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COMPOSITIONS OF PROJECTIONS IN BANACH SPACES AND RELATIONS BETWEEN APPROXIMATION PROPERTIES

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ABSTRACT. A necessary and sufficient condition for existence of a Banach space with a finite dimensional decomposition but without the π-property in terms of norms of compositions of projections is found.

The problem of existence of Banach spaces with the π-property but without a finite dimensional decomposition is one of the well-known open problems in Banach space theory. It was first studied by Johnson [**3**]. Casazza and Kalton [**2**] found important connections of this problem with other problems of Banach space theory. See in this connection the survey [**1**].

Recall the definitions. A separable Banach space X has the *π-property* if there is a sequence $T_n : X \to X$ of finite-dimensional projections such that, for all $x \in X$,

$$\lim_{n\to\infty} ||x - T_n x|| = 0.$$

If in addition the projections satisfy, for all $n, m \in \mathbf{N}$,

$$T_n T_m = T_{\min(m,n)},$$

then X has a *finite-dimensional decomposition.*

Problem 1. *Does every separable Banach space with the π-property have a finite-dimensional decomposition?*

The purpose of this paper is to find an equivalent reformulation of Problem 1 in terms of norms of compositions of projections. In the second part of the paper we discuss related problems on compositions of projections.

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The *relative projection constant* of a finite-dimensional subspace Y in a normed space X is defined by

$$\lambda(Y, X) = \inf\{||P|| : \ P : X \to X \text{ is a projection onto } Y\}.$$

In the case when $X = L_\infty(\mu)$, the constant $\lambda(Y, X)$ is also denoted $\lambda(Y)$ (it is well known that $\lambda(Y, L_\infty(\mu))$ depends on Y only, and not on the way in which Y is embedded into $L_\infty(\mu)$).

Theorem 1. *A separable Banach space X is a space with the π-property but without a finite-dimensional decomposition if and only if there exists an increasing sequence $\{X_i\}_{i=1}^\infty$ of finite-dimensional subspaces of X satisfying the conditions*:

(a) $\sup_i \lambda(X_i, X) < \infty$,

(b) $\operatorname{cl}(\cup_{i=1}^\infty X_i) = X$,

(c) *For every subsequence $\{X_{i_n}\}_{n=1}^\infty \subset \{X_i\}_{i=1}^\infty$ and every sequence $\{P_n\}_{n=1}^\infty$ of projections, $P_n : X_{i_{n+1}} \to X_{i_n}$, the following is true*:

$$\sup_{\substack{k,l \in \mathbf{N} \\ k<l}} ||P_k P_{k+1} \cdots P_{l-1} P_l|| = \infty. \tag{1}$$

Proof. The "only if" part of the theorem is a slight modification of Theorem 3 from Johnson [**3**]. We sketch its proof for convenience of the reader. Let X be a separable Banach space with the π-property but without a finite-dimensional decomposition. Using the standard perturbation argument (see, for example, [**4**]) we get that there exists an increasing sequence $\{X_i\}_{i=1}^\infty$ of finite-dimensional subspaces of X satisfying the conditions (a) and (b). Suppose that $\{X_i\}_{i=1}^\infty$ does not satisfy (c). Then there exists a subsequence $\{X_{i_n}\}_{n=1}^\infty \subset \{X_i\}_{i=1}^\infty$ and a sequence $\{P_n\}$ of projections; $P_n : X_{i_{n+1}} \to X_{i_n}$ such that

$$\sup_{\substack{k,l \in \mathbf{N} \\ k<l}} ||P_k P_{k+1} \cdots P_{l-1} P_l|| < \infty. \tag{2}$$

Let us define operators $T_k^n : X_{i_k} \to X_{i_n}$ by $T_k^n x = P_n P_{n+1} \cdots P_{k-1} x$ for $k > n$, $k, n \in \mathbf{N}$. Then the sequence $\{T_k^n x\}_{k=n+1}^\infty$ is eventually constant for every $x \in \cup_{n=1}^{\infty} X_{i_n}$. The inequality (2) implies that the sequence $\{T_k^n\}_{k=n+1}^{\infty}$ is uniformly bounded. Hence, it is strongly convergent. We denote its strong limit by T_n. It is easy to see that T_n is a continuous projection onto X_{i_n}. Therefore, $T_i T_j = T_j$ for $i \geq j$. Now let $i < j$. We have

$$T_i T_j x = s - \lim_{m \to \infty} (P_i \cdot \ldots \cdot P_{m-1})(T_j x) = P_i \cdot \ldots \cdot P_{j-1}(T_j x) = T_i x.$$

Hence, X has a finite-dimensional decomposition, contrary to the assumption.

We turn to the "if" part of the theorem. We assume that X contains an increasing sequence $\{X_i\}_{i=1}^{\infty}$ of finite dimensional subspaces satisfying the conditions (a)–(c). It is clear that X has the π-property. In order to show that X does not have a finite-dimensional decomposition, assume the contrary. Then X contains an increasing sequence $\{Z_i\}_{i=1}^{\infty}$ of finite-dimensional subspaces, such that

$$\mathrm{cl}\left(\bigcup_{i=1}^{\infty} Z_i\right) = X,$$

and there exist pairwise commuting projections $T_i : X \to Z_i$ with $\operatorname{im} T_i = Z_i$, for which $\sup_i ||T_i|| < \infty$.

We need the following analogue of [**5**, Proposition 1.a.9 (i)] for finite-dimensional decompositions (it can be proved using the same argument), see [**5**, Section 1.g] for terminology related to finite-dimensional decompositions.

Proposition 1. *Let $\{W_i\}_{i=1}^{\infty}$ be a finite-dimensional decomposition of X with the decomposition constant K. Let $E_i : W_i \to X$ be linear operators satisfying $||E_i w - w|| \leq \varepsilon_i ||w||$ for each $w \in W_i$, where $\varepsilon_i > 0$ are such that $\sum_{i=1}^{\infty} \varepsilon_i < 1/(2K)$. Then the spaces $\{E_i(W_i)\}_{i=1}^{\infty}$ also form a finite-dimensional decomposition of X.*

Let $U_i = (T_i - T_{i-1})X$ (we let $T_0 = 0$). Proposition 1 implies that we may assume without loss of generality that each U_i is contained in some X_{n_i}. Our next purpose is to show there exist a finite-dimensional decomposition $\{\widetilde{U}_i\}_{i=1}^{\infty}$ and a subsequence $\{\widetilde{X}_i\} \subset \{X_i\}$, such that for $\widetilde{Z}_i = \widetilde{U}_1 \oplus \cdots \oplus \widetilde{U}_i$ the condition

$$\widetilde{Z}_i \subset \widetilde{X}_i \subset \widetilde{Z}_{i+1} \quad \text{for all } i \in \mathbf{N} \tag{3}$$

is satisfied. Our proof of this fact uses induction and the following lemma.

Lemma 1. *Let $\{V_i\}_{i=1}^\infty$ be a finite-dimensional decomposition of a Banach space X, let H be a finite-dimensional subspace of X satisfying $V_i \subset H$ for $i = 1, \dots, k$, and let $\varepsilon > 0$. Then there exists a blocking $\{Y_i\}_{i=1}^\infty$ of the decomposition $\{V_i\}_{i=1}^\infty$, such that $Y_i = V_i$ for $i = 1, \dots k$, $Y_{k+j} = V_{m+j}$ for some $m \geq k$ and all $j \geq 2$, and $Y_{k+1} = V_{k+1} \oplus V_{k+2} \oplus \cdots \oplus V_{m+1}$; and there exists an operator $A : Y_{k+1} \to X$ satisfying the following three conditions*:

$$||Ay - y|| \leq \varepsilon ||y|| \text{ for all } y \in Y_{k+1}, \tag{4}$$

$$A(Y_{k+1}) \subset \text{lin}\left((V_1 \oplus \cdots \oplus V_{m+1}) \bigcup H\right), \tag{5}$$

$$H \subset V_1 \oplus V_2 \oplus \cdots \oplus V_k \oplus A(Y_{k+1}). \tag{6}$$

Proof of Lemma 1. Let $S_i : X \to V_1 \oplus \cdots \oplus V_i$ be the natural projections corresponding to the decomposition. Let $m \in \mathbf{N}$ be such that $m \geq k$ and

$$||S_{m+1}x - x|| \leq \delta ||x|| \quad \text{for all } x \in H, \tag{7}$$

where $\delta > 0$ is to be selected later. Let $U = S_{m+1}H$. Observe that $S_{m+1}|_{V_1 \oplus \cdots \oplus V_k}$ is the identity operator, and hence $V_1 \oplus \cdots \oplus V_k \subset U$. Using the standard perturbation argument, see [**6**, Proposition 5.3], we can estimate the projection constant of U in terms of δ and $\lambda(H, X)$ (when δ is small). Hence, $V_1 \oplus \cdots \oplus V_{m+1} = U \oplus C$ for some subspace C, where the norms of projections onto U and C are estimated in terms of δ and $\lambda(H, X)$. This fact and the estimate (7) allow us to claim that the operator $A : V_1 \oplus \cdots \oplus V_{m+1} \to X$ defined by $A(u+c) = S_{m+1}^{-1}(u) + c$ for $u \in U$, $c \in C$ satisfies (4) if $\delta > 0$ is selected to be small enough. The condition (5) follows immediately from the definition of A. To finish the proof it remains to observe that $Ax = x$ for $x \in V_1 \oplus \cdots \oplus V_k$. □

Now we use Lemma 1 to find $\{\widetilde{X}_i\}$ and $\{\widetilde{U}_i\}$. In each step we shall also find a new finite-dimensional decomposition $\{U_i^j\}_{i=1}^\infty$. Let $\varepsilon_i > 0$, $i = 2, 3 \dots$, be such that $\sum_{i=2}^\infty \varepsilon_i < 1/(2K)$.

In the first step we let $\widetilde{U}_1 = U_1$, $\widetilde{X}_1$ be any X_{n_1} satisfying the condition $U_1 \subset X_{n_1}$, and $\{U_i^1\}_{i=1}^{\infty} = \{U_i\}_{i=1}^{\infty}$.

In the second step we use Lemma 1 with $H = \widetilde{X}_1$, $k = 1$, $\varepsilon = \varepsilon_2$, and $\{V_i\}_{i=1}^{\infty} = \{U_i^1\}_{i=1}^{\infty}$. We let

$$\{U_i^2\}_{i=1}^{\infty} = \left\{U_1^1, A(Y_2), U_{m+2}^1, U_{m+3}^1, \dots\right\}.$$

By Proposition 1, $\{U_i^2\}_{i=1}^{\infty}$ is also a finite-dimensional decomposition. We let $\widetilde{U}_2 = A(Y_2)$, $\widetilde{X}_2$ be any X_{n_2} such that $n_2 > n_1$ and $\widetilde{U}_2 \subset X_{n_2}$. Such an n_2 exists by condition (5).

In the third step we use Lemma 1 with $H = \widetilde{X}_2$, $k = 2$, $\varepsilon = \varepsilon_3$, and $\{V_i\}_{i=1}^{\infty} = \{U_i^2\}_{i=1}^{\infty}$. Re-using the notation A, Y_i, m of Lemma 1 for different objects than in the previous step, we let

$$\{U_i^3\}_{i=1}^{\infty} = \left\{U_1^2, U_2^2, A(Y_3), U_{m+2}^2, U_{m+3}^2, \dots\right\}.$$

By Proposition 1 $\{U_i^3\}_{i=1}^{\infty}$ is also a finite dimensional decomposition. Here a bit more explanation is needed. Observe that $\{U_i^3\}_{i=1}^{\infty}$ is obtained from $\{U_i\}_{i=1}^{\infty}$ by making two blocks and perturbing them; one of them is perturbed no more than for ε_2 (in the sense of inequality (4)), the other for no more than ε_3, therefore we are in a position to apply Proposition 1.

We let $\widetilde{U}_3 = A(Y_3)$, $\widetilde{X}_3$ be any X_{n_3} satisfying $n_3 > n_2$ and $\widetilde{U}_3 \subset X_{n_3}$. Such an n_3 exists by condition (5).

We continue in an obvious way. The fact that condition (3) is satisfied is clear from the construction (see condition (6) in Lemma 1). It remains to check that $\{\widetilde{U}_i\}_{i=1}^{\infty}$ form a finite-dimensional decomposition of X. To see this observe that $\widetilde{U}_i$ are ε_i-perturbations of a blocking of $\{U_i\}_{i=1}^{\infty}$. Recalling the choice of ε_i and using Proposition 1, we get the desired statement.

Let $Q_n : X \to \widetilde{X}_n$ be some projections with $\sup_n ||Q_n|| < \infty$ and $\operatorname{im} Q_n = \widetilde{X}_n$. Let $R_n : X \to \widetilde{Z}_n$ be projections corresponding to the decomposition $\{\widetilde{U}_i\}_{i=1}^{\infty}$. We introduce new projections $P_n : X \to \widetilde{X}_n$ with $\operatorname{im} P_n = \widetilde{X}_n$ as:

$$P_n = R_n + (I - R_n)Q_n(R_{n+1} - R_n).$$

Let us show that P_n are projections onto $\widetilde{X}_n$ and $P_nP_{n+1} = P_n$.

If $x \in \widetilde{X}_n$, then $x = R_{n+1}x = R_n x + (R_{n+1} - R_n)x$. Since $(R_{n+1} - R_n)x \in \widetilde{X}_n$, then $Q_n(R_{n+1} - R_n)x = (R_{n+1} - R_n)x$. Hence,

$$x = R_n x + (I - R_n)Q_n(R_{n+1} - R_n)x.$$

Let us show that $\operatorname{im} P_n \subset \widetilde{X}_n$. Condition (3) implies that $\operatorname{im} R_n \subset \widetilde{X}_n$. Therefore, $(I - R_n)\widetilde{X}_n \subset \widetilde{X}_n$, and P_n is a projection onto $\widetilde{X}_n$.

Let us show that $P_n P_{n+1} = P_n$. In fact,

$$\begin{aligned} P_n P_{n+1} &= (R_n + (I - R_n)Q_n(R_{n+1} - R_n)) \\ &\quad \times (R_{n+1} + (I - R_{n+1})Q_{n+1}(R_{n+2} - R_{n+1})) \\ &= R_n + (I - R_n)Q_n(R_{n+1} - R_n) = P_n. \end{aligned}$$

It follows that $\{P_n\}$ is a uniformly bounded commuting sequence of projections onto $\{\widetilde{X}_n\}$. We get a contradiction with condition (1). □

Theorem 1 shows that one of the natural approaches to Problem 1 is to start with the following problem on composition of projections. A projection of a Banach space X onto its subspace Y is called *minimal*, if its norm is equal to $\lambda(Y, X)$, and *close-to-minimal*, if its norm is close to $\lambda(Y, X)$.

Consider a triple (X_1, X_2, X_3) of Banach spaces satisfying $X_1 \subset X_2 \subset X_3$. Assume that X_1 and X_2 are finite dimensional.

Problem 2. *Is it possible to find a close-to-minimal projection $P : X_3 \to X_1$ which can be factored as $P = P_1 P_2$, where $P_2 : X_3 \to X_2$ is a close-to-minimal projection onto X_2 and $P_1 : X_2 \to X_1$ is $P|_{X_2}$?*

Some related observations.

Proposition 2. *Each projection $P : X_3 \to X_1$ has a factorization of the form $P = P_1 P_2$, where $P_2 : X_3 \to X_2$ and $P_1 : X_2 \to X_1$ are projections.*

In fact, let $\ker P_1 = \ker P \cap X_2$. Let $\ker P_2$ be a complement of $\ker P_1$ in $\ker P$ (such a complement exists because $\ker P_1$ is finite dimensional).

Proposition 3. *There exist triples* (X_1, X_2, X_3) *and minimal projections* $P : X_3 \to X_1$ *which cannot be factored as* P_1P_2*, where* P_2 *is a minimal projection onto* X_2.

In the proof of this result and in further discussion it is convenient to use the notion of a sufficient enlargement. We denote the unit ball of a Banach space X by B_X; in the case when $X = l_p^n$, we use the notation B_p^n.

Definition 1. A bounded, closed, convex, 0-symmetric set A in a finite-dimensional normed space X is called a *sufficient enlargement* for X (or of B_X) if for an arbitrary isometric embedding $X \subset Y$ (Y is a Banach space) there exists a projection $P : Y \to X$ such that $P(B_Y) \subset A$. A *minimal sufficient enlargement* is defined to be a sufficient enlargement no proper subset of which is a sufficient enlargement.

It is easy to see that if X is a subspace of $L_\infty(\mu)$ and $P : L_\infty(\mu) \to X$ is a projection, then $\mathrm{cl}\,(P(B_{L_\infty(\mu)}))$ is a sufficient enlargement of B_X. See [**7, 8, 9**] for results on sufficient enlargements.

Proof of Proposition 3. Consider a triple of the form $l_2^k \subset l_2^n \subset L_\infty(\mu)$. The set $\lambda(l_2^n)B_2^n$ is a minimal sufficient enlargement of l_2^n, see [**8**, Section 3]. Therefore, if $P_2 : L_\infty(\mu) \to l_2^n$ is a minimal projection, then $\mathrm{cl}\,(P_2(B_{L_\infty(\mu)})) = \lambda(l_2^n)B_2^n$. Hence, for an arbitrary $P_1 : l_2^n \to l_2^k$, we have $\mathrm{cl}\,(P_1P_2(B_{L_\infty(\mu)})) = \mathrm{cl}\,(P_1(\lambda(l_2^n)B_2^n)) \supset \lambda(l_2^n)B_2^k$, where we have an equality instead of an inclusion if P_1 is orthogonal.

Of course, if k is much less than n, then $\lambda(l_2^k)$ is much less than $\lambda(l_2^n)$, and the projection P_1P_2 is far from being minimal. □

On the other hand, there exist $P_1 : l_2^n \to l_2^k$ and $P_2 : L_\infty(\mu) \to l_2^n$, such that P_1P_2 is a minimal projection and P_2 is a close-to-minimal projection. To show this we need the following observation about sufficient enlargements.

Lemma 2. *Let* X *and* Y *be two finite-dimensional normed spaces and* $X \oplus Y$ *be their direct sum.*

Suppose that $X \oplus Y$ is endowed with a norm $||\cdot||$ satisfying the conditions

$$||x|| \leq ||(x,y)||, \quad \textit{for all } (x,y) \in X \oplus Y \tag{1}$$

and

$$||y|| \leq ||(x,y)||, \quad \textit{for all } (x,y) \in X \oplus Y. \tag{2}$$

Let A_X be a sufficient enlargement of B_X and A_Y a sufficient enlargement of B_Y. Then the Minkowski sum $A_X + A_Y$ is a sufficient enlargement for $(X \oplus Y, \ ||\cdot||)$.

Proof. Let $X \oplus Y \subset Z$ be an isometric embedding. We show that there exists a projection $P_X : Z \to X$ such that $P_X(B_Z) \subset A_X$ and $P_X(Y) = \{0\}$. Let $\varphi_Y : Z \to Z/Y$ be a quotient mapping with $\ker \varphi_Y = Y$. By condition (1) the restriction $\varphi_Y|_X$ is an isometry. Hence, there is a projection $Q_X : Z/Y \to \varphi_Y(X)$ such that $Q_X(B_{Z/Y}) \subset \varphi_Y(A_X)$. Therefore, we may identify X with $\varphi_Y X$ and A_X with $\varphi_Y(A_X)$. We let $P_X = Q_X \varphi_Y$. It is clear that all of the conditions are satisfied.

In the same way, condition (2) implies that there exists a projection $P_Y : Z \to Y$ such that $P_Y(B_Z) \subset A_Y$ and $P_Y(X) = 0$.

Let $P : Z \to X \oplus Y$ be defined by $Pz = (P_X z, P_Y z)$.

It is easy to check that P is a projection onto $X \oplus Y$. In fact,

$$P(x,y) = (P_X(x,y), P_Y(x,y)) = (x,y).$$

Also $P(B_Z) \subset P_X(B_Z) + P_Y(B_Z) \subset A_X + A_Y$. $\square$

Now we are ready to construct projections P_1 and P_2 whose existence was claimed before Lemma 2. By Lemma 2 the set

$$A = \lambda(l_2^k) B_2^k + \lambda(l_2^{n-k}) B_2^{n-k}$$

is a sufficient enlargement for $l_2^n = l_2^k \oplus l_2^{n-k}$. Let $P_2 : L_\infty(\mu) \to l_2^n$ be a projection corresponding to this sufficient enlargement, that is, satisfying $P_2(B_{L_\infty(\mu)}) \subset A$. It is easy to see that the norm of this projection is $\leq ((\lambda(l_2^k))^2 + (\lambda(l_2^{n-k}))^2)^{1/2}$. Hence, it is not much more than $\lambda(l_2^n)$. In fact, $((\lambda(l_2^k))^2 + (\lambda(l_2^{n-k}))^2)^{1/2} < \sqrt{2}\lambda(l_2^n)$.

Remark. By [**8**, Theorem 5] the sufficient enlargement $A = \lambda(l_2^k)B_2^k + \lambda(l_2^{n-k})B_2^{n-k}$ is minimal. Hence, $\mathrm{cl}\,(P_2(B_{L_\infty(\mu)})) = A$ and $||P_2|| = ((\lambda(l_2^k))^2 + (\lambda(l_2^{n-k}))^2)^{1/2}$.

Is it always like this? More precisely,

Problem 3. *Does there exist a universal constant $C \in [1,\infty)$ such that for each triple $X_1 \subset X_2 \subset X_3$ of Banach spaces, with X_1 and X_2 finite dimensional, there exist projections $P_1 : X_2 \to X_1$ and $P_2 : X_3 \to X_2$, such that $||P_2|| \leq C\lambda(X_2, X_3)$ and $||P_1P_2|| = \lambda(X_1, X_3)$?*

Another version of this problem (which will be particularly interesting if Problem 3 has a negative answer):

Problem 4. *Do there exist universal constants $C_1, C_2 \in [1,\infty)$ such that for each triple $X_1 \subset X_2 \subset X_3$ of Banach spaces, with X_1 and X_2 finite dimensional, there exist projections $P_1 : X_2 \to X_1$ and $P_2 : X_3 \to X_2$, such that $||P_2|| \leq C_1\lambda(X_2, X_3)$ and $||P_1P_2|| = C_2\lambda(X_1, X_3)$?*

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