unique endemic equilibrium of the two cases is $(60, 49, 97.7)$ and $(92.5, 64.6, 161)$, respectively. Hence, we can conclude that the living population in the former case in the region ultimately is more than in the latter case. The dominant causation of the decline of the population in the region is the death from infection; therefore, the population dying from infection in the latter case is much more than in the former case.

3. The $SIQR$ model with classifying immigration. In the basic $SIQS$ model, we assume that the infected individuals transfer to removed class R , in which individuals have immunity from the epidemic. In this model, some of the susceptible individuals become infected and then some infected individuals remain in the I class for their entire infectious period before they gain immunity and enter the removed class, while other infected individuals are transferred into a quarantined class Q . Individuals in the infectious class including the

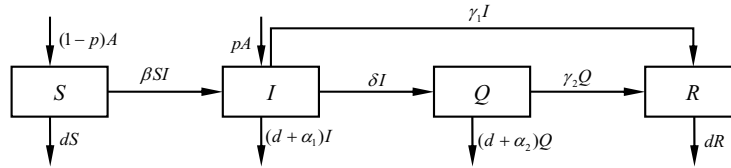


FIGURE 4. The general transfer diagram for the $SIQR$ model.

normal infectious class I and the class Q of quarantined individuals remain there until they are no longer infectious, at which time they return to the susceptible individuals. The transfer diagram is as in Figure 4.

The $SIQR$ model for transmission of infectious diseases has the following form

$$(6) \quad \begin{cases} S' = (1 - p)A - dS - \beta SI, \\ I' = pA + \beta SI - (\gamma_1 + d + \alpha_1 + \delta)I, \\ Q' = \delta I - (d + \alpha_2 + \gamma_2)Q, \\ R' = \gamma_1 I + \gamma_2 Q - dR, \end{cases}$$

where all of the constants are the same as the $SIQS$ model. So the total population N satisfies the following equation

$$N'(t) = A - dN - \alpha_1 I - \alpha_2 Q.$$

In the absence of disease, the population size N approaches the carrying capacity A/d . In the $SIQR$ model, the solution of (6) starting in R_+^4 either approaches, enters or remains in the subset of R_+^4 defined by $D = \{(S, I, Q, R) \in R_+^4 | 0 < S + I + Q + R \leq (A/d)\}$; thus, it suffices to consider solutions in region D . As before, the initial value problem is well posed both mathematically and epidemiologically in D . By setting the right side of each of the four differential equations in (6) equal to zero, we can easily obtain the unique positive equilibrium $P^*(S^*, I^*, Q^*, R^*)$, which is also the endemic equilibrium of system (6), where

$$S^* = \frac{2(1 - p)A(\gamma_1 + d + \delta + \alpha_1)}{A\beta + (\gamma_1 + d + \delta + \alpha_1)d + \Delta},$$

$$\begin{aligned}
 I^* &= \frac{A\beta - (\gamma_1 + d + \delta + \alpha_1)d + \Delta}{2\beta(\gamma_1 + d + \delta + \alpha_1)}, \\
 Q^* &= \frac{\delta[A\beta - (\gamma_1 + d + \delta + \alpha_1)d + \Delta]}{2\beta(\gamma_1 + d + \delta + \alpha_1)(d + \gamma_2 + \alpha_2)}, \\
 R^* &= \frac{[\gamma_1(\gamma_2 + d + \alpha_2) + \gamma_2\delta] \cdot [A\beta - (\gamma_1 + d + \delta + \alpha_1)d + \Delta]}{2d\beta(\gamma_2 + d + \alpha_2)(\gamma_1 + d + \delta + \alpha_1)},
 \end{aligned}$$

and

$$\Delta = \sqrt{[(\gamma_1 + d + \delta + \alpha_1)d - A\beta]^2 + 4pd\beta A(\gamma_1 + d + \delta + \alpha_1)}.$$

Theorem 2. *When $0 < p \leq 1$, the endemic equilibrium P^* of system (6) is globally asymptotically stable.*

Proof. As the method used in Lemma 1, the local asymptotic stability of the endemic equilibrium P^* of system (6) can be easily proved by analyzing the Jacobian matrix of system (6) at the endemic equilibrium P^* and the Routh-Hurwitz criterion. In order to prove the global stability, consider the former two equations of (6), which can compose an SI subsystem about system (6), for the absence of S and I in the two equations. By analogy with the method in Theorem 1, we rewrite the SI subsystem as follows:

$$(7) \quad \begin{cases} S' = S \left[-\frac{(1-p)}{SS^*}(S - S^*) - \beta(I - I^*) \right], \\ I' = I \left[-\frac{pA(I - I^*)}{I \cdot I^*} + \beta(S - S^*) \right]. \end{cases}$$

Consider the Liapunov function

$$V(S, I) = \int_{S^*}^S \frac{\beta u - (\gamma_1 + d + \alpha_1 + \delta) + ((pA)/I^*)}{u} du + \beta \int_{I^*}^I \frac{u - I^*}{u} du,$$

which is obviously positive definite and goes to infinity as $S \rightarrow +\infty$ or $I \rightarrow +\infty$. Using $(\gamma_1 + d + \alpha_1 + \delta) - ((pA)/I^*) = \beta S^*$, the Liapunov

derivative is

$$\begin{aligned} \frac{dV}{dt} &= \frac{\beta(S - S^*)}{S} \cdot S \left[-\frac{(1-p)}{SS^*}(S - S^*) - \beta(I - I^*) \right] \\ &\quad + \beta \frac{(I - I^*)}{I} \cdot I \left[-\frac{pA(I - I^*)}{I \cdot I^*} + \beta(S - S^*) \right] \\ &= -\frac{\beta(1-p)}{SS^*}(S - S^*)^2 - \frac{p\beta A(I - I^*)^2}{I \cdot I^*} \\ &\leq 0. \end{aligned}$$

Note that $V' = 0$ on the set R_+^2 where $(S, I) = (S^*, I^*)$. Because $S' \neq 0$ on $S = S^*$ unless $I = I^*$, the largest positively invariant subset is the equilibrium (S^*, I^*) , so that the Liapunov-Lasalle theorem implies that (S^*, I^*) is globally attractive in R_+^2 . For the local stability of (S^*, I^*, Q^*, R^*) , we can easily obtain the conclusion that (S^*, I^*) is locally stable. Therefore, (S^*, I^*) is globally stable in R_+^2 and $\lim_{t \rightarrow +\infty} S(t) = S^*$, $\lim_{t \rightarrow +\infty} I(t) = I^*$. From the third equation of (6), we can obtain

$$\begin{aligned} Q(t) &= \left[Q_0 + \int_{t_0}^t \delta I(\tau) \exp[(d + \alpha_2 + \gamma_2)(\tau - t_0)] d\tau \right] \\ &\quad \times \exp[-(d + \alpha_2 + \gamma_2)(t - t_0)]. \end{aligned}$$

By L'Hospital's rule, we obtain

$$\lim_{t \rightarrow +\infty} Q(t) = \lim_{t \rightarrow +\infty} \frac{\delta I(t)}{d + \alpha_2 + \gamma_2} = \frac{\delta I^*}{d + \alpha_2 + \gamma_2} = Q^*.$$

Similarly, solving for R using the fourth equation in (6) and L'Hospital's rule, we obtain $\lim_{t \rightarrow +\infty} R(t) = R^*$. So the P^* and (S^*, I^*, Q^*, R^*) is a global attractor. Combining with the P^* 's local asymptotic stability, the endemic equilibrium P^* is globally asymptotically stable in D . \square

Compared with the $SIQS$ model, the $SIQR$ model is a four-dimensional system. To reflect the situations that the trajectories of the system vary as time goes on comprehensively, the phase diagrams of system (6) are illustrated in Figure 5 and Figure 6.

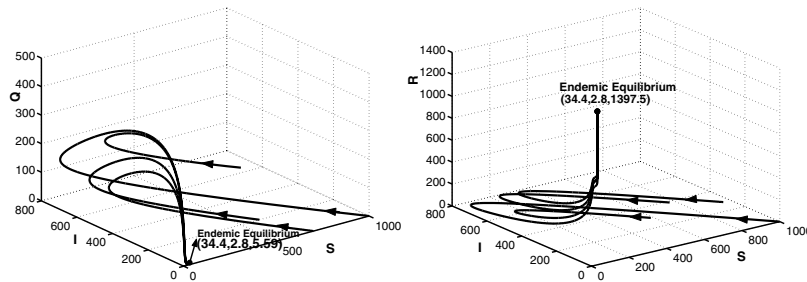


FIGURE 5. The phase diagram of system (6) in phase space (S, I, Q) and (S, I, R) where $A = 2$, $p = 0.5$, $\alpha_1 = 0.1$, $d = 0.001$, $\gamma_1 = 0.1$, $\beta = 0.01$, $\alpha_2 = 0.05$, $\delta = 0.5$, $\gamma_2 = 0.2$, and the endemic equilibrium is $P^*(34.4, 2.8, 5.59, 1397.5)$.

In the $SIQR$ model, the constants A , p , α_1 , d , γ_1 , β , α_2 , δ and γ_2 are chosen the same as the former $SIQS$ model to compare with the two models. In system (6), we choose respectively $(1000, 0, 0, 0)$, $(700, 0, 0, 300)$, $(600, 200, 0, 200)$ and $(500, 200, 200, 100)$ as initial points, which can represent four different cases. They are that initially there are no infectious people and no immunes in the region, there are no infectious people but some immunes, there are infectious individuals and none of them quarantined in the region, and there are infectious individuals and some of them are quarantined. In Figures 5 and 6, we can find that the trajectories from different initial points go to one point which is the endemic equilibrium, just as we found in Figures 3 and 4.

In the same way, the period when the epidemic affects the change of population in the region is divided into the following phases:

i) During the original time, the disease spreads rapidly in the region, so the infectious population, which includes the normal infectious population and the quarantined population, increases sharply, and the susceptible population declines, while some infectious individuals recover and get immunity from the epidemic.

ii) Along with the decrease of the infectious population and the increase of the removed population, the speed of the epidemic's spread goes down. At this moment, a large number of infectious individuals recover and enter the removed class R ; hence, the infectious population declines and the susceptible population ceases to decrease.

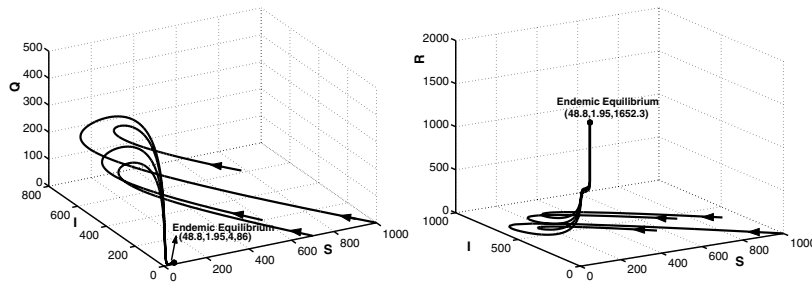


FIGURE 6. The phase diagram of system (6) in phase space (S, I, Q) and (S, I, R) where $A = 2$, $p = 0.5$, $\alpha_1 = 0.1$, $d = 0.001$, $\gamma_1 = 0.1$, $\beta = 0.01$, $\alpha_2 = 0.02$, $\delta = 0.8$, $\gamma_2 = 0.3$, and the endemic equilibrium is $P^*(48.8, 1.95, 4.86, 1652.3)$.

iii) In this section, the proportion of the removed people goes up, and the number of the susceptible people and the infectious people goes to stability, respectively, yet the proportion of them decreases.

iv) The increase of the removed individuals declines and goes to a fixed value ultimately. From the figures, the epidemic can't eliminate ultimately; however, in this model the infectious individuals, which account for less than 1 percent of the total population, are much less than that in $SIQS$ model. Just as the difference between Figure 3 and 4, in Figures 5 and 6 we know that better quarantined therapy can lead to more immune individuals and less infectious individuals.

4. Conclusion and discussion. This paper has considered the $SIQS$ model and the $SIQR$ model with constant classed immigration and simple mass action incidence rate. From all the constants, the amount of infectious individuals of the $SIQS$ model is larger than the $SIQR$ model, because in the latter one many recovered people transfer into the immune. Whether the therapy or the medicine can recover the patient drastically or not will determine the development of the epidemic. In the paper, we can easily see that if $0 < p \leq 1$ meant continuous entering of the infectious individuals, in theory it cannot become extinct and will form an endemic in the region, however effective the quarantined therapy is. Hence, the systems only have unique equilibrium whether isolated measures are taken or not; in other words, no matter whether the isolated measures are taken or

not, and how effective the quarantined therapy is, the analysis of the systems is definite. However, the quarantine does have its effect. It can reduce the size of the epidemic by advancing the quarantine therapy, which can reduce γ and α , and improve the quarantine ratio, which can increase δ . In [5], we know if $p = 0$, the analysis of stability of the two models have their basic reproduction number R_0 as well as the threshold of the systems, respectively, and if $R_0 < 1$, there is the unique disease-free equilibrium which is globally asymptotically stable, if $R_0 > 1$ the disease-free equilibrium is unstable and the epidemic equilibrium is globally asymptotically stable. That means if the infectious immigration is forbidden to enter, the epidemic can be eliminated in the region by better therapy, more effective quarantine measure and etc. Therefore, controlling the entrance of the infectious immigration is more important in the battle of epidemic prevention and cure. In the paper, the two models use the simple mass action incidence rate βSI , which is applied in the models with few people in the region. However, if there is a great population in the region, the standard mass action incidence rate $\beta SI/N$, or the quarantine-adjusted action incidence rate $\beta SI/(N - Q)$ referred to the work of Hethcote in 2002 is more valuable, and the latter one has a Hopf bifurcation when $p = 0$. The analysis of it is more complex than that in this paper. Assuming that the new action incidence rate $\beta(N - Q)SI$, where $\beta(x) > 0$, $\beta'(x) \leq 0$, $[\beta(x)x]' \geq 0$, and $[\beta'(x)]^2 + \{[\beta(x)x]'\}^2 \leq 0$, which is applied in the model with the quarantine. This action incidence rate combines the simple mass action incidence rate and $\beta SI/(N - Q)$, and it can be used more universally in models with quarantine. If the action incidence rate is used in the model of this paper, local stability can be obtained easily, but global stability needs a more tactful method to solve.

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