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# FOUR PERIODIC SOLUTIONS OF A GENERALIZED DELAYED PREDATOR-PREY SYSTEM ON TIME SCALES

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ABSTRACT. With the help of a continuation theorem based on Gaines and Mawhin's coincidence degree, easily verifiable criteria are established for the existence of four positive periodic solutions of a generalized delayed predator-prey system on time scales.

**1. Introduction.** It is well known that a very basic and important problem in the study of a population model with a periodic environment is the global existence and stability of a positive periodic solution. Many good results concerning the existence of at least one positive periodic solution have already been obtained and collected in some monographs (see, for example [**6, 7, 10, 11, 18**] and the references cited therein). However, the existence results of multiple periodic solutions for biological models are very scarce. Recently, Feng and Chen [**12**] studied the following two-predator and one prey system with nonmonotone functional response system:

$$
\begin{aligned}
x'(t) &= x(t)\left[a(t) - b(t)\int_{-\infty}^{t} K(t-s)x(s)\,ds \right.\\
&\qquad\left. - \frac{r(t)y^2(t)}{m^2y^2(t)+x^2(t)} - \frac{f(t)z^2(t)}{n^2z^2(t)+x^2(t)}\right],\\
y'(t) &= y(t)\left[\frac{r(t)x(t-\tau_1(t))y(t-\tau_1(t))}{m^2y^2(t-\tau_1(t))+x^2(t-\tau_1(t))} - d_1(t)\right],
\end{aligned}
\tag{1.1}
$$

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$$z'(t) = z(t)\left[\frac{f(t)x(t-\tau_2(t))z(t-\tau_2(t))}{n^2z^2(t-\tau_2(t))+x^2(t-\tau_2(t))} - d_2(t)\right].$$

By using the coincidence degree theory developed by Gaines and Mawhin [**13**], the existence of four periodic solutions for the delayed predator-prey system is established.

A natural question is whether the discrete analogy of system (1.1)

$$\begin{aligned} x(k+1) &= x(k)\exp\Bigg[a(k) - b(k)\sum_{s=-\infty}^{k} K(k-s)x(s) \\ &\qquad - \frac{r(k)y^2(k)}{m^2y^2(k)+x^2(k)} - \frac{f(k)z^2(k)}{n^2z^2(k)+x^2(k)}\Bigg], \\ y(k+1) &= y(k)\exp\left[\frac{r(k)x(k-\tau_1(k))y(k-\tau_1(k))}{m^2y^2(k-\tau_1(k))+x^2(k-\tau_1(k))} - d_1(k)\right], \\ z(k+1) &= z(k)\exp\left[\frac{f(k)x(k-\tau_2(k))z(k-\tau_2(k))}{n^2z^2(k-\tau_2(k))+x^2(k-\tau_2(k))} - d_2(k)\right], \end{aligned} \tag{1.2}$$

has four periodic solutions?

Recently, Bohner et al. [4] pointed out that it is unnecessary to explore the existence of periodic solutions of some continuous and discrete population models in separate ways. One can unify such studies in the sense of dynamic equations on general time scales. So, the second question is whether we can also unify the studies of multiple periodic solutions of such two-predator and one prey systems with nonmonotone functional response systems?

The theory of measure chains, which has recently received a lot of attention, see [**2**–**5, 8, 9, 15, 17, 19**], was introduced by Hilger in his Ph.D. thesis [**14**] in 1988 in order to unify continuous and discrete analysis.

Motivated by the above works, we consider the following system on time scales

$$
(1.3)\qquad
\begin{aligned}
u_1^{\Delta}(t) &= a(t) - b(t)\int_{-\infty}^{t+\kappa} K(t-s)e^{u_1(s)}\Delta s \\
&\qquad - \frac{r(t)e^{2u_2(t)}}{m^2e^{2u_2(t)} + e^{2u_1(t)}} - \frac{f(t)e^{2u_3(t)}}{n^2e^{2u_3(t)} + e^{2u_1(t)}}, \\
u_2^{\Delta}(t) &= \frac{r(t)e^{u_1(t-\tau_1(t))}e^{u_2(t-\tau_1(t))}}{m^2e^{2u_2(t-\tau_1(t))} + e^{2u_1(t-\tau_1(t))}} - d_1(t), \\
u_3^{\Delta}(t) &= \frac{f(t)e^{u_1(t-\tau_2(t))}e^{u_3(t-\tau_2(t))}}{n^2e^{2u_3(t-\tau_2(t))} + e^{2u_1(t-\tau_2(t))}} - d_2(t).
\end{aligned}
$$

It is clear that (1.3) becomes (1.1) when $\mathbf{T} = \mathbf{R}$. When $\mathbf{T} = \mathbf{N}$, let $x(t) = \exp\{u_1(t)\}$, $y(t) = \exp\{u_2(t)\}$ and $z(t) = \exp\{u_3(t)\}$. Then (1.3) becomes (1.2).

The main purpose of this paper is to study the existence of multiple positive periodic solutions of (1.3). The main results reveal that when we dealt with the existence of multiple positive periodic solutions of such two-predator and one prey systems with nonmonotone functional response systems it is also unnecessary to prove results for differential equations and separately again for difference equations. One can unify such problems in the frame of two-predator and one prey systems with nonmonotone functional response systems on time scales.

**2. Preliminaries.** In this section, we give a short introduction to time scales calculus and recall the continuation theorem from coincidence degree theory.

First, we present some foundational definitions and results; for proofs and further explanation and results, we refer to the paper by Hilger [**14**].

Let $\mathbf{T}$ be a time scale, i.e., $\mathbf{T}$ is a nonempty closed subset of $\mathbf{R}$.

**Definition 2.1.** We say that a time scale $\mathbf{T}$ is $\omega$ periodic, if $t \in \mathbf{T}$ implies $t + \omega \in \mathbf{T}$.

**Definition 2.2.** Let $\mathbf{T}$ be a time scale. For $t \in \mathbf{T}$, we define the forward jump operator $\sigma : \mathbf{T} \to \mathbf{T}$ by $\sigma(t) = \inf\{s \in \mathbf{T} : s > t\}$, and the backward jump operator $\rho : \mathbf{T} \to \mathbf{T}$ by $\rho(t) = \sup\{s \in \mathbf{T} : s < t\}$, while the graininess function $\mu : \mathbf{T} \to [0, +\infty)$ is defined by $\mu(t) = \sigma(t) - t$. If $\sigma(t) > t$, we say that $t$ is right-scattered, while if $\rho(t) < t$ we say that $t$ is left-scattered. Also, if $t < \sup \mathbf{T}$ and $\sigma(t) = t$, then $t$ is called right-dense, and if $t > \inf \mathbf{T}$ and $\rho(t) = t$, then $t$ is called left-dense.

**Definition 2.3.** Assume $f : \mathbf{T} \to \mathbf{R}$ is a function, and let $t \in \mathbf{T}$. Then, we define $f^{\Delta}(t)$ to be the number (provided it exists) with the property that, for any given $\varepsilon > 0$, there is a neighborhood $U$ of $t$, i.e., $U = (t - \delta, t + \delta) \cap \mathbf{T}$ for some $\delta > 0$, such that

$$\big|[f(\sigma(t)) - f(s)] - f^{\Delta}(t)[\sigma(t) - s]\big| \le \varepsilon\, |\sigma(t) - s| \text{ for all } s \in U.$$

In this case, $f^{\Delta}(t)$ is called the delta (or Hilger) derivative of $f$ at $t$. Moreover, $f$ is said to be a delta or Hilger differentiable on $\mathbf{T}$ if $f^{\Delta}(t)$ exists for all $t \in \mathbf{T}$. A function $F : \mathbf{T} \to \mathbf{R}$ is called an antiderivative of $f : \mathbf{T} \to \mathbf{R}$ provided $F^{\Delta}(t) = f(t)$ for all $t \in \mathbf{T}$. Then we define

$$\int_r^s f(t)\Delta t = F(s) - F(r) \text{ for } r, s \in \mathbf{T}.$$

**Definition 2.4.** A function $f : \mathbf{T} \to \mathbf{R}$ is called rd-continuous provided it is continuous at right-dense points in $\mathbf{T}$ and its left-sided limits exist (finite) at left-dense points in $\mathbf{T}$. The set of rd-continuous functions $f : \mathbf{T} \to \mathbf{R}$ will be denoted by $C_{rd}(\mathbf{T})$.

**Lemma 2.1.** *Every* rd-*continuous function has an antiderivative.*

**Lemma 2.2.** *If $a, b \in \mathbf{T}$, $\alpha, \beta \in \mathbf{R}$ and $f, g \in C_{rd}(\mathbf{T})$, then*

(a) $\int_a^b [\alpha f(t) + \beta g(t)]\Delta t = \alpha \int_a^b f(t)\Delta t + \beta \int_a^b g(t)\Delta t$;

(b) *if $f(t) \ge 0$ for all $a \le t < b$, then $\int_a^b f(t)\Delta t \ge 0$;*

(c) *if $|f(t)| \le g(t)$ on $[a, b) := \{t \in T : a \le t < b\}$, then $|\int_a^b f(t)\Delta t| \le \int_a^b g(t)\Delta t$.*

To facilitate the discussion below, we now introduce some notation to be used throughout this paper. Let

$$\kappa = \min\{[0,\infty)\cap \mathbf{T}\}, \qquad I_\omega = [\kappa, \kappa+\omega]\cap \mathbf{T},$$
$$g^M = \sup_{t\in\mathbf{T}} g(t), \qquad g^L = \inf_{t\in\mathbf{T}} g(t), \qquad \bar{g} = \frac{1}{\omega}\int_\kappa^{\kappa+\omega} g(s)\Delta s$$

where $g \in C_{rd}(\mathbf{T})$ is an $\omega$-periodic real function.

Next, for the reader's convenience, we shall summarize in the following a few concepts and results from [**13**] that will come into play later on.

Let $X$ and $Z$ be normed vector spaces, $L : \operatorname{Dom} L \subset X \to Z$ a linear mapping and $N : X \to Z$ a continuous mapping. The mapping $L$ will be called a *Fredholm mapping of index zero*, if $\dim \operatorname{Ker} L = \operatorname{Codim} \operatorname{Im} L < +\infty$ and $\operatorname{Im} L$ is closed in $Z$. If $L$ is a Fredholm mapping of index zero, there exist continuous projectors $P : X \to X$ and $Q : Z \to Z$ such that $\operatorname{Im} P = \operatorname{Ker} L$, $\operatorname{Im} L = \operatorname{Ker} Q = \operatorname{Im}(I - Q)$. It follows that $L|\operatorname{Dom} L \cap \operatorname{Ker} P : (I-P)X \to \operatorname{Im} L$ is invertible. We denote the inverse of that map by $K_P$. If $\Omega$ is an open bounded subset of $X$, the mapping $N$ will be called $L$-compact on $\overline{\Omega}$ if $QN(\overline{\Omega})$ is bounded and $K_P(I-Q)N : \overline{\Omega} \to X$ is compact. Since $\operatorname{Im} Q$ is isomorphic to $\operatorname{Ker} L$, there exists an isomorphism $J : \operatorname{Im} Q \to \operatorname{Ker} L$.

**Lemma 2.3** (continuation theorem). *Let $L$ be a Fredholm mapping of index zero, and let $N$ be $L$-compact on $\overline{\Omega}$. Suppose*

(a) *For each $\lambda \in (0,1)$, every solution $x$ of $Lx = \lambda N(x,\lambda)$ is such that $x \notin \partial\Omega$;*

(b) $QN(x,0) \neq 0$ *for each $x \in \partial\Omega \cap \operatorname{Ker} L$ and*

$$\deg\{JQN(.,0), \Omega \cap \operatorname{Ker} L, 0\} \neq 0.$$

*Then the equation $Lx = Nx$ has at least one solution lying in* $\operatorname{Dom} L \cap \overline{\Omega}$.

In the proof of our main result, we'll use the following lemma which can be found in [**3**, Lemma 1.4].

**Lemma 2.4.** *Let $t_1, t_2 \in I_\omega$ and $t \in \mathbf{T}$. If $g : \mathbf{T} \to \mathbf{R}$ is $\omega$ periodic, then*

$$g(t) \le g(t_1) + \int_\kappa^{\kappa+\omega} |g^\Delta(s)|\Delta s$$

*and*

$$g(t) \geq g(t_2) - \int_{\kappa}^{\kappa+\omega} |g^{\Delta}(s)|\Delta s.$$

**3. Existence of positive periodic solutions.** In this section, we investigate the existence conditions of four periodic solutions for system (1.3), where $a$, $b$, $c$, $r$, $f$, $d_1$, $d_2$, $\tau_1$, $\tau_2 \in C_{rd}(\mathbf{T})$, are all positive periodic functions with period $\omega > 0$, $\tau_1(t), \tau_2(t) \in \mathbf{T}$, for any $t \in \mathbf{T}$, $m$ and $n$ are two positive real constant and $K : [0, \infty) \cap \mathbf{T} \to [0, \infty) \cap \mathbf{T}$ is an $\omega$-periodic function such that $\int_{\kappa}^{\infty} K(s)\Delta s = 1$.

We make the following assumptions:

(A1) $\bar{r}e^{q_1} > 2me^{2\omega\bar{d}_1}\bar{d}_1 e^{p_1}$,

(A2) $\bar{f}e^{q_1} > 2ne^{2\omega\bar{d}_2}\bar{d}_1 e^{p_1}$,

(A3) $\bar{a} > \bar{r}/m^2 + \bar{f}/n^2$,

where $p_1 = \ln \bar{a}/\bar{b} + 2\omega\bar{a}$, $q_1 = \ln\{1/\bar{b}(\bar{a} - (\bar{r}/m^2) - (\bar{f}/n^2))\} - 2\omega\bar{a}$.

For convenience, we also introduce the following notation:

$$l_{\pm} = \frac{1}{2\bar{d}_1 m^2}\left(\bar{r}e^{p_1+2\omega\bar{d}_1} \pm \sqrt{(\bar{r})^2 e^{2p_1+4\omega\bar{d}_1} - 4(\bar{d}_1)^2 m^2 e^{2q_1}}\right);$$
$$u_{\pm} = \frac{1}{2\bar{d}_2 n^2}\left(\bar{f}e^{p_1+2\omega\bar{d}_1} \pm \sqrt{(\bar{f})^2 e^{2p_1+4\omega\bar{d}_1} - 4(\bar{d}_2)^2 n^2 e^{2q_1}}\right);$$
$$\gamma_{\pm} = \frac{1}{2\bar{d}_2 m^2}\left(\bar{r}e^{q_1-2\omega\bar{d}_1} \pm \sqrt{(\bar{r})^2 e^{2p_1-4\omega\bar{d}_1} - 4(\bar{d}_1)^2 m^2 e^{2p_1}}\right);$$
$$\mu_{\pm} = \frac{1}{2\bar{d}_2 n^2}\left(\bar{f}e^{q_1-2\omega\bar{d}_2} \pm \sqrt{(f^L)^2 e^{2q_1-4\omega\bar{d}_1} - 4(\bar{d}_2)^2 n^2 e^{2p_1}}\right);$$
$$x_{\pm} = \frac{\bar{a}}{2\bar{b}\bar{d}_1 m^2}\left(\bar{r} \pm \sqrt{(\bar{r})^2 - 4(\bar{d}_1)^2 m^2}\right);$$
$$y_{\pm} = \frac{\bar{a}}{2\bar{b}\bar{d}_2 n^2}\left(\bar{f} \pm \sqrt{(\bar{f})^2 - 4(\bar{d}_2)^2 n^2}\right).$$

**Lemma 3.1.** *Assume that* (A1)–(A3) *hold. Then*

(3.1)
$$l_- < x_- < \gamma_- < \gamma_+ < x_+ < l_+; \quad u_- < y_- < \mu_- < \mu_+ < y_+ < u_+.$$

*Proof.* It is clear that function $f_1(x,\delta) = x - \sqrt{x^2 - \delta}$ is a decrease function in $x$ for any $\delta > 0$. Noting that (A3) implies $e^{p_1} > \bar{a}/\bar{b} > e^{q_1}$, we have

$$\begin{aligned}
x_- &= \frac{1}{2\bar{d}_1 m^2}\left(\bar{r}\frac{\bar{a}}{\bar{b}} - \sqrt{(\bar{r})^2\left(\frac{\bar{a}}{\bar{b}}\right)^2 - 4(\bar{d}_1)^2 m^2 \left(\frac{\bar{a}}{\bar{b}}\right)^2}\right) \\
&> \frac{1}{2\bar{d}_1 m^2}\left(\bar{r}\frac{\bar{a}}{\bar{b}} - \sqrt{(\bar{r})^2\left(\frac{\bar{a}}{\bar{b}}\right)^2 - 4(\bar{d}_1)^2 m^2 e^{2q_1}}\right) \\
&= \frac{1}{2\bar{d}_1 m^2} f_1\left(\bar{r}\frac{\bar{a}}{\bar{b}}, 4(\bar{d}_1)^2 m^2 e^{2q_1}\right) \\
&> \frac{1}{2\bar{d}_1 m^2} f_1(\bar{r}e^{p_1 + 2\omega\bar{d}_1}, 4(\bar{d}_1)^2 m^2 e^{2q_1}) \\
&= \frac{1}{2\bar{d}_1 m^2}\left(\bar{r}e^{p_1+2\omega\bar{d}_1} - \sqrt{(\bar{r})^2 e^{2p_1+4\omega\bar{d}_1} - 4(\bar{d}_1)^2 m^2 e^{2q_1}}\right) \\
&= l_-.
\end{aligned}$$

Similarly, we can prove $x_- < \gamma_-$. It is obvious that $\gamma_- < \gamma_+$. Noting the fact that $f_2(x) = x + \sqrt{x^2 - \theta}$ is an increase function for any $\theta > 0$ and $e^{p_1} > \bar{a}/\bar{b} > e^{q_1}$, we can obtain $x_+ > \gamma_+$. Similarly, we can prove $x_+ < l_+$. So $l_- < x_- < \gamma_- < \gamma_+ < x_+ < l_+$ hold. A parallel relationship $u_- < y_- < \mu_- < \mu_+ < y_+ < u_+$ can be obtained in a similar way. □

Our main result is stated in the following theorem.

**Theorem 3.1.** *Assume that* (A1), (A2) *and* (A3) *hold. Then system* (1.3) *has at least four positive periodic solutions.*

As a direct corollary of Theorem 3.1, we have the following theorems.

**Theorem 3.2.** *Assume that* (A1), (A2) *and* (A3) *hold, where* $\bar{g} = 1/\omega \int_0^\omega g(t)\,dt$ *for any continuous* $\omega$ *periodic function* $\{g(x)\}$. *Then system* (1.1) *has at least four positive periodic solutions.*

**Theorem 3.3.** *Assume that* (A1), (A2) *and* (A3) *hold where* $\bar{g} = 1/\omega \sum_{k=0}^{\omega-1} g(k)$ *for any* $\omega$ *periodic sequence* $\{g(k)\}$. *Then system* (1.2) *has at least four positive periodic solutions.*

*Remark* 3.1. It is clear that conditions (A1) and (A2) in Theorem 3.2 are weaker than conditions (H1) and (H2) of [**12**].

*Proof of Theorem* 3.1. In order to apply Lemma 2.3 (continuation theorem) to (1.3), we first define

$$X = Z = \Big\{ u = (u_1, u_2, u_3)^T \in C(\mathbf{T}, \mathbf{R}^3) : u(t+\omega) = u(t),\ \forall t \in \mathbf{T} \Big\}$$

and

$$\|u\| = \|(u_1(t), u_2(t), u_3(t))^T\| = \max_{t\in I_\omega} |u_1(t)| + \max_{t\in I_\omega} |u_2(t)| + \max_{t\in I_\omega} |u_3(t)|$$

for any $u \in X$ (or $Z$). Then $X$ and $Z$ are Banach spaces with the norm $\|\cdot\|$. Let

$$N(u,\lambda) = \begin{pmatrix} a(t) - b(t)\int_{-\infty}^{t} K(t-s)e^{u_1(s)}\Delta s - \frac{\lambda r(t)e^{2u_2(t)}}{m^2e^{2u_2(t)}+e^{2u_1(t)}} \\ - \frac{\lambda f(t)e^{2u_3(t)}}{n^2e^{2u_3(t)}+e^{2u_1(t)}} \\ \frac{r(t)e^{u_1(t-\tau_1(t))}e^{u_2(t-\tau_1(t))}}{m^2e^{2u_2(t-\tau_1(t))}+e^{2u_1(t-\tau_1(t))}} - d_1(t), \\ \frac{f(t)e^{u_1(t-\tau_2(t))}e^{u_3(t-\tau_2(t))}}{n^2e^{2u_3(t-\tau_2(t))}+e^{2u_1(t-\tau_2(t))}} - d_2(t) \end{pmatrix}, \quad u \in X,$$

$$\begin{aligned} Lu &= (u_1^{\Delta}, u_2^{\Delta}, u_3^{\Delta})^T, \\ Pu &= (\bar{u}_1, \bar{u}_2, \bar{u}_3)^T, \quad u \in X; \\ Qz &= (\bar{z}_1, \bar{z}_2, \bar{z}_3)^T, \quad z \in Z. \end{aligned}$$

It follows that

$$\begin{aligned} \operatorname{Ker} L &= \{(u_1, u_2, u_3) \in X : (u_1(t), u_2(t), u_3(t)) \\ &\qquad \equiv (h_1, h_2, h_3) \in \mathbf{R}^3,\ \text{for } t \in \mathbf{T}\}, \\ \operatorname{Im} L &= \{z \in Z : \bar{z} = 0\} \end{aligned}$$

is closed in $Z$,

$$\dim \operatorname{Ker} L = 3 = \operatorname{codim} \operatorname{Im} L,$$

and $P$ and $Q$ are continuous projectors such that

$$\operatorname{Im} P = \operatorname{Ker} L, \quad \operatorname{Ker} Q = \operatorname{Im} L = \operatorname{Im}(I - Q).$$

Therefore, $L$ is a Fredholm mapping of index zero. Furthermore, the generalized inverse (to $L$) $K_P : \operatorname{Im} L \to \operatorname{Ker} P \cap \operatorname{Dom} L$ reads

$$K_P(z) = \int_\kappa^t z(s)\Delta s - \frac{1}{\omega}\int_\kappa^{\kappa+\omega}\int_\kappa^t z(s)\Delta s\Delta t.$$

Thus,

$$QNu\bigg(\frac{1}{\omega}\int_\kappa^{\kappa+\omega} F_1(s)\Delta s, \frac{1}{\omega}\int_\kappa^{\kappa+\omega} F_2(s)\Delta s, \frac{1}{\omega}\int_\kappa^{\kappa+\omega} F_3(s)\Delta s\bigg)^T,$$

$$K_P(I-Q)Nu = \begin{pmatrix} \int_\kappa^t F_1(s)\Delta s - 1/\omega\int_\kappa^{\kappa+\omega}\int_\kappa^t F_1(s)\Delta s\Delta t \\ +\Big(t-\kappa-1/\omega\int_\kappa^{\kappa+\omega}(t-\kappa)\Delta t\Big)\overline{F}_1 \\ \int_\kappa^t F_2(s)\Delta s - 1/\omega\int_\kappa^{\kappa+\omega}\int_\kappa^t F_2(s)\Delta s\Delta t \\ +\Big(t-\kappa-1/\omega\int_\kappa^{\kappa+\omega}(t-\kappa)\Delta t\Big)\overline{F}_2 \\ \int_\kappa^t F_3(s)\Delta s - 1/\omega\int_\kappa^{\kappa+\omega}\int_\kappa^t F_3(s)\Delta s\Delta t \\ +\Big(t-\kappa-1/\omega\int_\kappa^{\kappa+\omega}(t-\kappa)\Delta t\Big)\overline{F}_3 \end{pmatrix}$$

where

$$\begin{aligned} F_1(s) &= a(s) - b(s)\int_{-\infty}^s K(s-t)e^{u_1(t)}\Delta t \\ &\qquad - \frac{\lambda r(s)e^{2u_2(s)}}{m^2e^{2u_2(s)}+e^{2u_1(s)}} - \frac{\lambda f(s)e^{2u_3(s)}}{n^2e^{2u_3(s)}+e^{2u_1(s)}}; \\ F_2(s) &= \frac{r(s)e^{u_1(s-\tau_1(s))}e^{u_2(s-\tau_1(t))}}{m^2e^{2u_2(s-\tau_1(t))}+e^{2u_1(s-\tau_1(t))}} - d_1(s); \\ F_3(s) &= \frac{f(s)e^{u_1(s-\tau_2(s))}e^{u_3(s-\tau_2(s))}}{n^2e^{2u_3(s-\tau_2(s))}+e^{2u_1(s-\tau_2(s))}} - d_2(s). \end{aligned}$$

Obviously, $QN$ and $K_P(I-Q)N$ are continuous. It is not difficult to show that $\overline{K_P(I-Q)N(\overline{\Omega})}$ is compact for any open bounded set $\Omega \subset X$ by using the Arzela-Ascoli theorem. Moreover, $QN(\overline{\Omega})$ is clearly bounded. Thus, $N$ is $L$-compact on $\overline{\Omega}$ with any open bounded set $\Omega \subset X$.

Now we reach the position of searching for an appropriate open bounded subset $\Omega$ for the application of the continuation theorem

(Lemma 2.3). Corresponding to the operator equation $Lu = \lambda Nu$, $\lambda \in (0,1)$, we have

$$\begin{aligned} u_1^{\Delta}(t) &= \lambda\Bigg[a(t) - b(t)\int_{-\infty}^{t} K(t-s)e^{u_1(s)}\Delta s \\ &\qquad - \frac{\lambda r(t)e^{2u_2(t)}}{m^2e^{2u_2(t)} + e^{2u_1(t)}} - \frac{\lambda f(t)e^{2u_3(t)}}{n^2e^{2u_3(t)} + e^{2u_1(t)}}\Bigg], \\ u_2^{\Delta}(t) &= \lambda\Bigg[\frac{r(t)e^{u_1(t-\tau_1(t))}e^{u_2(t-\tau_1(t))}}{m^2e^{2u_2(t-\tau_1(t))} + e^{2u_1(t-\tau_1(t))}} - d_1(t)\Bigg], \\ u_3^{\Delta}(t) &= \lambda\Bigg[\frac{f(t)e^{u_1(t-\tau_2(t))}e^{u_3(t-\tau_2(t))}}{n^2e^{2u_3(t-\tau_2(t))} + e^{2u_1(t-\tau_2(t))}} - d_2(t)\Bigg]. \end{aligned} \tag{3.2}$$

Assume that $u(t) = (u_1(t), u_2(t), u_3(t)) \in X$ is a solution of (3.2) for a certain $\lambda \in (0,1)$. Integrating (3.1) from $\kappa$ to $\kappa + \omega$, we obtain

$$\begin{cases} \int_{\kappa}^{\kappa+\omega} \Big\{a(t) - b(t)\int_{-\infty}^{t} K(t-s)e^{u_1(s)}\Delta s \\ \qquad - \frac{\lambda r(t)e^{2u_2(t)}}{m^2e^{2u_2(t)}+e^{2u_1(t)}} - \frac{\lambda f(t)e^{2u_3(t)}}{n^2e^{2u_3(t)}+e^{2u_1(t)}}\Big\}\Delta t = 0, \\ \int_{\kappa}^{\kappa+\omega} \Big\{\frac{r(t)e^{u_1(t-\tau_1(t))}e^{u_2(t-\tau_1(t))}}{m^2e^{2u_2(t-\tau_1(t))}+e^{2u_1(t-\tau_1(t))}} - d_1(t)\Big\}\Delta t = 0, \\ \int_{\kappa}^{\kappa+\omega} \Big\{\frac{f(t)e^{u_1(t-\tau_2(t))}e^{u_3(t-\tau_2(t))}}{n^2e^{2u_3(t-\tau_2(t))}+e^{2u_1(t-\tau_2(t))}} - d_2(t)\Big\}\Delta t = 0. \end{cases}$$

That is,

$$\begin{aligned} \int_{\kappa}^{\kappa+\omega} \Bigg\{b(t)\int_{-\infty}^{t} K(t-s)e^{u_1(s)}\Delta s \\ + \frac{\lambda r(t)e^{2u_2(t)}}{m^2e^{2u_2(t)} + e^{2u_1(t)}} + \frac{\lambda f(t)e^{2u_3(t)}}{n^2e^{2u_3(t)} + e^{2u_1(t)}}\Bigg\}\Delta t = \bar{a}\omega, \end{aligned} \tag{3.3}$$

$$\int_{\kappa}^{\kappa+\omega} \Bigg\{\frac{r(t)e^{u_1(t-\tau_1(t))}e^{u_2(t-\tau_1(t))}}{m^2e^{2u_2(t-\tau_1(t))} + e^{2u_1(t-\tau_1(t))}}\Bigg\}\Delta t = \bar{d}_1\omega, \tag{3.4}$$

and

$$\int_{\kappa}^{\kappa+\omega} \Bigg\{\frac{f(t)e^{u_1(t-\tau_2(t))}e^{u_3(t-\tau_2(t))}}{n^2e^{2u_3(t-\tau_2(t))} + e^{2u_1(t-\tau_2(t))}}\Bigg\}\Delta t = \bar{d}_2\omega. \tag{3.5}$$

From (3.2)–(3.5), we have

$$\begin{aligned}(3.6)\quad &\int_{\kappa}^{\kappa+\omega}|u_1^{\Delta}(t)|\Delta t\\ &=\lambda\int_{\kappa}^{\kappa+\omega}\bigg|a(t)-b(t)\int_{-\infty}^{t}K(t-s)e^{u_1(s)}\Delta s\\ &\qquad-\frac{\lambda r(t)e^{2u_2(t)}}{m^2e^{2u_2(t)}+e^{2u_1(t)}}-\frac{\lambda f(t)e^{2u_3(t)}}{n^2e^{2u_3(t)}+e^{2u_1(t)}}\bigg|\Delta t\\ &<\int_{\kappa}^{\kappa+\omega}a(t)\Delta t+\int_{\kappa}^{\kappa+\omega}\bigg[b(t)\int_{-\infty}^{t}K(t-s)e^{u_1(s)}\Delta s\\ &\qquad+\frac{\lambda r(t)e^{2u_2(t)}}{m^2e^{2u_2(t)}+e^{2u_1(t)}}+\frac{\lambda f(t)e^{2u_3(t)}}{n^2e^{2u_3(t)}+e^{2u_1(t)}}\bigg]\Delta t\\ &=2\bar{a}\omega;\end{aligned}$$

similarly,

(3.7)

$$\begin{aligned}\int_{\kappa}^{\kappa+\omega}|u_2^{\Delta}(t)|\Delta t&=\lambda\int_{\kappa}^{\kappa+\omega}\left|\frac{r(t)e^{u_1(t-\tau_1(t))}e^{u_2(t-\tau_1(t))}}{m^2e^{2u_2(t-\tau_1(t))}+e^{2u_1(t-\tau_1(t))}}-d_1(t)\right|\Delta t\\ &<2\bar{d}_1\omega\end{aligned}$$

and

(3.8)

$$\begin{aligned}\int_{\kappa}^{\kappa+\omega}|u_3^{\Delta}(t)|\Delta t&=\lambda\int_{\kappa}^{\kappa+\omega}\left|\frac{f(t)e^{u_1(t-\tau_1(t))}e^{u_3(t-\tau_2(t))}}{n^2e^{2u_3(t-\tau_2(t))}+e^{2u_1(t-\tau_1(t))}}-d_2(t)\right|\Delta t\\ &<2\bar{d}_2\omega.\end{aligned}$$

We note that $(u_1(t),u_2(t),u_3(t))^T\in X$. Then there exist $\xi_i,\eta_i\in I_\omega$ such that

$$(3.9)\qquad x_i(\xi_i)=\max_{t\in I_\omega}x_i(t),\qquad x_i(\eta_i)=\min_{t\in I_\omega}x_i(t),\quad i=1,2,3.$$

Then, by (3.3) and (3.9), we have

$$\begin{aligned}\bar{a}\omega&\geq e^{u_1(\eta_1)}\int_{\kappa}^{\kappa+\omega}b(t)\int_{-\infty}^{t}K(t-s)\Delta s\Delta t=\bar{b}e^{u_1(\eta_1)}\omega\\ \bar{a}\omega&\leq e^{u_1(\xi_1)}\int_{\kappa}^{\kappa+\omega}b(t)\int_{-\infty}^{t}K(t-s)\Delta s\Delta t-\frac{\bar{r}}{m^2}-\frac{\bar{f}}{n^2}\\ &=\bar{b}e^{u_1(\xi_1)}-\frac{\bar{r}}{m^2}-\frac{\bar{f}}{n^2},\end{aligned}$$

that is,

$$u_1(\eta_1) < \ln \frac{\bar{a}}{\bar{b}}, \quad u_1(\xi_1) > \ln \frac{1}{\bar{b}}\left(\bar{a} - \frac{\bar{r}}{m^2} - \frac{\bar{f}}{n^2}\right). \tag{3.10}$$

The above inequalities, together with (3.6) and Lemma 1.4, lead to

$$u_1(t) \le u_1(\xi_1) + \int_\kappa^{\kappa+\omega} |u_1^\Delta(t)|\Delta t \le \ln \frac{\bar{a}}{\bar{b}} + 2\bar{a}\omega := p_1. \tag{3.11}$$

$$\begin{aligned} u_1(t) &\ge u_1(\xi_1) - \int_\kappa^{\kappa+\omega} |u_1^\Delta(t)|\Delta t \\ &\ge \ln \frac{1}{\bar{b}}\left(\bar{a} - \frac{\bar{r}}{m^2} - \frac{\bar{f}}{n^2}\right) - 2\bar{a}\omega := q_1. \end{aligned} \tag{3.12}$$

Equations (3.4) and (3.9) imply that

$$\bar{d}_1\omega \le \frac{\bar{r}e^{p_1}e^{u_2(\xi_2)}\omega}{m^2e^{2u_2(\eta_2)} + e^{2q_1}}.$$

That is,

$$u_2(\xi_2) \ge \ln\left\{\frac{\bar{d}_1}{\bar{r}e^{p_1}}(m^2e^{2u_2(\eta_2)} + e^{2q_1})\right\}.$$

This, together with (3.7), gives

$$\begin{aligned} u_2(t) &\ge u_2(\xi_2) - \int_\kappa^{\kappa+\omega} |u_2^\Delta(t)|\Delta t \\ &> \ln\left\{\frac{\bar{d}_1}{\bar{r}e^{p_1}}(m^2e^{2u_2(\eta_2)} + e^{2q_1})\right\} - 2\bar{d}_1\omega. \end{aligned}$$

In particular, we have

$$u_2(\eta_2) > \ln\left\{\frac{\bar{d}_1}{\bar{r}e^{p_1}}(m^2e^{2u_2(\eta_2)} + e^{2q_1})\right\} - 2\bar{d}_1\omega$$

or

$$\bar{d}_1m^2e^{2u_2(\eta_2)} - \bar{r}e^{p_1+2\bar{d}_1\omega}e^{u_2(\eta_2)} + \bar{d}_1e^{2q_1} < 0. \tag{3.13}$$

Noting that condition (A1) implies $\bar{r}e^{p_1+2\bar{d}_1\omega} > 2\bar{d}_1 m e^{q_1}$, we have

$$\ln l_- < u_2(\eta_2) < \ln l_+. \tag{3.14}$$

From (3.7) and (3.14), it follows that

$$u_2(t) < u_2(\eta_2) + \int_\kappa^{\kappa+\omega} |u_2^\Delta(t)|\Delta t < \ln l_+ + 2\omega\bar{d}_1 := H_{21}, \ \forall t \in I_\omega. \tag{3.15}$$

From (3.5), we have a parallel argument to (3.13) which gives

$$\bar{d}_2 n^2 e^{2u_3(\eta_3)} - \bar{f}e^{p_1+2\bar{d}_2\omega}e^{u_3(\eta_3)} + \bar{d}_2 e^{2q_1} < 0.$$

Noting that (A2) implies $\bar{r}e^{p_1+2\bar{d}_2\omega} > 2\bar{d}_2 n e^{q_1}$, we have

$$\ln u_- < u_3(\eta_3) < \ln u_+. \tag{3.16}$$

This, together with (3.8), leads to

$$u_3(t) \le u_3(\eta_3) + \int_\kappa^{\kappa+\omega} |u_3^\Delta(t)|\Delta t < \ln u_+ + \bar{d}_2\omega := H_{31}. \tag{3.17}$$

From (3.10), we have

$$|u_1(t) < \max\left\{\left|\ln\frac{\bar{a}}{\bar{b}}\right|, \left|\ln\frac{1}{\bar{b}}\left(\bar{a} - \frac{\bar{r}}{m^2} - \frac{\bar{f}}{n^2}\right)\right|\right\} := R_1.$$

On the other hand, (3.4) implies

$$\frac{\bar{r}e^{q_1}e^{u_2(\eta_2)}}{m^2 e^{2u_2(\xi_2)} + e^{2p_1}} < \bar{d}_1,$$

that is,

$$u_2(\eta_2) < \ln\left(\frac{\bar{d}_1 m^2 e^{2u_2(\xi_2)}}{\bar{r}e^{q_1}} + \frac{\bar{d}_1 e^{2p_1}}{\bar{r}e^{q_1}}\right).$$

Thus,

$$\begin{aligned} u_2(t) &\le u_2(\eta_2) + \int_\kappa^{\kappa+\omega} |u_2^\Delta(t)|\Delta t \\ &< \ln\left(\frac{\bar{d}_1 m^2 e^{2u_2(\xi_2)}}{\bar{r}e^{q_1}} + \frac{\bar{d}_1 e^{2p_1}}{\bar{r}e^{q_1}}\right) + 2\omega\bar{d}_1, \quad \forall t \in I_\omega. \end{aligned}$$

In particular,

$$u_2(\xi_2) < \ln\left(\frac{\bar{d}_1 m^2 e^{2u_2(\xi_2)}}{\bar{r}e^{q_1}} + \frac{\bar{d}_1 e^{2p_1}}{\bar{r}e^{q_1}}\right) + 2\omega\bar{d}_1$$

or

$$\bar{d}_1 m^2 e^{u_2(\xi_2)} - \bar{r}e^{q_1 - 2\omega\bar{d}_1} e^{u_2(\xi_2)} + e^{2p_1}\bar{d}_1 > 0.$$

Thus,

$$u_2(\xi_2) > \ln\gamma_+ \text{ or } u_2(\xi_2) < \ln\gamma_-. \tag{3.18}$$

From (3.5), a parallel argument to (3.18) gives

$$u_3(\xi_3) > \ln\mu_+ \text{ or } u_3(\xi_3) < \ln\mu_-. \tag{2.18}$$

It is clear that $\ln l_\pm$, $\ln\gamma_\pm$, $\ln\mu_\pm$, $\ln u_\pm$, $H_{31}$, $H_{21}$ and $R_1$ are all independent of $\lambda$. Choose a positive number $C$ such that $C > |\ln\bar{a}/\bar{b}|$. Let

$$\begin{aligned}
\Omega_1 = \{&(u_1,u_2,u_3)^T \in X : |u_1(t)| < R_1 + C,\\
&u_2(t) \in (\ln l_-, \ln l_+), u_3(t) \in (\ln\mu_-, \ln\mu_+)\},\\
\Omega_2 = \{&(u_1,u_2,u_3)^T \in X : |u_1(t)| < R_1 + C, u_2(t) \in (\ln l_-, \ln\gamma_-),\\
&\min\{u_3(t)\} \in (\ln u_-, \ln u_+), \max\{u_3(t)\} \in (\ln\mu_+, H_{31})\},\\
\Omega_3 = \{&(u_1,u_2,u_3)^T \in X : |u_1(t)| < R_1 + C, \min\{u_2(t)\} \in (\ln l_-, \ln l_+),\\
&\max\{u_2(t)\} \in (\ln\gamma_+, H_{21}), u_3(t) \in (\ln u_-, \ln\mu_-)\},\\
\Omega_4 = \{&(u_1,u_2,u_3)^T \in X : |u_1(t)| < R_1 + C, \min\{u_2(t)\} \in (\ln l_-, \ln l_+),\\
&\max\{u_2(t)\} \in (\ln\gamma_+, H_{21}), \min\{u_3(t)\} \in (\ln u_-, \ln u_+),\\
&\max\{u_3(t)\} \in (\ln\mu_+, H_{31})\}.
\end{aligned}$$

Then $\Omega_i$, $i = 1,2,3,4$, are bounded open subsets of $X$, and $\Omega_i \cap \Omega_j = \varnothing$, $i \neq j$, $i,j = 1,2,3,4$. So $\Omega_i$ satisfies condition (a) in Lemma 1.3 for $i = 1,2,3,4$.

Now let us consider $QN(0,u)$ with $u = (u_1,u_2,u_3)^T \in R^3$. Note that

$$QN(0,u) = \begin{pmatrix} \bar{a} - \bar{b}e^{u_1} \\ \dfrac{\bar{r}e^{u_1+u_2}}{m^2 e^{2u_2} + e^{2u_1}} - \bar{d}_1 \\ \dfrac{\bar{f}e^{u_1+u_3}}{n^2 e^{2u_3} + e^{2u_1}} - \bar{d}_2 \end{pmatrix}.$$

Because of (A1) and (A2), it is easy to solve that $QN(0,(u_1,u_2,u_3)) = 0$ has four distinct solutions

$$U_1 = \left(\ln\frac{\bar a}{\bar b}, \ln x_-, \ln y_-\right), \qquad U_2 = \left(\ln\frac{\bar a}{\bar b}, \ln x_-, \ln y_+\right),$$
$$U_3 = \left(\ln\frac{\bar a}{\bar b}, \ln x_+, \ln y_-\right), \qquad U_4 = \left(\ln\frac{\bar a}{\bar b}, \ln x_+, \ln y_+\right),$$

and $U_i \in \Omega_i$, $i = 1,2,3,4$. Thus, when $u \in \partial\Omega_i \cap \operatorname{Ker} L = \partial\Omega_i \cap R^3$, $QN(0,u) \neq 0$. Furthermore, in view of the assumptions in Theorem 3.1, a direct calculation produces

$$\deg\{JQN, \Omega_i \cap \operatorname{Ker} L, 0\} = (-1)^{i+1} \neq 0.$$

Here $J$ can be the identity mapping since $\operatorname{Im} P = \operatorname{Ker} L$. By now we have proved that $\Omega_i$ verifies all the requirements of Lemma 3.3. Hence $Lz = Nz$ has four solutions $u^i(t) = (u_1^i(t), u_2^i(t), u_3^i(t)))$, $i = 1,2,3,4$ in $\operatorname{Dom} L \cap \overline{\Omega}_i$, respectively. Obviously, $u^i(t)$ and $u^j(t)$ are different, $i \neq j$, $i,j = 1,2,3,4$, i.e., equation (1.3) has at least four $\omega$-periodic solutions. □

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## REFERENCES

**1.** R. Agarwal and M. Bohner, *Basic calculus on time scales and some of its applications*, Results Math. **35** (1999), 3–22.

**2.** M. Bohner and P.W. Eloe, *Higher order dynamic equations on measure chains*; *Wronskians, disconjugacy and interpolating families of functions*, J. Math. Anal. Appl. **246** (2000), 639–656.

**3.** M. Bohner, M. Fan and J.M. Zhang, *Periodicity of scalar dynamic equations and applications to population models*, J. Math. Anal. Appl. **330** (2007), 1–9.

**4.** M. Bohner, M. Fan, J.M. Zhang and M. Bochner, *Existence of periodic solutions in predator-prey and competition dynamic systems*, Nonlinear Anal.: Real World Appl., in press.

**5.** M. Bohner and A. Peterson, *Dynamic equations on time scales*: *An introduction with applications*, Birkhauser, Boston, 2001.

**6.** F.D. Chen, Xiaoxing Chen, Jinde Cao and Anping Chen, *Positive periodic solutions of a class of non-autonomous single species population model with delays and feedback control*, Acta Math. Appl. Sinica, **21** (2005), 1319–1336.

**7.** Xiaoxing Chen, *Periodicity in a nonlinear discrete predator-Prey system with state dependent delays*, Nonlinear Anal.: Real World Appl., in press.

**8.** L. Erbe and A. Peterson, *Oscillation criteria for second order matrix dynamic equation on time scale*, J. Comput. Appl. Math. **14** (2002), 169–186.

**9.** ———, *Positive solution for a nonlinear differential equation on a measure chain*, Math. Comput. Modeling **32** (2000), 571–585.

**10.** M. Fan and S. Agarwal, *Periodic solutions of nonautonomous discrete predator-prey system of Lotka-Volterra type*, Appl. Anal. **81** (2002), 801–812.

**11.** M. Fan and Q. Wang, *Periodic solutions of a class of nonautonomous discrete time semi-ratio-dependent predator-prey systems*, Discrete Continuous Dynamic Systems **4** (2004), 563–574.

**12.** J.F. Feng and S.H. Chen, *Four periodic solutions of a generalized delayed predator-prey system*, Appl. Math. Computing **181** (2006), 932–939.

**13.** R.E. Gaines and J.L. Mawhin, *Coincidence degree and nonlinear differential equations*, Springer-Verlag, Berlin, 1977.

**14.** S. Hilger, *Analysis on measure chains*—it A unified approach to continuous and discrete calculus, Results Math. **18** (1990), 18–56.

**15.** Eric R. Kaufmann and Youssef N. Raffoul, *Periodic solutions for a neutral nonlinear dynamical equation on a time scale*, J. Math. Anal. Appl. **319** (2006), 315–325.

**16.** B. Kaymakçalan, V. Lakshmikantham and S. Sivasundaram, *Dynamic systems on measure chains*, Kluwer Academic, Dordrecht, 1996.

**17.** Y.K. Li, *Periodic solutions for delay Lotka-Volterra competition systems*, J. Math. Anal. Appl. **246** (2000), 230–244.

**18.** B.G. Zhang and D. Xinghua, *Oscillation of delay differential equations on time scales*, Math. Comput. Modelling **36** (2002), 1307–1318.

**19.** J.D. Zhao and W.C. Chen, *The qualitative analysis of N-species nonlinear prey-competition systems*, Appl. Math. Computing **149** (2004), 567–576.

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