

A GOLDBACH-WARING PROBLEM FOR UNEQUAL POWERS OF PRIMES

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ABSTRACT. Let p_i , $1 \leq i \leq 4$ be prime numbers. It is proved that all but $\ll x^{(1633/1680)+\varepsilon}$ even integers $n \leq x$ can be written as $n = p_1^2 + p_2^3 + p_3^4 + p_4^5$.

1. Introduction and statement of results. In 1952, Prachar [11] considered the following diophantine equation of the Goldbach-Waring type:

$$(1.1) \quad n = p_1^2 + p_2^3 + p_3^4 + p_4^5$$

for prime numbers p_i , $1 \leq i \leq 4$. Defining $E(x)$ as the set of all positive even integers not larger than a real number x that cannot be written as in (1.1), Pracher showed that

$$(1.2) \quad E(x) \ll x(\log x)^{-c},$$

for some constant $c > 0$. Roth [15] had earlier achieved a similar result where the p_i are allowed to take any integer value but do not necessarily have to be prime numbers. Using the approach of Montgomery and Vaughan, in [1] the logarithmic gain in the bound for $E(x)$ in (1.2) was replaced by a small power of x , i.e.,

$$E(x) \ll x^{1-\delta},$$

for very small $\delta > 0$. Using new techniques [9, 10] that allow to increase the size of the *major arcs* in the circle method and apply new estimates for the contribution of the minor arcs [7, 8] the upper bound for $E(x)$ was further reduced in [2, 3, 13] with the best bound given in [14] as:

$$(1.3) \quad E(x) \ll x^{(47/48)+\varepsilon}.$$

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In this paper, we further improve on these results by proving the following:

Theorem 1.

$$E(x) \ll x^{(1633/1680)+\varepsilon}.$$

We see that $1633/1680 = 1 - 47/1680$ and $47/48 = 1 - 1/48$. As $47/1680 > 1/36$, we see that Theorem 1 improves over (1.3) by a factor $> 4/3$.

We apply the circle method to prove Theorem 1. The improvement of Theorem 1 over (1.3) is mainly due to a new estimate for the integral over the *minor arcs*. Whereas previous work [14] used an estimate due to Roth [15], we develop a new estimate for the integral under consideration. We combine this estimate with a new estimate for exponential sums over powers of primes due to Kumchev [8]. On the *major arcs*, the iterative procedure introduced in [9] is combined with a new bound for Dirichlet polynomials [4]. This combination of techniques allows us to increase the modules of the *major arcs* compared to the procedure in [13], but we achieve *major arcs* slightly smaller than in [14].

However, the calculation of the integral over the *major arcs* is not the decisive factor to determine the size of the exceptional set $E(x)$. Indeed, the existing bounds for exponential sums over powers of primes [8] do not allow us to fully use the size of the *major arcs* as determined in this paper or in [14] in order to reduce the upper bound for $E(x)$. As we will explain later, this implies that any improvement on Theorem 1 requires further improvements of the estimates for the contribution of the integral over the *minor arcs*.

2. Notation and structure of the proof. k always denotes an integer $k \in \{2, 3, 4, 5\}$, by p we denote a prime number and L denotes $\log n$. c is an effective positive constant and ε will denote an arbitrarily small positive number; both of them may take different values at different occasions. $[a_1, \dots, a_n]$ denotes the least common multiple of the integers $a_1, \dots, a_n$. We define

$$r \sim R \iff R/2 < r \leq R,$$

$$\sum_{\chi \bmod q}^* = \sum_{\substack{\chi \bmod q \\ \chi \text{ primitive}}}^*, \quad \sum_{1 \leq a \leq q}^* = \sum_{\substack{1 \leq a \leq q \\ (a, q) = 1}}^q.$$

We set

$$(2.1) \quad P = n^{(1/12) - \varepsilon}, \quad Q = nP^{-1}L^{-E} \quad (E > 0 \text{ will be defined later}).$$

We define for any characters $\chi, \chi_j \pmod{q}$, $q \leq P$ and a fixed integer n :

$$C_k(a, \chi) = \sum_{l=1}^q \chi(l) e\left(\frac{al^k}{q}\right), \quad C_k(a, \chi_0) = C_k(a, q).$$

$$Z(q, \chi_2, \chi_3, \chi_4, \chi_5) = \sum_{h=1}^q {}^* e\left(\frac{-hn}{q}\right) \prod_{k=2}^5 C_k(h, \chi_k),$$

$$Y(q) = Z(q, \chi_0, \chi_0, \chi_0, \chi_0), \quad A(q) = \frac{Y(q)}{\phi^4(q)}.$$

When the variable n is fixed, we will always write $A(q)$ and neglect the dependency of $A(q)$ on n . Otherwise, we will write $A(q, n)$. We set $N_k :=]\sqrt[k]{n/4}, \leq \sqrt[k]{n}]$.

$$\begin{aligned} s(p) &= 1 + \sum_{\alpha \geq 1} A(p^\alpha), \quad S_k(\lambda) = \sum_{n \in N_k} \Lambda(n) e(n^k \lambda), \\ S_k(\lambda, \chi) &= \sum_{n \in N_k} \Lambda(n) \chi(n) e(n^k \lambda), \quad T_k(\lambda) = \sum_{n \in N_k} e(n^k \lambda), \\ W_k(\lambda, \chi) &= S_k(\lambda, \chi) - E_0 T_k(\lambda, \chi), \\ E_0 &= \begin{cases} 1 & \text{if } \chi = \chi_0, \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

Using the circle method we define the major arcs M and minor arcs m as follows:

$$\begin{aligned} M &= \sum_{q \leq P} \sum_{a=1}^q {}^* I(a, q), \quad I(a, q) = \left[\frac{a}{q} - \frac{1}{Qq}, \frac{a}{q} + \frac{1}{Qq} \right], \\ m &= \left[\frac{1}{Q}, 1 + \frac{1}{Q} \right] \setminus M. \end{aligned}$$

Let

$$R(n) = \sum_{\substack{n \in N_k, k \in \{2, \dots, 5\} \\ n_2^2 + \dots + n_5^2 = n}} \Lambda(n_2) \dots \Lambda(n_5).$$

Then we find

$$\begin{aligned} (2.2) \quad R(n) &= \int_{1/Q}^{1+(1/Q)} e(-n\alpha) \prod_{k=2}^5 S_k(\alpha) d\alpha \\ &= \left(\int_M + \int_m \right) e(-n\alpha) \prod_{k=2}^5 S_k(\alpha) d\alpha =: R_1(n) + R_2(n). \end{aligned}$$

3. The integral over the minor arcs. For the estimation of the integral over the minor arcs, we use the following estimate:

Lemma 3.1.

$$(3.1) \quad \int_m |S_2(\alpha) S_3(\alpha) S_4(\alpha) S_5(\alpha)|^2 \ll n^{(47/30) - (47/1680) + \varepsilon}.$$

Proof. We denote the lefthand side of (3.1) by G . By Cauchy's inequality,

$$\begin{aligned} (3.2) \quad G &\leq \left(\int_0^1 |S_2(\alpha) S_4^2(\alpha)|^2 d\alpha \right)^{1/2} \left(\int_0^1 |S_2(\alpha) S_3(\alpha) S_5(\alpha)|^2 d\alpha \right)^{1/2} \\ &\quad \times \max_{\alpha \in m} |S_3(\alpha)| \max_{\alpha \in m} |S_5(\alpha)| := A_1^{1/2} A_2^{1/2} \max_{\alpha \in m} |S_3(\alpha)| \max_{\alpha \in m} |S_5(\alpha)|. \end{aligned}$$

The value of A_1 equals the number of integer solutions of the equation:

$$a_1^2 - a_2^2 + b_1^4 - b_2^4 + b_3^4 - b_4^4,$$

where $a_i \in N_2$, $b_i \in N_4$. The solutions are of three kinds:

1. $a_1 \neq a_2$,
2. $a_1 = a_2$, $b_1 \neq b_2$,
3. $a_1 = a_2$, $b_1 = b_2$, $b_3 = b_4$.

Thus,

$$(3.3) \quad A_1 \ll N_4^{4+\varepsilon} + N_2 N_4^{2+\varepsilon} + N_2 N_4^2 \ll n^{1+\varepsilon}.$$

The value of A_2 equals the number of integer solutions of the equation:

$$a_1^2 - a_2^2 + b_1^3 - b_2^3 + c_1^5 - c_2^5,$$

where $a_i \in N_2$, $b_i \in N_3$, $c_i \in N_5$. The solutions are of three kinds:

1. $a_1 \neq a_2$,
2. $a_1 = a_2$, $b_1 \neq b_2$,
3. $a_1 = a_2$, $b_1 = b_2$, $c_1 = c_2$.

Thus,

$$(3.4) \quad A_2 \ll N_3^2 N_5^{2+\varepsilon} + N_2 N_5^{2+\varepsilon} + N_2 N_3 N_5 \ll n^{16/15+\varepsilon}.$$

For the reminder of the proof of Lemma 3.1, we quote two estimates from [8, Theorem 3]:

Lemma 3.2. a) Suppose $\alpha \in \mathbf{R}$ and there exist $a \in \mathbf{N}$ and $q \in \mathbf{N}$ satisfying

$$(3.5) \quad 1 \leq q \leq n^{4/7}, \quad (a, q) = 1, \quad |q\alpha - a| < n^{-4/7}.$$

Then, for any fixed ε , one has

$$S_3(\alpha) \ll n^{(1/3)-(1/42)+\varepsilon} + \frac{q^\varepsilon n^{1/3} L^c}{(q + n|q\alpha - a|)^{1/2}}.$$

b) Suppose $\alpha \in \mathbf{R}$ and there exist $a \in \mathbf{N}$ and $q \in \mathbf{N}$ satisfying

$$(3.6) \quad 1 \leq q \leq n^{119/216}, \quad (a, q) = 1, \quad |q\alpha - a| < n^{-119/216}.$$

Then, for any fixed ε , one has

$$S_5(\alpha) \ll n^{(1/5)-(1/240)+\varepsilon} + \frac{q^\varepsilon n^{1/5} L^c}{(q + n|q\alpha - a|)^{1/2}}.$$

We now consider an $\alpha \in m$ for which by Dirichlet's lemma on rational approximation we find integers a and q satisfying (3.5). By the definition of the minor arcs m , we see that either $q > P$ or $q \leq P$ and $n|q\alpha - a| \geq P$. Thus, by Lemma 3.2,

$$(3.7) \quad S_3(\alpha) \ll n^{(1/3)-(1/42)}.$$

Applying (3.6) in a similar way,

$$(3.8) \quad S_5(\alpha) \ll n^{(1/5)-(1/240)}.$$

Summarizing (3.2), (3.3), (3.4), (3.7) and (3.8), we get

$$G \ll n^{(47/30)-(47/1680)}. \quad \square$$

We derive from Lemma 3.1 in the standard way that

$$(3.9) \quad R_2(n) \ll n^{17/60} L^{-A}$$

for any $A > 0$ and all but $\ll x^{(1633/1680)+\varepsilon}$ even integers $3x/4 \leq n < x$. In Section 6, we will show that for any given $A > 0$,

$$(3.10) \quad R_1(n) = \frac{1}{120} P_0 \sum_{q \geq 1} A(n, q) + O\left(n^{17/60} L^{-A}\right),$$

where

$$(3.11) \quad n^{17/60} \ll P_0 := \sum_{\substack{m_2+m_3+m_4+m_5=n \\ m_k \in N_k}} \prod_{k=2}^5 \frac{1}{m^{1-(1/k)}} \ll n^{17/60}.$$

Now Theorem 1 follows from (2.2), (3.9), (3.10), (3.11) and Lemma 4.3 below.

4. Preliminary lemmas. We will make use of the following lemmas:

Lemma 4.1. *Let $f(x)$, $g(x)$ and $f'(x)$ be three real differentiable and monotonic functions in the interval $[a, b]$. If $|f'(x)| \leq \theta < 1$, $g(x), g'(x) \ll 1$, then*

$$\sum_{a < n \leq b} g(n)e(f(n)) = \int_a^b g(x)e(f(x)) dx + O\left(\frac{1}{1-\theta}\right).$$

Proof. See Lemma 4.8 in [16].

Lemma 4.2. *For primitive characters $\chi_i \bmod r_i$, $i = 1, 2, 3, 4$, and the principal characters $\chi_0 \bmod q$, we have*

$$\sum_{\substack{q \leq P \\ r|q}} \frac{|Z(q, \chi_0 \chi_1, \chi_0 \chi_2, \chi_0 \chi_3, \chi_0 \chi_4)|}{\phi^4(q)} \ll r^{-1+\varepsilon} (\log P)^c,$$

where $r = [r_1, r_2, r_3, r_4]$.

Proof. This is Lemma 3.3 in [2].

Lemma 4.3. *For any even integer n , the following is true:*

a)

$$\sum_{q > x} |A(n, q)| \ll x^{-(1/2)+\varepsilon} d(n).$$

b)

$$\mathcal{S}(n) := \sum_{q \geq 1} A(n, q) \gg (\log \log n)^{-c}.$$

Proof. This is Lemma 5.3 in [13].

5. Main lemmas. We define the following quantities for $2 \leq k \leq 5$:

$$I_k(g) = \sum_{r \leq P} [g, r]^{-1+\varepsilon} \sum_{\chi \pmod{r}}^* \max_{|\lambda| \leq 1/rQ} |W_k(\lambda, \chi)|,$$

$$J_k(g) = \sum_{r \leq P} [g, r]^{-1+\varepsilon} \sum_{\chi \pmod{r}}^* \left(\int_{-1/rQ}^{1/rQ} |W_k(\lambda, \chi)|^2 d\lambda \right)^{1/2}.$$

The proof of Theorem 1 will make use of the following lemmas:

Lemma 5.1. *If $P \leq n^{9/100-\varepsilon}$ and $g > 1$, then*

$$I_k(g) \ll g^{-1+\varepsilon} n^{(1/k)-(1/2)} L^c.$$

Lemma 5.2. *If $P \leq n^{9/100-\varepsilon}$ and $g > 1$, then*

$$J_k(g) \ll g^{-1+\varepsilon} n^{1/k} L^c.$$

Lemma 5.3. *If $P \leq n^{1/12-\varepsilon}$, then*

$$J_k(1) \ll n^{1/k} L^{-A},$$

for any $A > 0$.

5.1 Proof of Lemma 5.1. In order to prove the lemma we show that

$$(5.1) \quad \sum_{r \sim R} [g, r]^{-1+\varepsilon} \sum_{\chi \pmod{r}}^* \left(\int_{-1/rQ}^{1/rQ} |W_k(\lambda, \chi)|^2 d\lambda \right)^{1/2} \\ \ll g^{-1+\varepsilon} n^{(1/k)-(1/2)} L^c$$

for $R \leq P$. Applying Lemma 1, [5], we see

$$(5.2) \quad \int_{-1/rQ}^{1/rQ} |W_k(\lambda, \chi)|^2 d\lambda \ll (QR)^{-2} \\ \cdot \int_{(n/4)-QR}^n \left| \sum_{X < m^k \leq X+Y} \Lambda(m) \chi(m) - E_0(\chi) \sum_{X < m^k \leq X+Y} 1 \right|^2 dt,$$

where $X = \max(t, n/4)$, $X + Y = \min(t + Qr, n)$ where $E_0(\phi) = 1$ if $\chi = \chi_0$ and $= 0$ otherwise. We first consider the case $R = 1$. A trivial estimate shows that the righthand side of (5.2) can be estimated by $\ll (QR)^{-2} n (QR)^2 n^{(2/k)-2} = n^{(2/k)-1}$, which is sufficient in view of (5.1). For $R > 1$, we argue as in [4]. Applying Perron's summation formula that the inner sum of (5.2) can be written as

$$(5.3) \quad S := \frac{1}{2\pi i} \int_{b-iT}^{b+iT} F_k(s, \chi) \frac{(X+Y)^{s/k} - X^{s/k}}{s} ds + O(T^{-1}nL^2),$$

for $T = n^2$ and $0 < b < L^{-1}$, with

$$(5.4) \quad F_k(s, \chi) = \sum_{X^{1/k} \leq m \leq (X+Y)^{1/k}} \Lambda(m) \chi(m) m^{-s}.$$

Using trivial estimates, we see that for $s = \rho + it$, $0 < b < L^{-1}$,

$$\frac{(X+Y)^{s/k} - X^{s/k}}{s} \ll \min(T_0^{-1}, (|t|+1)^{-1})$$

for $T_0 = n(QR)^{-1}$. Thus, for $b \downarrow 0$, S is bounded by

$$(5.5) \quad \begin{aligned} S &\ll \int_{-T}^T |F_k(it, \chi)| \frac{dt}{T_0 + |t|} + \frac{L^2}{n} \\ &\ll L \max_{T_0 \leq T_1 \leq T} \frac{1}{T_1} \int_{-T_1}^{T_1} |F_k(it, \chi)| dt + \frac{L^2}{n}. \end{aligned}$$

Thus, we see from (5.2) and (5.5) that the lefthand side of (5.1) is upper-bounded by

$$(5.6) \quad \begin{aligned} &\ll L \max_{T_0 \leq T_1 \leq T} (QR)^{-1} T_1^{-1} n^{1/2} \sum_{r \sim R} [g, r]^{-1+\varepsilon} \\ &\quad \sum_{\chi \pmod{r}}^* \int_{-T_1}^{T_1} |F_k(it, \chi)| dt \\ &\quad + L^3 (QR)^{-1} n^{-1/2} g^{-1} R^2. \end{aligned}$$

In view of (5.1), we see that the second term in (5.6) is permissible for $P \leq n^{9/100-\varepsilon}$. We note that $g, r = gr$. Thus, the first term of

(5.6) is

$$(5.7) \quad \ll \max_{T_0 \leq T_1 \leq T} g^{-1+\varepsilon} L(QR)^{-1} T_1^{-1} n^{1/2} \sum_{\substack{d \leq R \\ d|g}} \left(\frac{R}{d}\right)^{-1+\varepsilon} \\ \sum_{\substack{r \sim R \\ d|r}} \sum_{\chi \pmod{r}}^* \int_{-T_1}^{T_1} |F_k(it, \chi)| \, dt.$$

In order to estimate the expression (5.7), we apply Theorem 1.1 from [4]:

Lemma 5.4. *For $F_k(s, \chi)$ defined as in (5.4), there is*

$$\sum_{\substack{r \sim R \\ m|r}} \sum_{\chi} \int_{-T_1}^{T_1} |F_k(it, \chi)| \ll \left(n^{1/k} + \frac{R^2 T_1}{m} n^{11/20k} \right).$$

Applying the lemma and the estimate of divisor function $\sum_{d|g} 1 \ll g^\varepsilon$ to (5.7) yields

$$\begin{aligned} &\ll \max_{T_0 \leq T_1 \leq T} \max_{\substack{d|g \\ d \leq R}} g^{-1+\varepsilon} L(QR)^{-1} T_1^{-1} n^{1/2} \\ &\times \left(\frac{R}{d}\right)^{-1+\varepsilon} \left(n^{1/k} + \frac{R^2 T_1}{d} n^{11/20k} \right) \\ &\ll g^{-1+\varepsilon} n^{(1/k)-(1/2)} L^c, \end{aligned}$$

for $2 \leq k \leq 5$ and $P \leq n^{9/100-\varepsilon}$, where we recall that $Q = nP^{-1}L^{-E}$.

5.2 Proof of Lemma 5.2. To prove the lemma it is enough to show that

$$(5.8) \quad \max_{R \leq P/2} \max_{0 \leq \Delta \leq (1/RQ)} \max_{\Delta \leq |\lambda| \leq 2\Delta} \sum_{r \sim R} [g, r]^{-1+\varepsilon} \sum_{\chi}^* |W_k(\lambda, \chi)| \\ \ll g^{-1+\varepsilon} n^{1/k} L^C.$$

We recall the definitions (5.3) and (5.4) from the previous section and in the definition (5.4) change X to $n/4$ and $X+Y$ to n . Then we follow the argument in [4] and see by partial summation that

$$(5.9) \quad W_k(\lambda, \chi) = \frac{1}{2\pi i} \int_{b-iT}^{b+iT} F_k(s, \chi) V_k(s, \lambda) ds + O(1),$$

where $0 < b \leq L^{-1}$ and $T = n^{10}$,

$$V_k(s, \lambda) = \int_{n/4}^n w^{s-1} e(\lambda w^k) dw.$$

Using well-known estimates for exponential integrals as given in [16, Chapter 4], we see for $\Delta \leq |\lambda| \leq 2\Delta$:

$$(5.10) \quad V_k(\sigma + it, \lambda) \ll n^{\sigma/k} \min \left\{ T_0^{-1/2}, \max_{n/4 \leq w \leq n} |t + 2k\pi\lambda w|^{-1} \right\},$$

where $T_0 = n/QR$. We combine (5.9) and (5.10) and let $b \downarrow 0$, such that

$$W_k(\lambda, \chi) \ll T_0^{1/2} \int_{-T}^T |F_k(it, \chi)| \frac{dt}{T_0 + |t|} + O(1).$$

Thus, the lefthand side of (5.8) is upper bounded by

$$(5.11) \quad \ll \max_{R \leq P/2} \max_{T_0 \leq T_1 \leq T} L T_0^{1/2} T_1^{-1} \sum_{\substack{r \sim R \\ g|r}} [g, r]^{-1+\varepsilon} \sum_{\chi} \int_{-T_1}^{T_1} |F_k(it, \chi)| dt + \frac{R^2}{g^{1-\varepsilon}}.$$

The term $R^2/g^{1-\varepsilon}$ is permissible for $R \leq n^{9/100-\varepsilon}$ in view of (5.8). For the first term, we apply Lemma 5.4 similarly to the previous section:

$$\begin{aligned} & \ll \max_{R \leq P/2} \max_{T_0 \leq T_1 \leq T} \max_{\substack{d|g \\ d \leq R}} L T_0^{1/2} T_1^{-1} g^{-1+\varepsilon} \left(\frac{R}{d} \right)^{-1+\varepsilon} \\ & \quad \times \sum_{\substack{r \sim R \\ d|r}} \sum_{\chi} \int_{-T_1}^{T_1} |F_k(it, \chi)| dt \\ & \ll g^{-1+\varepsilon} n^{1/k} L^c, \end{aligned}$$

for $P \leq n^{9/100+\varepsilon}$.

5.3 Proof of Lemma 5.3. We treat separately the cases $R \leq L^V$ and $R \geq L^V$ for a sufficiently large V to be determined later. In the second case, we argue as in the previous section using Lemma 5.4 for $g = 1$ and use $R \geq L^V$. In the case $R \leq L^V$, we argue as in [9] and use the zero expansion of the von Mangoldt-function [12]:

$$\sum_{m \leq u} \Lambda(m) \chi(m) = E_0(\chi)u - \sum_{|\operatorname{Im} \rho| \leq T} \frac{u^\rho}{\rho} + O\left(\left(\frac{u}{T} + 1\right) \log^2(uT)\right),$$

where ρ runs over the nontrivial zeros of the L -function corresponding to $\chi \bmod r$ with $|\operatorname{Im} \rho| \leq T$. Thus, for $T = n^{(5/12k) - \varepsilon_1}$ and $\beta = \operatorname{Re} \rho$,

$$\begin{aligned} W_k(\lambda, \chi) &= \int_{\sqrt[k]{n/4}}^{\sqrt[k]{n}} e(u^k \lambda) d\left\{ \sum_{n \leq u} \Lambda(m) \chi(m) - E_0(\chi)u \right\} \\ &= \int_{\sqrt[k]{n/4}}^{\sqrt[k]{n}} e(u^k \lambda) \sum_{|\operatorname{Im} \rho| \leq n^{(5/12k) - \varepsilon_1}} u^{\rho-1} du \\ &\quad + O(n^{(7/12k) + \varepsilon_1} (1 + |\lambda|n) L^2) \\ &\ll n^{1/k} \sum_{|\operatorname{Im} \rho| \leq n^{(5/12k) - \varepsilon_1}} n^{(\beta-1)/k} + O(n^{1/k} L^{-A}), \end{aligned}$$

because of $P = n^{(1/12) - \varepsilon}$. We now use the fact that $L(\sigma + it, \chi)$ with $\chi \bmod r$ and $r \leq L^V$ has no zeros in the region, see [12, VIII Satz 6.2],

$$(5.12) \quad \sigma \geq 1 - \delta(T) := 1 - \frac{c_0}{\log r + (\log(T+2))^{4/5}}, \quad |t| \leq T,$$

where c_0 is an absolute constant. We also appeal to a well-known zero density estimate [6]:

Lemma 5.5. *Let $N^*(\alpha, T, q)$ denote the number of zeros $\sigma + it$ of all L -functions to primitive characters modulo q within the region $\sigma \geq \alpha$, $|t| \leq T$. Then for a positive integer m :*

$$\sum_{\substack{q \leq P \\ m|q}} N^*(\alpha, T, q) \ll (\log PT)^c \left(\frac{P^2 T}{m} \right)^{((12/5) + \varepsilon)(1-\alpha)}.$$

Using (5.12) and Lemma 5.5, we obtain for $\delta(T) = \delta(n^{(5/12k)-\varepsilon_1}) = cL^{-4/5}$:

$$\begin{aligned}
& \max_{R \leq L^V} \sum_{r \sim R} r^{-1+\varepsilon} \sum_{\chi \pmod r}^* \max_{|\lambda| \leq 1/rQ} |W_k \lambda, \chi| \\
& \ll n^{1/k} L^V \sum_{r \leq L^V} \sum_{\chi \pmod r}^* \sum_{|\operatorname{Im} \rho| \leq n^{(5/12k)-\varepsilon_1}} n^{(\beta-1)/k} + n^{1/k} L^{-A+2V} \\
& \ll n^{1/k} L^{V+c} \max_{0 \leq \beta \leq 1-\delta(n^{(5/12k)-\varepsilon_1})} (n^{(5/12k)-\varepsilon_1})^{((12/5)+\varepsilon)(1-\beta)} n^{(\beta-1)/k} \\
& \quad + n^{1/k} L^{-A+2V} \\
& \ll n^{1/k} n^{-\delta(n^{(5/12k)-\varepsilon_1})\varepsilon_1} + n^{1/k} L^{-A+2V} \\
& \ll n^{1/k} L^{-A},
\end{aligned}$$

for any $\varepsilon_1 = 2\varepsilon$ and $A =: A + 2V > 0$.

6. Treatment of the major arcs. Splitting the summation over n in residue classes modulo q , we obtain

$$S_k\left(\frac{a}{q} + \lambda\right) = \frac{C_k(a, q)}{\phi(q)} T_k(\lambda) + \frac{1}{\phi(q)} \sum_{\chi \pmod q} C_k(a, \chi) W_k(\lambda, \chi) + O(L^2).$$

Thus, we obtain from (2.2)

$$(6.1) \quad R_1(n) = R_1^m(n) + R_1^e(n) + O(n^{17/60} L^{-A}) \quad (\text{for any } A > 0),$$

where

$$\begin{aligned}
(6.2) \quad R_1^m(n) &= \sum_{q \leq P} \frac{1}{\phi^4(q)} \sum_{1 \leq a \leq q}^* \int_{-1/Qq}^{1/Qq} \prod_{k=2}^5 C_k(a, q) T_k(\lambda) e\left(-\frac{a}{q}n - \lambda n\right) d\lambda, \\
R_1^e(n) &= \sum_{k=2}^5 \sum_{q \leq P} \frac{1}{\phi^4(q)} \sum_{1 \leq a \leq q}^* \int_{-1/Qq}^{1/Qq} \prod_{\substack{l=2 \\ l \neq k}}^5 C_l(a, q) T_l(\lambda) \\
&\quad \times \sum_{\chi \pmod q} C_k(a, q) W_k(\lambda, \chi) e\left(-\frac{a}{q}n - \lambda n\right) d\lambda
\end{aligned}$$

$$\begin{aligned}
& + \sum_{\substack{k,l=2 \\ k < l}}^5 \sum_{q \leq P} \frac{1}{\phi^4(q)} \sum_{1 \leq a \leq q}^* \int_{-1/Qq}^{1/Qq} \prod_{m \in \{k,l\}} C_m(a, q) T_m(\lambda) \\
& \quad \times \prod_{\substack{o=2 \\ o \neq k,l}}^5 \sum_{\chi \bmod q} C_o(a, \chi) W_o(\lambda, \chi) e\left(-\frac{a}{q}n - \lambda n\right) d\lambda \\
& + \sum_{k=2}^5 \sum_{q \leq P} \frac{1}{\phi^4(q)} \sum_{1 \leq a \leq q}^* \int_{-1/Qq}^{1/Qq} C_k(a, q) T_k(\lambda) \\
& \quad \times \prod_{\substack{l=2 \\ l \neq k}}^5 \sum_{\chi \bmod q} C_l(a, q) W_l(\lambda, \chi) e\left(-\frac{a}{q}n - \lambda n\right) d\lambda \\
& + \sum_{q \leq P} \frac{1}{\phi^4(q)} \sum_{1 \leq a \leq q}^* \int_{-1/Qq}^{1/Qq} \prod_{k=2}^5 \sum_{\chi \bmod q} C_k(a, \chi) W_k(\chi, \lambda) \\
& \quad e\left(-\frac{a}{q}n - \lambda n\right) d\lambda, \\
& =: S_1 + S_2 + S_3 + S_4.
\end{aligned}$$

We first calculate $R_1^m(n)$. Applying Lemma 4.1 yields

$$\begin{aligned}
T_k(\lambda) &= \int_{\sqrt[k]{n/4}}^{\sqrt[k]{n}} e(\lambda u^k) du + O(1) = \frac{1}{k} \int_{n/4}^n v^{(1/k)-1} e(\lambda v) dv + O(1) \\
&= \frac{1}{k} \sum_{n/4 < m \leq n} \frac{e(\lambda m)}{m^{1-(1/k)}} + O(1).
\end{aligned}$$

Substituting this in $R_1^m(n)$ we see

$$\begin{aligned}
R_1^m(n) &= \frac{1}{120} \sum_{q \leq P} A(q) \int_{-1/Qq}^{1/Qq} \prod_{k=2}^5 \left(\sum_{n/4 < m \leq n} \frac{e(\lambda m)}{m^{1-(1/k)}} \right) e(-n\lambda) d\lambda \\
&+ O\left(\sum_{p=2}^5 \sum_{q \leq P} |A(q)| \prod_{\substack{k=2 \\ k \neq p}}^5 \int_{1/Qq}^{-1/Qq} \right. \\
&\quad \left. \times \max\left(\left| \sum_{n/4 < m \leq n} \frac{e(\lambda m)}{m^{1-(1/k)}} \right|, 1 \right) d\lambda \right).
\end{aligned}$$

Using Lemmas 4.2 and 4.3 as well as the trivial bound

$$(6.3) \quad \sum_{n/4 < m \leq n} \frac{e(\lambda m)}{m^{1-(1/k)}} \ll \min \left(\sqrt[k]{n}, \frac{1}{n^{1-(1/k)} |\lambda|} \right),$$

we obtain

$$(6.4) \quad \begin{aligned} R_1^m(n) &= \frac{1}{120} \sum_{q \leq P} A(q) \int_{-1/2}^{1/2} \prod_{k=2}^5 \left(\sum_{n/4 < m \leq n} \frac{e(\lambda m)}{m^{1-(1/k)}} \right) e(-n\lambda) d\lambda \\ &\quad + O \left(\sum_{q \leq P} |A(q)| \int_{1/Qq}^{1/2} \frac{1}{n^{3-(17/60)} |\lambda|^4} d\lambda \right) + O(n^{(17/60)} L^{-A}) \\ &= \frac{1}{120} P_0 \sum_{q \leq P} A(q) + O((PQ)^3 n^{(17/60)-3} L^c) \\ &\quad + O(n^{17/60} L^{-A}) \\ &= \frac{1}{120} P_0 \sum_{q \geq 1} A(q) + O(n^{17/60} L^{-A}), \end{aligned}$$

where P_0 is defined as in (3.11) and E is chosen sufficiently large in $Q = nP^{-1}L^{-E}$. Applying Lemma 4.2, we can estimate S_4 in the following way:

$$(6.5) \quad \begin{aligned} |S_4| &= \left| \sum_{q \leq P} \frac{1}{\phi^4(q)} \sum_{\chi_1 \bmod q} \sum_{\chi_2 \bmod q} \sum_{\chi_3 \bmod q} \sum_{\chi_4 \bmod q} Z(q, \chi_1, \chi_2, \chi_3, \chi_4) \right. \\ &\quad \left. \times \int_{-1/Qq}^{1/Qq} \prod_{j=1}^4 W_{j+1}(\lambda, \chi_j) e(-n\lambda) d\lambda \right| \\ &\leq \sum_{r_1 \leq P} \sum_{r_2 \leq P} \sum_{r_3 \leq P} \sum_{\substack{r_4 \leq P \\ [r_1, r_2, r_3, r_4] \leq P}} \\ &\quad \times \sum_{\chi_1 \bmod r_1}^* \sum_{\chi_2 \bmod r_2}^* \sum_{\chi_3 \bmod r_3}^* \sum_{\chi_4 \bmod r_4}^* \\ &\quad \times \int_{-1/Q[r_1, r_2, r_3, r_4]}^{1/Q[r_1, r_2, r_3, r_4]} \prod_{j=1}^4 |W_{j+1}(\lambda, \chi_j)| d\lambda \end{aligned}$$

$$\begin{aligned}
& \times \sum_{\substack{q \leq P \\ [r_1, r_2, r_3, r_4] | q}} \frac{|Z(q, \chi_1 \chi_0, \chi_2 \chi_0, \chi_3 \chi_0 \chi_4 \chi_0)|}{\phi^4(q)} \\
& \ll L^c \sum_{r_1 \leq P} \sum_{r_2 \leq P} \sum_{r_3 \leq P} \sum_{r_4 \leq P} [r_1, r_2, r_3, r_4]^{-1+\varepsilon} \\
& \times \sum_{\chi_1 \bmod r_1}^* \sum_{\chi_2 \bmod r_2}^* \sum_{\chi_3 \bmod r_3}^* \\
& \quad \times \sum_{\chi_4 \bmod r_4}^* \int_{-1/Q[r_1, r_2, r_3, r_4]}^{1/Q[r_1, r_2, r_3, r_4]} \prod_{j=1}^4 |W_{j+1}(\lambda, \chi_j)| d\lambda \\
& \leq L^c \sum_{r_1 \leq P} \sum_{\chi_1 \bmod r_1}^* \max_{|\lambda| \leq 1/r_1 Q} |W_2(\lambda, \chi_1)| \\
& \times \sum_{r_2 \leq P} \sum_{\chi_2 \bmod r_2}^* \max_{|\lambda| \leq 1/r_2 Q} \sum_{r \leq P} \sum_{\chi}^* |W_3(\lambda, \chi_2)| \\
& \times \sum_{r_3 \leq P} \sum_{\chi_3 \bmod r_3}^* \left(\int_{-1/r_3 Q}^{1/r_3 Q} |W_4(\lambda, \chi_3)|^2 d\lambda \right)^{1/2} \\
& \times \sum_{r_4 \leq P} \sum_{\chi_4 \bmod r_4}^* [r_1, r_2, r_3, r_4]^{-1+\varepsilon} \\
& \times \left(\int_{-1/Q r_4}^{1/Q r_4} |W_5(\lambda, \chi_4)|^2 d\lambda \right)^{1/2}.
\end{aligned}$$

Since $[r_1, r_2, r_3, r_4] = [[r_1, r_2, r_3], r_4]$, we use Lemma 5.1 to estimate the sum over r_4 :

$$\begin{aligned}
& \sum_{r_4 \leq P} [r_1, r_2, r_3, r_4]^{-1+\varepsilon} \sum_{\chi_4 \bmod r_4}^* \left(\int_{-1/r_4 Q}^{1/r_4 Q} |W_5(\lambda, \chi_4)|^2 d\lambda \right)^{1/2} \\
& = I_5([r_1, r_2, r_3]) \ll [r_1, r_2, r_3]^{-1+\varepsilon} n^{-3/10+\varepsilon} L^c.
\end{aligned}$$

Applying Lemma 5.1 again we see that the contribution of this quantity to the sum over r_3 is:

$$\begin{aligned}
&\ll n^{-3/10+\varepsilon} L^c \sum_{r_3 \leq P} [r_1, r_2, r_3]^{-1+\varepsilon} \\
&\quad \sum_{\chi_3 \bmod r_3}^* \left(\int_{-1/r_3 Q}^{1/r_3 Q} |W_4(\lambda, \chi_3)| d\lambda \right)^{1/2} \\
&\leq n^{-3/10+\varepsilon} I_4([r_1, r_2]) L^c \ll [r_1, r_2]^{-1+\varepsilon} n^{-11/20+\varepsilon} L^c.
\end{aligned}$$

Applying Lemma 5.2, we see

$$\begin{aligned}
&\ll n^{-11/20+\varepsilon} L^c \sum_{r_2 \leq P} [r_1, r_2]^{-1+\varepsilon} \\
&\quad \sum_{\chi_2 \bmod r_2}^* \max_{|\lambda| \leq 1/r_2 Q} |W_3(\lambda, \chi_2)| \\
&\leq n^{-11/20+\varepsilon} L^c J_3([r_1]) \ll [r_1]^{-1+\varepsilon} n^{-13/60} L^c.
\end{aligned}$$

Finally, we apply Lemma 5.3 to estimate the sum over r_1 as follows:

$$\begin{aligned}
S_4 &\ll n^{-13/60} L^c \sum_{r_1 \leq P} r_1^{-1+\varepsilon} \sum_{\chi_1 \bmod r_1}^* \max_{|\lambda| \leq 1/r_1 Q} |W_2(\lambda, \chi_1)| \\
&= n^{-13/60} L^c J_2(1) \ll n^{17/60} L^{-A}.
\end{aligned}$$

For the estimation of the sums $\sum_2 - \sum_4$ we note that

$$T = \max_{|\lambda| \leq 1/Q} |T_k(\lambda)| \ll n^{1/k},$$

and using (6.3) we get:

$$\left(\int_{-1/Q}^{1/Q} |T_k(\lambda)|^2 d\lambda \right)^{1/2} \ll n^{(1/k)-(1/2)}.$$

Using these estimates and the Lemmas 5.1–5.3, we argue similarly to the estimation of S_4 and obtain:

$$(6.6) \quad S_1 + S_2 + S_3 \ll n^{17/60} L^{-A}.$$

Thus, we see from (6.1), (6.2), (6.4), (6.5) and (6.6):

$$R_1^m(n) = \frac{1}{120} P_0 \sum_{q \geq 1} A(q) + O(n^{17/60} L^{-A}),$$

i.e., we proved (3.10). $\square$

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