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A STUDY ON QUASI POWER INCREASING SEQUENCES

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ABSTRACT. In the present paper, using a quasi β-power increasing sequence instead of an almost increasing sequence, a result of Bor and Seyhan [**5**] concerning the $\varphi - |C,\alpha|_k$ summability factors has been proved under weaker conditions. Also, this theorem generalizes some well-known results.

1. Introduction. A positive sequence (b_n) is said to be almost increasing if there exists a positive increasing sequence c_n and two positive constants A and B such that $Ac_n \leq b_n \leq Bc_n$, see [**1**]. Let (φ_n) be a sequence of complex numbers, and let $\sum a_n$ be a given infinite series with partial sums (s_n). We denote by z_n^α and t_n^α the nth Cesàro means of order α, with $\alpha > -1$, of the sequence (s_n) and (na_n), respectively, i.e.,

$$z_n^\alpha = \frac{1}{A_n^\alpha} \sum_{v=0}^{n} A_{n-v}^{\alpha-1} s_v, \tag{1}$$

$$t_n^\alpha = \frac{1}{A_n^\alpha} \sum_{v=1}^{n} A_{n-v}^{\alpha-1} v a_v, \tag{2}$$

where

$$A_n^\alpha = O(n^\alpha), \quad \alpha > -1, \quad A_0^\alpha = 1 \text{ and } A_{-n}^\alpha = 0 \text{ for } n > 0. \tag{3}$$

The series $\sum a_n$ is said to be summable $\varphi - |C,\alpha|_k$, $k \geq 1$ and $\alpha > -1$, if (see [**2**])

$$\sum_{n=1}^{\infty} |\varphi_n (z_n^\alpha - z_{n-1}^\alpha)|^k < \infty. \tag{4}$$

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But since $t_n^\alpha = n(z_n^\alpha - z_{n-1}^\alpha)$, see [**7**], condition (4) can also be written as

$$\sum_{n=1}^{\infty} n^{-k} |\varphi_n t_n^\alpha|^k < \infty. \tag{5}$$

In the special case when $\varphi_n = n^{1-(1/k)}$, respectively $\varphi_n = n^{\delta+1-(1/k)}$, $\varphi - |C,\alpha|_k$ summability is the same as $|C,\alpha|_k$, respectively $|C,\alpha;\delta|_k$, summability.

Quite recently Bor and Seyhan [**5**] have proved the following theorem.

Theorem A. *Let (X_n) be an almost increasing sequence, and let there be sequences (β_n) and (λ_n) such that*

$$|\Delta\lambda_n| \le \beta_n, \tag{6}$$

$$\beta_n \longrightarrow 0 \ \textit{as}\ n \to \infty, \tag{7}$$

$$\sum_{n=1}^{\infty} n\,|\Delta\beta_n|\,X_n < \infty, \tag{8}$$

$$|\lambda_n|\,X_n = O(1) \ \textit{as}\ n \to \infty. \tag{9}$$

*If there exists an $\varepsilon > 0$ such that the sequence $(n^{\varepsilon-k}|\varphi_n|^k)$ is nonincreasing and, if the sequence (w_n^α), defined by, see [**9**],*

$$w_n^\alpha = \begin{cases} |t_n^\alpha| & \alpha = 1 \\ \max_{1\le v\le n} |t_v^\alpha| & 0 < \alpha < 1 \end{cases} \tag{10}$$

satisfies the condition

$$\sum_{n=1}^{m} n^{-k} (|\varphi_n| w_n^\alpha)^k = O(X_m) \ \textit{as}\ m \to \infty, \tag{11}$$

then the series $\sum a_n\lambda_n$ is summable $\varphi - |C,\alpha|_k$, $k \ge 1$, $0 < \alpha \le 1$ and $k\alpha + \varepsilon > 1$.

2. The main result. The aim of this paper is to generalize Theorem A under weaker conditions for $\varphi - |C,\alpha|_k$ summability. For this we need the concept of quasi β-power increasing sequence. A positive sequence (γ_n) is said to be a quasi β-power increasing sequence if there exists a constant $K = K(\beta,\gamma) \geq 1$ such that

$$Kn^\beta \gamma_n \geq m^\beta \gamma_m \tag{12}$$

holds for all $n \geq m \geq 1$. It should be noted that every almost increasing sequence is a quasi β-power increasing sequence for any nonnegative β, but the converse need not be true as can be seen by taking the example, say $\gamma_n = n^{-\beta}$ for $\beta > 0$. So we are weakening the hypotheses of the theorem replacing an almost increasing sequence by a quasi β-power increasing sequence. Now, we shall prove the following theorem:

Theorem. *Let (X_n) be a quasi β-power increasing sequence for some $0 < \beta < 1$. If the sequences (λ_n) and (β_n) satisfy the conditions from* (6) *to* (11) *of Theorem* A, *then the series $\sum a_n\lambda_n$ is summable $\varphi - |C,\alpha|_k$, $k \geq 1$, $0 < \alpha \leq 1$ and $k\alpha + \varepsilon > 1$.*

Remark. If we take (X_n) as an almost increasing sequence, then we obtain Theorem A.

We need the following lemmas for the proof of our theorem.

Lemma 1 [**6**]. *If $0 < \alpha \leq 1$ and $1 \leq v \leq n$, then*

$$\left|\sum_{p=0}^{v} A_{n-p}^{\alpha-1} a_p\right| \leq \max_{1\leq m\leq v}\left|\sum_{p=0}^{m} A_{m-p}^{\alpha-1} a_p\right|. \tag{13}$$

Lemma 2 [**8**]. *Under the conditions on (X_n), (β_n) and (λ_n) as taken in the statement of the theorem, the following conditions hold, when* (8) *is satisfied*:

$$n\beta_n X_n = O(1) \text{ as } n \longrightarrow \infty \tag{14}$$

$$\sum_{n=1}^{\infty} \beta_n X_n < \infty. \tag{15}$$

3. Proof of the theorem. Let (T_n^α) be the nth (C,α), with $0<\alpha\leq 1$, mean of the sequence $(na_n\lambda_n)$. Then, by (2), we have

$$T_n^\alpha = \frac{1}{A_n^\alpha}\sum_{v=1}^{n} A_{n-v}^{\alpha-1} v a_v \lambda_v. \tag{16}$$

Using Abel's transformation, we get

$$T_n^\alpha = \frac{1}{A_n^\alpha}\sum_{v=1}^{n-1} \Delta\lambda_v \sum_{p=1}^{v} A_{n-p}^{\alpha-1} p a_p + \frac{\lambda_n}{A_n^\alpha}\sum_{v=1}^{n} A_{n-v}^{\alpha-1} v a_v,$$

so that, making use of Lemma 1, we have

$$\begin{aligned} |T_n^\alpha| &\leq \frac{1}{A_n^\alpha}\sum_{v=1}^{n-1} |\Delta\lambda_v| \left|\sum_{p=1}^{v} A_{n-p}^{\alpha-1} p a_p\right| + \frac{|\lambda_n|}{A_n^\alpha}\left|\sum_{v=1}^{n} A_{n-v}^{\alpha-1} v a_v\right| \\ &\leq \frac{1}{A_n^\alpha}\sum_{v=1}^{n-1} A_v^\alpha w_v^\alpha |\Delta\lambda_v| + |\lambda_n| w_n^\alpha \\ &= T_{n,1}^\alpha + T_{n,2}^\alpha, \quad \text{say.} \end{aligned}$$

Since

$$\left|T_{n,1}^\alpha + T_{n,2}^\alpha\right|^k \leq 2^k\left(\left|T_{n,1}^\alpha\right|^k + \left|T_{n,2}^\alpha\right|^k\right),$$

to complete the proof of the theorem, it is sufficient to show that

$$\sum_{n=1}^{\infty} n^{-k}\left|\varphi_n T_{n,r}^\alpha\right|^k < \infty \text{ for } r=1,2, \text{ by (5).}$$

Now, when $k>1$, applying Hölder's inequality with indices k and k', where $1/k+1/k'=1$, we get that

$$\begin{aligned} \sum_{n=2}^{m+1} n^{-k}\left|\varphi_n T_{n,1}^\alpha\right|^k &\leq \sum_{n=2}^{m+1} n^{-k}(A_n^\alpha)^{-k}|\varphi_n|^k\left\{\sum_{v=1}^{n-1} A_v^\alpha w_v^\alpha |\Delta\lambda_v|\right\}^k \\ &\leq \sum_{n=2}^{m+1} n^{-k} n^{-\alpha k}|\varphi_n|^k\left\{\sum_{v=1}^{n-1} v^{\alpha k}(w_v^\alpha)^k |\beta_v|\right\} \\ &\quad \times\left\{\sum_{v=1}^{n-1}|\beta_v|\right\}^{k-1} \end{aligned}$$

$$
\begin{aligned}
&= O(1)\sum_{v=1}^{m} v^{\alpha k}(w_v^{\alpha})^k \left|\beta_v\right| \sum_{n=v+1}^{m+1} \frac{n^{-k}\left|\varphi_n\right|^k}{n^{\alpha k}} \\
&= O(1)\sum_{v=1}^{m} v^{\alpha k}(w_v^{\alpha})^k \beta_v \sum_{n=v+1}^{m+1} \frac{n^{\varepsilon-k}\mid\varphi_n\mid^k}{n^{\alpha k+\varepsilon}} \\
&= O(1)\sum_{v=1}^{m} v^{\alpha k}(w_v^{\alpha})^k \beta_v v^{\varepsilon-k}\left|\varphi_v\right|^k \sum_{n=v+1}^{m+1} \frac{1}{n^{\alpha k+\varepsilon}} \\
&= O(1)\sum_{v=1}^{m} v^{\alpha k}(w_v^{\alpha})^k \beta_v v^{\varepsilon-k}\left|\varphi_v\right|^k \int_v^{\infty} \frac{dx}{x^{\alpha k+\varepsilon}} \\
&= O(1)\sum_{v=1}^{m} v\beta_v v^{-k}(w_v^{\alpha}\left|\varphi_v\right|)^k \\
&= O(1)\sum_{v=1}^{m-1} \Delta(v\beta_v)\sum_{r=1}^{v} r^{-k}(w_r^{\alpha}\left|\varphi_r\right|)^k \\
&\quad + O(1)m\beta_m\sum_{v=1}^{m} v^{-k}(w_v^{\alpha}\left|\varphi_v\right|)^k \\
&= O(1)\sum_{v=1}^{m-1} \left|\Delta(v\beta_v)\right| X_v + O(1)m\beta_m X_m \\
&= O(1)\sum_{v=1}^{m-1} \left|(v+1)\Delta\beta_v - \beta_v\right| X_v + O(1)m\beta_m X_m \\
&= O(1)\sum_{v=1}^{m-1} v\left|\Delta\beta_v\right| X_v + O(1)\sum_{v=1}^{m-1}\left|\beta_v\right| X_v \\
&\quad + O(1)m\beta_m X_m \\
&= O(1) \quad \text{as} \quad m\to\infty,
\end{aligned}
$$

by virtue of the hypotheses of the theorem and Lemma 2.

Finally, we have that

$$
\begin{aligned}
\sum_{n=1}^{m} n^{-k}\left|\varphi_n T_{n,2}^{\alpha}\right|^k &= O(1)\sum_{n=1}^{m}\left|\lambda_n\right| n^{-k}(w_n^{\alpha}\left|\varphi_n\right|)^k \\
&= O(1)\sum_{n=1}^{m-1}\Delta\left|\lambda_n\right|\sum_{v=1}^{n} v^{-k}(w_v^{\alpha}\left|\varphi_v\right|)^k
\end{aligned}
$$

$$
\begin{aligned}
&+ O(1)\,|\lambda_m| \sum_{n=1}^{m} n^{-k} (w_n^{\alpha}\,|\varphi_n|)^k \\
&= O(1) \sum_{n=1}^{m-1} |\Delta\lambda_n|\, X_n + O(1)\,|\lambda_m|\, X_m \\
&= O(1) \sum_{n=1}^{m-1} \beta_n X_n + O(1)\,|\lambda_m|\, X_m = O(1) \\
&\quad \text{as} \quad m \to \infty,
\end{aligned}
$$

by virtue of the hypotheses of the theorem and Lemma 2.

Therefore, we get that

$$
\sum_{n=1}^{m} n^{-k} \left|\varphi_n T_{n,r}^{\alpha}\right|^k = O(1) \text{ as } m \to \infty, \text{ for } r = 1, 2.
$$

This completes the proof of the theorem.

4. Special cases. 1. If we take (X_n) as a positive nondecreasing sequence and $\varepsilon = 1$, $\varphi_n = n^{\delta+1-(1/k)}$, with $0 \le \delta < 1$, then we get a result due to Bor [**3**].

2. If we take $\varepsilon = 1$ and $\varphi_n = n^{1-(1/k)}$, respectively $\varepsilon = 1$, $\alpha = 1$ and $\varphi_n = n^{1-(1/k)}$ in this theorem, then we get a new result related to $|C,\alpha|_k$, respectively $|C,1|_k$, summability factors.

3. If we take $\varepsilon = 1$ and $\varphi_n = n^{\delta+1-(1/k)}$, then we get a result due to Bor [**4**] concerning the $|C,\alpha;\delta|_k$ summability factors.

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