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A GENERALIZATION OF WOLSTENHOLME'S HARMONIC SERIES CONGRUENCE

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ABSTRACT. Let A, B be two nonzero integers. Define the Lucas sequences $\{u_n\}_{n=0}^{\infty}$ and $\{v_n\}_{n=0}^{\infty}$ by

$$u_0 = 0, \quad u_1 = 1, \quad u_n = Au_{n-1} - Bu_{n-2} \text{ for } n \geq 2$$

and

$$v_0 = 2, \quad v_1 = A, \quad v_n = Av_{n-1} - Bv_{n-2} \text{ for } n \geq 2.$$

For any $n \in \mathbf{Z}^+$, let w_n be the largest divisor of u_n prime to $u_1, u_2, \dots, u_{n-1}$. We prove that for any $n \geq 5$

$$\sum_{j=1}^{n-1} \frac{v_j}{u_j} \equiv \frac{(n^2-1)\Delta}{6} \cdot \frac{u_n}{v_n} \pmod{w_n^2},$$

where $\Delta = A^2 - 4B$.

1. Introduction. Let A, B be two nonzero integers. Define the Lucas sequence $\{u_n\}_{n=0}^{\infty}$ by

$$u_0 = 0, \quad u_1 = 1 \quad \text{and} \quad u_n = Au_{n-1} - Bu_{n-2} \quad \text{for } n \geq 2.$$

Also its companion sequence $\{v_n\}_{n=0}^{\infty}$ is given by

$$v_0 = 2, \quad v_1 = A \quad \text{and} \quad v_n = Av_{n-1} - Bv_{n-2} \quad \text{for } n \geq 2.$$

Let $\Delta = A^2 - 4B$ be the discriminant of $\{u_n\}_{n=0}^{\infty}$ and $\{v_n\}_{n=0}^{\infty}$. It is easy to show that

$$v_n = \alpha^n + \beta^n$$

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and

$$u_n = \sum_{j=0}^{n-1} \alpha^j \beta^{n-1-j} = \begin{cases} n\alpha^{n-1} & \text{if } \Delta = 0, \\ (\alpha^n - \beta^n)/(\alpha - \beta) & \text{otherwise,} \end{cases}$$

where

$$\alpha = \frac{1}{2}(A + \sqrt{\Delta}), \qquad \beta = \frac{1}{2}(A - \sqrt{\Delta}).$$

Let $p \geq 5$ be a prime. The well-known Wolstenholme's harmonic series congruence asserts that

$$\sum_{j=1}^{p-1} \frac{1}{j} \equiv 0 \pmod{p^2}. \tag{1.1}$$

In [**3**], Kimball and Webb proved a generalization of (1.1) involving the Lucas sequences. Let r be the rank of apparition of p in the sequence $\{u_n\}_{n=0}^{\infty}$, i.e., r the least positive integer such that $p \mid u_r$. Kimball and Webb showed that

$$\sum_{j=1}^{r-1} \frac{v_j}{u_j} \equiv 0 \pmod{p^2} \tag{1.2}$$

provided that $\Delta = 0$ or $r = p \pm 1$.

In this paper we will extend the result of Kimball and Webb to arbitrary Lucas sequences. For any positive integer n, let w_n be the largest divisor of u_n prime to $u_1, u_2, \ldots, u_{n-1}$. Here w_n was firstly introduced by Hu and Sun [**2**] in an extension of the Lucas congruence for Lucas's u-nomial coefficients.

Theorem 1.1. *Let $n \geq 5$ be a positive integer. Then*

$$\sum_{j=1}^{n-1} \frac{v_j}{u_j} \equiv \frac{(n^2-1)\Delta}{6} \cdot \frac{u_n}{v_n} \pmod{w_n^2}. \tag{1.3}$$

It is easy to check that either all u_n are odd when $n \geq 1$, or one of $u_2 = A$ and $u_3 = A^2 - B$ is even. So w_n is odd for any $n > 3$. Also we can verify that either u_n is prime to 3 for each $n \geq 1$, or 3 divides one of u_2, u_3 and $u_4 = A^3 - 2AB$. Hence, $3 \nmid w_n$ provided that $n > 4$. Finally, we mention that w_n is always prime to v_n when $n \geq 3$. Indeed, since

$$u_n = Au_{n-1} - Bu_{n-2} \quad \text{and} \quad (w_n, Au_{n-1}) = (w_n, u_2 u_{n-1}) = 1,$$

we have w_n and B are co-prime. And, from

$$v_n = u_{n+1} - Bu_{n-1} = Au_n - 2Bu_{n-1},$$

it follows that $(w_n, v_n) = (w_n, 2Bu_{n-1}) = 1$.

The Fibonacci numbers $F_0, F_1, \ldots$ are given by

$$F_0 = 0,\ F_1 = 1 \quad \text{and} \quad F_n = F_{n-1} + F_{n-2} \quad \text{for } n \geq 2.$$

And the Lucas numbers $L_0, L_1, \ldots$ are given by

$$L_0 = 2,\ L_1 = 1 \quad \text{and} \quad L_n = L_{n-1} + L_{n-2} \quad \text{for } n \geq 2.$$

Then, by Theorem 1.1, we immediately have

Corollary 1.2. *Let $p \geq 5$ be a prime. Let n be the least positive integer such that $p \mid F_n$. Then we have*

$$\sum_{j=1}^{n-1} \frac{L_j}{F_j} \equiv \frac{5(n^2-1)}{6} \cdot \frac{F_n}{L_n} \pmod{p^2}. \tag{1.4}$$

The proof of Theorem 1.1 will be given in the next section.

2. Proof of Theorem 1.1. For any $n \in \mathbf{N}$, the q-integer $[n]_q$ is given by

$$[n]_q = \frac{1-q^n}{1-q} = 1 + q + \cdots + q^{n-1}.$$

Now we consider $[n]_q$ as the polynomial in the variable q. Recently Shi and Pan [**5**] established a q-analogue of (1.1) for prime $p \geq 5$:

$$\sum_{j=1}^{p-1} \frac{1}{[j]_q} \equiv \frac{p-1}{2}(1-q) + \frac{p^2-1}{24}(1-q)^2[p]_q \pmod{[p]_q^2}. \tag{2.1}$$

Let

$$\Phi_n(q) = \prod_{\substack{1 \le k \le n \\ (k,n)=1}} (q - \zeta_n^k)$$

be the nth cyclotomic polynomial, where $\zeta_n = e^{2\pi i/n}$. We know that $\Phi_n(q)$ is a polynomial with integral coefficients, and clearly $\Phi_n(q)$ is prime to $[j]_q$ for each $1 \le j < n$. Indeed, using a similar method, we can easily extend (2.1) as follows:

Proposition 2.1. *Let n be a positive integer. Then*

(2.2)

$$24\sum_{j=1}^{n-1} \frac{1}{[j]_q} \equiv 12(n-1)(1-q) + (n^2-1)(1-q)^2[n]_q \pmod{\Phi_n(q)^2}.$$

For the proof of (2.1) and (2.2), the reader may refer to [**5**]. From (2.2), we deduce that

$$\begin{aligned} 12\sum_{j=1}^{n-1} \frac{1+q^j}{[j]_q} &= 12\sum_{j=1}^{n-1} \frac{2-(1-q^j)}{[j]_q} \\ &= 24\sum_{j=1}^{n-1} \frac{1}{[j]_q} - 12(n-1)(1-q) \\ &\equiv (n^2-1)(1-q)^2[n]_q \pmod{\Phi_n(q)^2}. \end{aligned}$$

And the above congruence can be rewritten as

$$\left(12\sum_{j=1}^{n-1} \frac{1+q^j}{[j]_q} - (n^2-1)(1-q)(1-q^n)\right) \prod_{j=1}^{n-1} [j]_q \equiv 0 \pmod{\Phi_n(q)^2}.$$

Since $\Phi_n(q)$ is a primitive polynomial, by Gauss's lemma, cf. [**4**, Chapter IV, Theorem 2.1 and Corollary 2.2], there exists a polynomial $G(q)$ with integral coefficients such that

$$\left(12\sum_{j=1}^{n-1} \frac{1+q^j}{[j]_q} - (n^2-1)(1-q)(1-q^n)\right) \prod_{j=1}^{n-1} [j]_q = G(q)\Phi_n(q)^2. \tag{2.3}$$

Proof of Theorem 1.1. When $\Delta = 0$, the theorem reduces to Wolstenholme's congruence (1.1). So below we assume that $\Delta \neq 0$, i.e., $\alpha \neq \beta$. Let p be a prime with $p \mid w_n$, and let m be the integer such that $p^m \mid w_n$ but $p^{m+1} \nmid w_n$. Obviously, we only need to show that

$$\sum_{j=1}^{n-1} \frac{v_j}{u_j} \equiv \frac{(n^2-1)\Delta}{6} \cdot \frac{u_n}{v_n} \pmod{p^{2m}}$$

for each such p and m.

Let $\mathbf{K} = \mathbf{Q}(\sqrt{\Delta})$, and let $\mathcal{O}_{\mathbf{K}}$ be the ring of algebraic integers in $\mathbf{K}$. Clearly $\alpha, \beta \in \mathcal{O}_{\mathbf{K}}$.

Let (p) denote the ideal generated by p in $\mathcal{O}_{\mathbf{K}}$. We know that if

$$\left(\frac{\Delta}{p}\right) = -1,$$

then (p) is prime in $\mathcal{O}_{\mathbf{K}}$, where

$$\left(\frac{\cdot}{p}\right)$$

is the Legendre symbol. Also, there exist two distinct prime ideals $\mathfrak{p}$ and $\mathfrak{p}'$ such that $(p) = \mathfrak{p}\mathfrak{p}'$ provided that

$$\left(\frac{\Delta}{p}\right) = 1.$$

Finally, when $p \mid \Delta$, (p) is the square of a prime ideal $\mathfrak{p}$. The reader can find the details in [**1**]. Let

$$\mathfrak{P} = \begin{cases} (p) & \text{if } (\Delta/p) = -1 \quad \text{or} \quad 0, \\ \mathfrak{p} & \text{if } (\Delta/p) = 1. \end{cases}$$

Obviously, either α or β is prime to $\mathfrak{P}$, otherwise we must have $\mathfrak{P}$ is not prime to u_j for any $j \geq 2$, which implies that $p \mid u_j$. Without loss of generality, we may assume that β is prime to $\mathfrak{P}$.

Lemma 2.2. *Let p be a prime and $k \in \mathbf{Z}$. Suppose that*

$$\left(\frac{\Delta}{p}\right) = 1$$

and $(p) = \mathfrak{p}\mathfrak{p}'$. Then for any $m \in \mathbf{Z}^+$, $\mathfrak{p}^m \mid k$ implies that $p^m \mid k$.

Proof. Observe that $\sigma : \sqrt{\Delta} \mapsto -\sqrt{\Delta}$ is an automorphism over $\mathbf{K}$. Also we know that $\sigma(\mathfrak{p}) = \mathfrak{p}'$. Hence,

$$\mathfrak{p}'^m = \sigma(\mathfrak{p}^m) \mid \sigma(k) = k.$$

Since $\mathfrak{p}$ and $\mathfrak{p}'$ are distinct prime ideals, by the unique factorization theorem, we have $(p)^m = \mathfrak{p}^m\mathfrak{p}'^m$ divides k. □

Now it suffices to prove that

$$\sum_{j=1}^{n-1} \frac{v_j}{u_j} \equiv \frac{(n^2-1)\Delta}{6} \cdot \frac{u_n}{v_n} \pmod{\mathfrak{P}^{2m}}.$$

For any $l \in \mathbf{Z}^+$, let

$$\Phi_l(\alpha, \beta) = \beta^{\varphi(l)}\Phi_l(\alpha/\beta) = \prod_{\substack{1 \leq d \leq l \\ (d,l)=1}} (\alpha - \zeta_l^d \beta),$$

where φ is the Euler totient function. Apparently, $\Phi_l(\alpha, \beta) \in \mathcal{O}_{\mathbf{K}}$. Notice that

$$u_l = \frac{\alpha^l - \beta^l}{\alpha - \beta} = \prod_{\substack{1<d \\ d|l}} \beta^{\varphi(d)}\Phi_d(\alpha/\beta) = \prod_{\substack{1<d \\ d|l}} \Phi_d(\alpha, \beta).$$

Hence u_l is always divisible by $\Phi_l(\alpha, \beta)$. Then we have w_n divides

$$\Phi_n(\alpha, \beta) = \frac{u_n}{\prod_{\substack{1<d<n \\ d|n}} \Phi_d(\alpha, \beta)}$$

since w_n is prime to u_d whenever $1 \leq d < n$.

Substituting α/β for q in (2.3), and noting that

$$u_j = \beta^{j-1}\frac{1-(\alpha/\beta)^j}{1-\alpha/\beta} = \beta^{j-1}[j]_{\alpha/\beta} \text{ and } v_j = \beta^j(1 + (\alpha/\beta)^j),$$

we obtain that

$$\left(12\beta^{-1}\sum_{j=1}^{n-1} \frac{v_j}{u_j} - (n^2-1)\beta^{-n-1}(\alpha-\beta)^2 u_n\right) \prod_{j=1}^{n-1} \beta^{1-j} u_j$$
$$= \beta^{-2\varphi(n)} G(\alpha, \beta)\Phi_n(\alpha, \beta)^2.$$

As $(w_n, 6) = 1$ and $\mathfrak{P}$ is prime to β, we conclude that

$$\sum_{j=1}^{n-1} \frac{v_j}{u_j} - \frac{(n^2-1)\Delta}{12}\beta^{-n}u_n \equiv 0 \pmod{(\mathfrak{P}^m)^2}.$$

Finally, since

$$\alpha^n = \frac{1}{2}(v_n + u_n\sqrt{\Delta}) \quad \text{and} \quad \beta^n = \frac{1}{2}(v_n - u_n\sqrt{\Delta}),$$

we have

$$\alpha^n \equiv \beta^n \equiv v_n/2 \pmod{w_n}.$$

All is done. □

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