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EXISTENCE OF ALMOST PERIODIC SOLUTIONS FOR IMPULSIVE CELLULAR NEURAL NETWORKS

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ABSTRACT. In this paper we present results on the existence of almost periodic solutions for impulsive neural networks. By means of estimated for the Cauchy matrix sufficient conditions for existence and exponential stability of these equations are obtained.

1. Introduction. Many physical systems are characterized by the fact that at certain moments of time they experience a sudden change of their state. These systems are subject to short-term perturbations which are often assumed to be in the form of impulses in the modeling process. Adequate mathematical models of such processes are the impulsive differential equations in the form

$$\begin{cases} \dot{x}(t) = F(t, x(t)) & t \neq \tau_k, \\ \Delta x(t) = x(t+0) - x(t-0) = I_k(x(t)) & t = \tau_k,\ k \in \mathbf{Z}, \end{cases}$$

where t belongs to the interval $J \subset \mathbf{R}$, $F : J \times \mathbf{R}^n \to \mathbf{R}^n$, the sequence $\{\tau_k\}$ has no finite accumulation point and $I_k : \mathbf{R}^n \to \mathbf{R}^n$.

The theory of these differential equations goes back to the works of Mil'man and Myshkis [**7**]. In recent years impulsive differential equations have been intensively researched (see the monographs of Samoilenko and Perestyuk [**8**] and Lakshmikantham et al. [**6**]). Recently, some qualitative properties (oscillation, asymptotic behavior and stability) are investigated by several authors, see [**3, 9**].

In this paper we obtain some sufficient conditions to ensure that for Hopfield neural networks, see [**5**], with distributed delays and impulses at fixed moments of time where there exist unique almost periodic solutions. It is well known that neural networks have successful applications in many fields such as optimization, associative memory, signal and image processing.

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The main results related to the study of the existence of almost periodic solutions for systems with impulse effects have been obtained in [**1, 2, 10–13**].

2. Preliminary notes. Let $\mathbf{R}^n$ be the n-dimensional Euclidean space with elements $x = \operatorname{col}(x_1, x_2, \dots, x_n)$ and norm $|x| = \max_i\{|x_i|\}$, $\mathbf{R} = (-\infty, \infty)$, $\mathbf{R}_+ = [0, +\infty)$, Ω a domain in $\mathbf{R}^n$, $\Omega \neq \varnothing$.

By B, $B = \{\{\tau_k\}_{k=-\infty}^{\infty} : \ \tau_k \in \mathbf{R},\ \tau_k < \tau_{k+1},\ k \in \mathbf{Z}, \lim_{k\to\pm\infty} \tau_k = \pm\infty\}$ we denote the set of all sequences unbounded and strictly increasing.

We shall investigate the problem of existence of almost periodic solutions of the system of impulsive Hopfield neural networks with distributed delays

(1)
$$\begin{cases} \dot{x}_i(t) = \sum_{j=1}^n a_{ij}(t)x_j(t) + \sum_{j=1}^n \beta_{ij}(t) f_j(\mu_j \int_0^\infty k_{ij}(u)x_j(t-u)\,du) \\ \qquad\quad +\gamma_i(t), \quad t \neq \tau_k,\ i = 1, 2, \dots, n, \\ \Delta x(t) = A_k x(t) + I_k(x(t)) + \gamma_k,\ t = \tau_k, \quad k \in \mathbf{Z}, \end{cases}$$

where

(i) $t \in \mathbf{R}$, $a_{ij}(t), \beta_{ij}(t) \in C(\mathbf{R}, \mathbf{R})$, $f_j(t) \in C(\mathbf{R}, \mathbf{R})$, $\mu_j \in \mathbf{R}_+$, $k_{ij}(t) \in C(\mathbf{R}_+, \mathbf{R}_+)$, $\gamma_i(t) \in C(\mathbf{R}, \mathbf{R})$, $i = 1, 2, \dots, n$, $j = 1, 2, \dots, n$;

(ii) $A_k \in \mathbf{R}^{n\times n}$, $I_k(x) \in C(\Omega, \mathbf{R}^n)$, $\gamma_k \in \mathbf{R}^n$, $\{\tau_k\} \in B$, $k \in \mathbf{Z}$, $\Delta x(t) = x(t+0) - x(t-0)$.

Let $PC(J, \mathbf{R}^n)$, $J \subset \mathbf{R}$, be the space of all piecewise continuous functions $x : J \to \mathbf{R}^n$ with points of discontinuity of first kind τ_k in which it is left continuous, i.e., the following relations hold

$$x(\tau_k - 0) = x(\tau_k),\ x(\tau_k + 0) = x(\tau_k) + \Delta x(\tau_k), \quad k \in \mathbf{Z}.$$

Recall [**3**] it follows that the solution $x(t)$ of (1) is from $PC(J, \mathbf{R}^n)$.

The initial condition associated with (1) is of the form

$$x(t) = \phi_0(t), \quad t \in \mathbf{R}, \tag{2}$$

where $\phi_0(t) \in PC(\mathbf{R}, \mathbf{R}^n)$ is almost periodic function with points of discontinuity of the first kind τ_k, $k \in \mathbf{Z}$.

Since the solutions of (1), (2) are piecewise functions we adopt the following definitions for almost periodicity.

Definition 1 [**8**]. The set of sequences $\{\tau_k^j\}$, $\tau_k^j = \tau_{k+j} - \tau_k$, $k \in \mathbf{Z}$, $j \in \mathbf{Z}$, $\{\tau_k\} \in B$ is said to be *uniformly almost periodic* if for arbitrary $\varepsilon > 0$ there exists a relatively dense set of ε-almost periods common for any sequences.

Definition 2 [**8**]. The function $\varphi \in PC(\mathbf{R}, \mathbf{R}^n)$ is said to be *almost periodic*, if:

a) the set of sequences $\{\tau_k^j\}$, $\tau_k^j = \tau_{k+j} - \tau_k$, $k \in \mathbf{Z}$, $j \in \mathbf{Z}$, $\{\tau_k\} \in B$ is uniformly almost periodic.

b) For any $\varepsilon > 0$ there exists a real number $\delta > 0$ such that if the points t' and t'' belong to one and the same interval of continuity of $\varphi(t)$ and satisfy the inequality $|t' - t''| < \delta$, then $|\varphi(t') - \varphi(t'')| < \varepsilon$.

c) For any $\varepsilon > 0$ there exists a relatively dense set T such that if $\tau \in T$, then $|\varphi(t + \tau) - \varphi(t)| < \varepsilon$ for all $t \in \mathbf{R}$ satisfying the condition $|t - \tau_k| > \varepsilon$, $k \in \mathbf{Z}$.

The elements of T are called *ε-almost periods.*

Together with system (1) we consider the linear system

$$
\begin{cases} \dot{x}(t) = A(t)x(t) & t \neq \tau_k, \\ \Delta x(t) = A_k x(t) & t = \tau_k,\ k \in \mathbf{Z}, \end{cases} \tag{3}
$$

where $t \in \mathbf{R}$, $A(t) = (a_{ij}(t))$, $i = 1, 2, \dots n$, $j = 1, 2, \dots, n$.

We introduce the following conditions:

H1. $A(t) \in C(\mathbf{R}, \mathbf{R}^n)$ and is almost periodic in the sense of Bohr.

H2. $\det(E + A_k) \neq 0$ and the sequence $\{A_k\}$, $k \in \mathbf{Z}$ is almost periodic, $E \in \mathbf{R}^{n\times n}$.

H3. The set of sequences $\{\tau_k^j\}$, $\tau_k^j = \tau_{k+j} - \tau_k$, $k \in \mathbf{Z}$, $j \in \mathbf{Z}$, $\{\tau_k\} \in B$ is uniformly almost periodic and there exists $\theta > 0$ such that $\inf_k \tau_k^1 = \theta > 0$.

Recall [**6**] that if $U_k(t, s)$ is the Cauchy matrix for the system

$$
\dot{x}(t) = A(t)x(t), \quad \tau_{k-1} < t \leq \tau_k, \quad \{\tau_k\} \in B,
$$

then the Cauchy matrix for the system (3) is in the form

$$W(t,s)=\begin{cases} U_k(t,s) \\ \qquad \tau_{k-1}<s\le t\le \tau_k, \\ U_{k+1}(t,\tau_k+0)(E+A_k)U_k(t,s), \\ \qquad \tau_{k-1}<s\le \tau_k<t\le \tau_{k+1}, \\ U_{k+1}(t,\tau_k+0)(E+A_k)U_k(\tau_k,\tau_k+0) \\ \quad \dots(E+A_i)U_i(\tau_i,s), \\ \qquad \tau_{i-1}<s\le \tau_i<\tau_k<t\le \tau_{k+1}. \end{cases}$$

and the solutions of (3) are written in the form

$$x(t;t_0,x_0)=W(t,t_0)x_0.$$

Lemma 1 [**8**]. *Let the following conditions be fulfilled*:

1. *Conditions* H1–H3 *are fulfilled.*

2. *For the Cauchy matrix* $W(t,s)$ *of the system* (3) *there exist positive constants* K *and* λ *such that*

$$|W(t,s)|\le Ke^{-\lambda(t-s)},\quad t\ge s,\quad t,s\in\mathbf{R}.$$

Then, for any $\varepsilon>0$, $t\in\mathbf{R}$, $s\in\mathbf{R}$, $t\ge s$, $|t-\tau_k|>\varepsilon$, $|s-\tau_k|>\varepsilon$, $k\in\mathbf{Z}$ *there exists a relatively dense set* T *of* ε*-almost periods of the matrix* $A(t)$ *and a positive constant* Γ *such that for* $\tau\in T$ *it follows*

$$|W(t+\tau,s+\tau)-W(t,s)|\le \varepsilon\Gamma e^{-(\lambda/2)(t-s)}.$$

Introduce the following conditions:

H4. *The functions* $\beta_{ij}(t)$, $i=1,2,\dots,n$, $j=1,2,\dots,n$, *are almost periodic in the sense of Bohr,and*

$$0<\sup_{t\in\mathbf{R}}|\beta_{ij}(t)|=\overline{\beta}_{ij}<\infty.$$

H5. *The functions* $f_j(t)$ *are almost periodic in the sense of Bohr, and*

$$0<\sup_{t\in\mathbf{R}}|f_j(t)|<\infty,\quad f_j(0)=0,$$

and there exists $L_1 > 0$ such that for $t, s \in \mathbf{R}$

$$\max_j |f_j(t) - f_j(s)| < L_1|t - s|, \quad j = 1, 2, \dots, n.$$

H6. *The functions $k_{ij}(t)$ satisfy*

$$\int_0^\infty k_{ij}(s)\, ds = 1, \qquad \int_0^\infty s k_{ij}(s)\, ds < \infty, \quad i, j = 1, 2, \dots, n.$$

H7. *The functions $\gamma_i(t)$, $i = 1, 2, \dots, n$, are almost periodic in the sense of Bohr, $\{\gamma_k\}_{k\in\mathbf{Z}}$ is an almost periodic sequence and there exists $C_0 > 0$ such that*

$$\max\left\{ \max_i |\gamma_i(t)|, \max_k |\gamma_k| \right\} \le C_0.$$

H8. *The sequence of functions $I_k(x)$ is almost periodic uniformly with respect to $x \in \Omega$, and there exists $L_2 > 0$ such that*

$$|I_k(x) - I_k(y)| \le L_2|x - y|$$

for $k \in \mathbf{Z}$, $x, y \in \Omega$.

Lemma 2 [**8**]. *Let the conditions* H1–H5, H7 *be fulfilled. Then, for each $\varepsilon > 0$ there exist ε_1, $0 < \varepsilon_1 < \varepsilon$ and relatively dense sets T of real numbers and Q of whole numbers, such that the following relations are fulfilled*:

(a) $|A(t + \tau) - A(t)| < \varepsilon$, $t \in \mathbf{R}$, $\tau \in T$;

(b) $|\beta_{ij}(t + \tau) - \beta_{ij}(t)| < \varepsilon$, $t \in \mathbf{R}$, $\tau \in T$, $|t - \tau_k| > \varepsilon$, $k \in \mathbf{Z}$, $i, j = 1, 2, \dots, n$;

(c) $|f_j(t + \tau) - f_j(t)| < \varepsilon$, $t \in \mathbf{R}$, $\tau \in T$, $|t - \tau_k| > \varepsilon$, $k \in \mathbf{Z}$, $j = 1, 2, \dots, n$;

(d) $|A_{k+q} - A_k| < \varepsilon$, $q \in Q$, $k \in \mathbf{Z}$;

(e) $|\gamma_j(t + \tau) - \gamma_j(t)| < \varepsilon$, $t \in \mathbf{R}$, $\tau \in T$, $|t - \tau_k| > \varepsilon$, $k \in \mathbf{Z}$, $j = 1, 2, \dots, n$;

(f) $|\gamma_{k+q} - \gamma_k| < \varepsilon$, $q \in Q$, $k \in \mathbf{Z}$;

(g) $|\overline{\tau}_k^q - \tau| < \varepsilon_1$, $q \in Q$, $\tau \in T$, $k \in \mathbf{Z}$.

Lemma 3 [**8**]. *Let the set of sequences $\{\tau_k^j\}$ be uniformly almost periodic. Then for each $p > 0$ there exists a positive integer N such that on each interval of length p no more than N elements of the sequence $\{\tau_k\}$, i.e.,*

$$i(s,t) \leq N(t-s) + N,$$

where $i(s,t)$ is the number of points τ_k in the interval (s,t).

3. Main results.

Theorem 1. *Let the following conditions be fulfilled*:

1. *Conditions* H1–H8 *are fulfilled.*
2. *The number*

$$r = K\left\{ \max_i \lambda^{-1} L_1 \sum_{j=1}^{n} \overline{\beta}_{ij}\mu_j + \frac{L_2}{1-e^{-\lambda}} \right\} < 1.$$

Then:

1. *There exists a unique almost periodic solution $x(t)$ of* (1).
2. *If the following inequalities hold*

$$1 + KL_2 < e, \quad \lambda - KL_1 \max_i \sum_{j=1}^{n} \overline{\beta}_{ij}\mu_j - N\ln(1+KL_2) > 0,$$

then the solution $x(t)$ is exponentially stable.

Proof of assertion 1. We denote with D, $D \subset PC(\mathbf{R}, \mathbf{R}^n)$ the set of all almost periodic functions $\varphi(t)$ satisfying the inequality $||\varphi|| < \overline{K}$, $||\varphi|| = \sup_{t\in\mathbf{R}} |\varphi(t)|$, $\overline{K} = KC_0((1/\lambda) + (1/1-e^{-\lambda}))$.

Set

$$F(t,x) = \mathrm{col}\left\{F_1(t,x), F_2(t,x), \dots, F_n(t,x)\right\},$$
$$\gamma(t) = \mathrm{col}\,(\gamma_1(t), \gamma_2(t), \dots, \gamma_n(t)),$$

where

$$F_i(t,x) = \sum_{j=1}^{n} \beta_{ij}(t) f_j(\mu_j \int_0^\infty k_{ij}(u) x_j(t-u)\,du), \quad i = 1, 2, \dots, n.$$

Define in D an operator S,

$$(4) \qquad \begin{aligned} S\varphi = &\int_{-\infty}^{t} W(t,s)[F(s,\varphi(s)) + \gamma(s))]\,ds \\ &+ \sum_{\tau_k < t} W(t,\tau_k)[I_k(\varphi(\tau_k),) + \gamma_k], \end{aligned}$$

and subset D^*, $D^* \subset D$,

$$D^* = \left\{ \varphi \in D : ||\varphi - \varphi_0|| \le \frac{r\overline{K}}{1-r} \right\},$$

where

$$\varphi_0 = \int_{-\infty}^{t} W(t,s)\gamma(s)\,ds + \sum_{t_k < t} W(t,\tau_k)\gamma_k.$$

We have

(5)

$$\begin{aligned} ||\varphi_0|| &= \sup_{t\in\mathbf{R}} \left\{ \max_i \left(\int_{-\infty}^{t} |W(t,s)||\gamma_i(s)|\,ds \right) + \sum_{\tau_k < t} |W(t,\tau_k)||\gamma_k| \right\} \\ &\le \sup_{t\in\mathbf{R}} \left\{ \max_i \left(\int_{-\infty}^{t} K e^{-\lambda(t-s)} |\gamma_i(s)|\,ds \right) \right. \\ &\quad \left. + \sum_{\tau_k < t} K e^{-\lambda(t-\tau_k)} |\gamma_k| \right\} \\ &\le K \left(\frac{C_0}{\lambda} + \frac{C_0}{1 - e^{-\lambda}} \right) = \overline{K}. \end{aligned}$$

Then, for arbitrary $\varphi \in D^*$ from (4) and (5) we have

$$||\varphi|| \le ||\varphi - \varphi_0|| + ||\varphi_0|| \le \frac{r\overline{K}}{1-r} + \overline{K} = \frac{\overline{K}}{1-r}.$$

Now we prove that S is self-mapping from D^* to D^*.

For arbitrary $\varphi \in D^*$ it follows

(6)

$$\begin{aligned}||S\varphi - \varphi_0|| = \sup_{t\in\mathbf{R}} \Big\{ \max_i \Big(\int_{-\infty}^t |W(t,s)| \sum_{j=1}^n |\beta_{ij}(s)||f_j \\ \times (\mu_j \int_0^\infty k_{ij}(u)\varphi_j(s-u)\,du)|\,ds\Big) \\ + \sum_{\tau_k<t} |W(t,\tau_k)||I_k(\varphi(\tau_k))|\Big\} \\ \le \Big\{ \max_i \Big(\int_{-\infty}^t Ke^{-\lambda(t-s)} \sum_{j=1}^n \overline{\beta}_{ij} L_1 \mu_j\,ds\Big) \\ + \sum_{\tau_k<t} Ke^{-\lambda(t-\tau_k)} L_2 \Big\} ||\varphi|| \\ \le K\Big\{ \max_i \lambda^{-1} L_1 \sum_{j=1}^n \overline{\beta}_{ij}\mu_j + \frac{L_2}{1-e^{-\lambda}}\Big\}||\varphi|| \\ = r||\varphi|| \le \frac{r\overline{K}}{1-r}.\end{aligned}$$

Let $\tau \in T$, $q \in Q$ where the sets T and Q are determined in Lemma 2. Then

(7)

$$\begin{aligned}||S\varphi(t+\tau) - S\varphi(t)|| \\ \le \sup_{t\in\mathbf{R}} \Big\{ \max_i \Big(\int_{-\infty}^t |W(t+\tau, s+\tau) - W(t,s)| \\ \times \Big| \sum_{j=1}^n \beta_{ij}(s+\tau) f_j\Big(\mu_j \int_0^\infty k_{ij}(u)\varphi_j(s+\tau-u)\,du\Big)\Big| ds \\ + \int_{-\infty}^t |W(t,s)|| \sum_{j=1}^n \beta_{ij}(s+\tau) f_j \\ \times \Big(\mu_j \int_0^\infty k_{ij}(u)\varphi_j(s+\tau-u)\,du\Big)\end{aligned}$$

$$
\begin{aligned}
&- \sum_{j=1}^{n} \beta_{ij}(s) f_j\Big(\mu_j \int_0^{\infty} k_{ij}(u)\varphi_j(s-u)\,du\Big)|\,ds\Big) \\
&+ \sum_{\tau_k<t} |W(t+\tau,\tau_{k+q}) - W(t,\tau_k)||I_{k+q}(\varphi(\tau_{k+q}))| \\
&+ \sum_{\tau_k<t} |W(t,\tau_k)||I_{k+q}(\varphi(\tau_{k+q}) - I_k(\varphi(\tau_k)))|\Big\} \le \varepsilon C_1
\end{aligned}
$$

where

$$
C_1 = \frac{L_1}{\lambda}\bigg(\max_i \bigg(\sum_{j=1}^{n}(2\Gamma + K)\overline{\beta}_{ij}\mu_j\bigg) + K\bigg) + \frac{L_2 \Gamma N}{1-e^{-\lambda}}.
$$

From (6) and (7) we obtain that $S\varphi \in D^*$.

Let $\varphi \in D^*$, $\psi \in D^*$. We get

(8)

$$
\begin{aligned}
||S\varphi - S\psi|| \le \sup_{t\in\mathbf{R}} \Big\{ &\max_i \bigg(\int_{-\infty}^{t} |W(t,s)| \bigg[\sum_{j=1}^{n} |\beta_{ij}(s)| \bigg| f_j \\
&\times \Big(\mu_j \int_0^{\infty} k_{ij}(u)\varphi_j(s-u)\,du\Big) \\
&- f_j\Big(\mu_j \int_0^{\infty} k_{ij}(u)\psi_j(s-u)\,du\Big)\bigg|\bigg]\,ds\bigg) \\
&+ \sum_{\tau_k<t} |W(t,\tau_k)||I_k(\varphi(\tau_k)) - I_k(\psi(\tau_k))|\Big\} \\
\le K\Big(&\max_i \Big(\lambda^{-1} L_1 \sum_{j=1}^{n} \overline{\beta}_{ij}\mu_j\Big) + \frac{L_2}{1-e^{-\lambda}}\Big)||\varphi - \psi|| \\
= r||&\varphi - \psi||.
\end{aligned}
$$

Then from (8) it follows that S is contracting operator in D^*. So there exists a unique almost periodic solution of (1)

Proof of assertion 2. Let $y(t)$ be an arbitrary solution of (1) with initial condition $y(t_0 + 0, t_0, \varpi_0) = \varpi_0$, $\varpi_0 \in PC(t_0)$. Then from (3) we obtain

$$\begin{aligned} y(t)-x(t) &= W(t,t_0)(\varpi_0-\varphi_0) \\ &\quad + \int_{t_0}^{t} W(t,s)[F(s,y(s))-F(s,x(s))]\,ds \\ &\quad + \sum_{t_0<\tau_k<t} W(t,\tau_k)[I_k(y(\tau_k)-I_k(x(\tau_k))]. \end{aligned}$$

Then

$$\begin{aligned} |y(t)-x(t)| &\le Ke^{-\lambda(t-t_0)}|\varpi_0-\varphi_0| \\ &\quad + \max_i \Big(\int_{t_0}^{t} Ke^{-\lambda(t-s)} L_1 \sum_{j=1}^{n} \overline{\beta}_{ij}\mu_j |y_i(s)-x_i(s)|\,ds \Big) \\ &\quad + \sum_{t_0<\tau_k<t} Ke^{-\lambda(t-\tau_k)} L_2 |y(\tau_k)-x(\tau_k)|. \end{aligned}$$

Set $u(t) = |y(t)-x(t)|e^{\lambda t}$ and from Gronwall-Bellman's lemma [**8**] we have

$$\begin{aligned} |y(t)-x(t)| &\le K|\varpi_0-\varphi_0|(1+KL_2)^{i(t_0,t)} \\ &\quad \times \exp\Big(-\lambda + KL_1 \max_i \sum_{j=1}^{n} \overline{\beta}_{ij}\mu_j\Big)(t-t_0). \end{aligned}$$

Thus the proof of Theorem 1 is complete. □

We note that the main inequalities which are used in the proof of Theorem 1 are connecting with the properties of the matrix $W(t,s)$ for system (3). Now we will consider some special cases in which these properties are accomplished.

Together with system (1) we shall consider the following systems of impulsive differential equations with perturbations on the linear part.

$$(9)\qquad \begin{cases} \dot{x}_i(t) = \sum_{j=1}^{n}(a_{ij}(t)+p_{ij}(t))x_j(t) \\ +\sum_{j=1}^{n}\beta_{ij}(t)f_j(\mu_j\int_0^\infty k_{ij}(u)x_j(t-u)du)+\gamma_i(t), \\ \qquad t\ne\tau_k, \quad i=1,2,\dots,n, \\ \Delta x(t) = (A_k+P_k)x(t)+I_k(x(t))+\gamma_k \quad t=\tau_k,\ k\in\mathbf{Z}, \end{cases}$$

where $P(t) = (p_{ij}(t)) \in C(\mathbf{R}, \mathbf{R}^n)$, $i = 1, 2, \dots, n$, $j = 1, 2, \dots, n$, $P_k \in \mathbf{R}^{n\times n}$, $k \in \mathbf{Z}$, and let $Q(t, s)$ denote the Cauchy matrix for system

$$\begin{cases} \dot{x}(t) = (A(t) + P(t))x(t) & t \neq \tau_k, \\ \Delta x(t) = (A_k + P_k)x(t) & t = \tau_k,\ k \in \mathbf{Z}, \end{cases} \tag{10}$$

Introduce the following conditions.

H9. $P(t) \in C(\mathbf{R}, \mathbf{R}^n)$ and is almost periodic in the sense of Bohr.

H10. $\det(E + P_k) \neq 0$ and the sequence $\{P_k\}_{k\in\mathbf{Z}}$ is almost periodic.

Lemma 4 [**5**]. *Let conditions* H9 *and* H10 *be fulfilled*:

Then there exist positive constants d_1 *and* d_2 *such that*

$$\sup_{t\in(t_0,\infty)} |P(t)| < d_1, \qquad \sup_{\tau_k\in(t_0,\infty)} |P_k| < d_2.$$

Lemma 5 [**12**]. *Let the following conditions be fulfilled*:

1. *The conditions* H1–H3, H9, H10 *are fulfilled.*
2. *The following relation holds true*

$$|W(t, s)| \leq Ke^{-\lambda(t-s)}, \quad s < t, \quad t, s \in \mathbf{R},$$

where $K \leq 1$, $\lambda = \text{const} > 0$.

Then

1. *If*

$$\nu = -\lambda + Kd + N(1 + Kd) > 0,$$

where $d = \max(d_1, d_2)$*, it follows that for each* $\varepsilon > 0$*,* $t \in \mathbf{R}$*,* $s \in \mathbf{R}$ *there exists a relatively dense set* T *of* ε-almost periods, common for $A(t)$ *and* $P(t)$ *such that for each* $\tau \in T$ *the following inequality holds true*

$$|Q(t + \tau, s + \tau) - Q(t, s)| < \varepsilon\overline{\Gamma}e^{-(\nu/2)(t-s)}, \tag{11}$$

where $\overline{\Gamma} = 1/\lambda 2Ke^{N\ln(1+Kd)}(1 + N + (Nd/\lambda))$.

2. *If*

$$\int_{t_0}^{\infty} |P(\theta)|\, d\theta + \sum_{t_0<\tau_k} |P_k| \leq H,$$

where $H > 0$ *it follows that for each* $\varepsilon > 0$, $t \in \mathbf{R}$, $s \in \mathbf{R}$, *there exists a relatively dense set* T *of* ε*-almost periods, common for* $A(t)$ *and* $P(t)$ *such that for each* $\tau \in T$ *the following inequality holds true*

$$|Q(t+\tau, s+\tau) - Q(t,s)| < \varepsilon\overline{\Gamma}e^{-(\lambda/2)(t-s)}, \tag{12}$$

where $\overline{\Gamma} = 2/\lambda K e^{KH}(1 + N + (2N/\lambda))$.

Theorem 2. *Let the following conditions be fulfilled*:

1. *The conditions* H1–H11 *are fulfilled.*
2. *For the system* (1) *there exists a unique almost periodic solution.*

Then there exists a constant d_0 *such that for* $d \in (0, d_0]$ *for the system* (9), *there exists a unique almost periodic solution.*

Proof. Recall [**9**] it follows that for the matrix $Q(t,s)$ we have

$$|Q(t,s)| \leq Ke^{-(\lambda - Kd)(t-s)+i(s,t)}.$$

Then the proof follows immediately from Lemma 4, (11) and the proof of Theorem 1. □

Theorem 3. *Let the following conditions be fulfilled*:

1. *The conditions* H1–H11 *are fulfilled.*
2. *For the system* (1), *there exists a unique almost periodic solution.*
3. *There exists a constant* $H > 0$, *such that*

$$\int_{t_0}^{\infty} |P(\theta)|\, d\theta + \sum_{t_0<\tau_k} |P_k| < H.$$

Then there exists a constant $H_0 > 0$ *such that for* $H \in (0, H_0]$ *for the system* (9) *there exists a unique almost periodic solution.*

Proof. Recall [**9**] it follows that for the matrix $Q(t,s)$ now we have

$$|Q(t,s)| \leq Ke^{KH}e^{-(\lambda-KH)(t-s)}.$$

Then the proof follows immediately from Lemma 4, (12) and the proof of Theorem 1. □

Corollary 1. *Let the following conditions be fulfilled*:

1. *The conditions* H1–H8 *are fulfilled.*
2. *For the system* (1) *there exists a unique almost periodic solution.*
3. $P(t) = P$, $P_k = \Lambda$, *where* P *and* Λ *are a constant matrix such that*

$$|P| + |\Lambda| \leq \overline{d}, \quad \overline{d} > 0.$$

Then there exists a constant $d_0 > 0$, $d_0 \leq \overline{d}$ *such that for* $d \in (0, d_0]$ *for the system* (9), *there exists a unique almost periodic solution.*

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