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BEST APPROXIMATION BY SMOOTH FUNCTIONS IN THE NONPERIODIC CASE

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ABSTRACT. Let W_n be the set of those functions $f \in C([0,1])$ which have absolutely continuous $(n-1)$th derivatives and nth derivatives with essential suprema bounded by one. Let $n \in \mathbf{N}$. The paper states, if $f \in C([0,1]) \setminus W_n$, then there is a separating measure $\lambda \in C([0,1])^\star$ for f and W_n which has finite support and for all $f \in C([0,1]) \setminus W_n$ there is no bound for $|\operatorname{supp} \lambda|$.

1. Introduction. Let $C([0,1])$ be the space of continuous real valued functions defined on the interval $[0,1]$, equipped with the uniform norm, and let W_n be the set of those functions $f \in C([0,1])$ which have absolutely continuous $(n-1)$th derivatives and whose nth derivatives satisfy the condition $\|f^{(n)}\|_\infty \le 1$.

The central results of [**1**] concerning best approximation from W_n in $C([0,1])$ have as a corollary the fact that if $f \in C([0,1]) \setminus W_n$ then there exists a separating measure $\lambda \in C([0,1])^\star \cong \mathcal{M}([0,1])$, the space of real valued regular Borel measures λ on $[0,1]$ for f and W_n which has finite support. In [**4**, Theorem 3.1.7], there is a direct proof of periodic case, and now we give a direct proof of the nonperiodic case (Theorem 2.6). Both of them are a result of [**1**, Theorem 1].

Although in [**5**] it has been stated that, if $\mathcal{M}$ is a finite-dimensional subspace of $C([0,1])$ then there exists a separating measure λ for $f \in C([0,1]) \setminus \mathcal{M}$ and $\mathcal{M}$, $|\operatorname{supp} \lambda| \le \dim \mathcal{M} + 1$, that is, for all $f \in C([0,1]) \setminus \mathcal{M}$, there exists a separating measure λ for f and $\mathcal{M}$ such that $|\operatorname{supp} \lambda|$ is bounded by $\dim \mathcal{M} + 1$. But, in Section 3 (Theorem 3.3), we shall show that no such result holds for best approximation by W_n. That is, for any $m \in \mathbf{N}$, there exists $f \in C([0,1]) \setminus W_n$ such that $|\operatorname{supp} \lambda| \ge m$ for any separating measures λ for f and W_n.

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Require theorems. The uniform norm is defined by

$$\|f\| = \max_{t\in[0,1]} |f(t)| \quad \text{for all} \quad f \in C([0,1]),$$

and

$$d(f, W_n) = \inf_{g\in W_n} \|f - g\| \quad \text{for} \quad f \in C([0,1])$$

is called the distance from f to W_n; if the inf attains at $g_0 \in W_n$ we say that g_0 is a best approximation to f from W_n. The set W_n is a nonempty convex boundedly compact subset of $C([0,1])$; hence, W_n is a proximinal subset of the space $C([0,1])$. That is, for each $f \in C([0,1])$, there exists a best uniform approximation g_0 from W_n. If $f \in C([0,1])\backslash W_n$, then the set W_n and the open ball $B(f,\, d(f, W_n))$ are convex and disjoint and can be separated by a nonzero linear functional $\lambda \in C([0,1])^\star \cong \mathcal{M}([0,1])$.

Let $\varphi_1(x) = \chi_{[0,1]}$. Define φ_n for $n \geq 2$ to be the convolution powers of φ_1, that is,

$$\varphi_n = \varphi_{n-1} * \varphi_1.$$

If $f \in L^1([0,1])$, then $(\varphi_1 * f)(x) = \int_{[0,1]} \varphi_1(x-y) f(y)\, dy = \int_0^x f(y)\, dy$ and $\varphi_n * f$ is an nth integral of f. Now consider the kernel

$$K_n(x,y) = \frac{(x-y)_+^{n-1}}{(n-1)!} \quad \text{for} \quad n > 1;$$

it follows that $K_n(x,y) = \varphi_n(x-y)$ for $n > 1$ and so $\varphi_n * f = K_n * f$ for $n > 1$.

If $\lambda \in \mathcal{M}([0,1])$, then

$$(\varphi_1 * \lambda)(x) = \int_{[0,1]} \varphi_1(x-y)\, d\lambda(y) = \lambda([0,x]),$$

and $\varphi_n * \lambda$ is an $(n-1)$th integral of $\lambda([0,\cdot])$. If λ is a measure defined by $d\lambda(y) = f(y)\, dy$, where f is Lebesgue integrable on $[0,1]$, then $\varphi_n * \lambda$ is just $\varphi_n * f$.

The following theorem of [**2**, Theorem 2.2] is required.

Theorem 1.1. *Let $\lambda \in \mathcal{M}([0,1])$ and $n \geq 2$. If μ is the function defined by*

$$\mu(y) = \int_{[0,1]} K_n(x,y)\, d\lambda(x),$$

then μ is $n-2$ times differentiable, $\mu^{(n-2)}$ has left and right derivatives $\mu_-^{(n-1)}$ and $\mu_+^{(n-1)}$ at every point and

$$\mu_-^{(n-1)}(y) = (-1)^{n-1}\lambda([y,1]),$$
$$\mu_+^{(n-1)}(y) = (-1)^{n-1}\lambda((y,1]).$$

The following simple results will be required.

Proposition 1.2. *Suppose $\lambda \in \mathcal{M}([0,1])$, and let $\mu \in C([0,1])$, $n \in \mathbf{N}$,*

$$\mu(y) = \int_{[0,1]} \varphi_n(x-y)\,d\lambda(x).$$

(i) *If $a < b$, then* $\operatorname{supp}\lambda \cap (a,b) = \varnothing$ *if and only if the restriction of μ to (a,b) is a polynomial of degree $\leq n-1$.*

(ii) *If* $\operatorname{supp}\lambda^+ \cap \operatorname{supp}\lambda^- = \varnothing$, *then μ is a piecewise monotonic function.*

Proof. (i) If $n = 1$, then

$$\mu(y) = \int_{[0,1]} \varphi_1(x-y)\,d\lambda(x) = \int_y^1 d\lambda(x) = \lambda([y,1]),$$

so μ is constant on (a,b) if and only if $\operatorname{supp}\lambda \cap (a,b) = \varnothing$. For $n \geq 2$, by Theorem 1.1, there exist $\mu_+^{(n-1)}$ and $\mu_-^{(n-1)}$ at every point and

$$\mu_-^{(n-1)}(y) = (-1)^{n-1}\lambda([y,1]),$$
$$\mu_+^{(n-1)}(y) = (-1)^{n-1}\lambda((y,1]),$$

so $\mu^{(n-1)}$ exists and is constant on (a,b) if and only if $\operatorname{supp}\lambda \cap (a,b) = \varnothing$. This proves (i).

(ii) Suppose $\operatorname{supp}\lambda^+ \cap \operatorname{supp}\lambda^- = \varnothing$; thus, $\lambda([0,\cdot])$ is piecewise monotonic on $[0,1]$. It follows that $\varphi_n * \lambda$, which is a repeated integral of λ, is also piecewise monotonic. □

2. Separating measures. The following theorem is the general result of [**1**, Theorem 1] which, applied to the subset W_n of $C([0,1])$, contains the following preliminary characterization theorem as a special case.

Theorem 2.1. *Let $n \in \mathbf{N}$. Suppose $f \in C([0,1]) \setminus W_n$, $g_0 \in W_n$, $\lambda \in C([0,1])^\star \setminus \{0\}$ and let w_n be defined by*

$$w_n(\lambda) = \varphi_n * \lambda.$$

Then the two conditions

I(a) *g_0 is a best approximation to f from W_n,*

I(b) *$\lambda(g) < \lambda(h)$, for all $g \in W_n$ and $h \in B(f, d(f, W_n))$,*

together are equivalent to the three conditions

II(i) *$\lambda(p) = 0$, for all $p \in P_{n-1}$, i.e., $\lambda \in P_{n-1}^\perp$, where P_{n-1} is the set of polynomials of degree not greater than $n-1$,*

II(ii) *$g_0^{(n)}(y) = \operatorname{sgn} w_n(\lambda)(y)$ for almost every y in $[0,1] \setminus w_n(\lambda)^{-1}(0)$,*

II(iii) *$\lambda(f - g_0) = \|\lambda\| \|f - g_0\|$ or, equivalently,*

$$\begin{aligned} \operatorname{supp} \lambda^+ &\subseteq (f - g_0)^{-1}(\|f - g_0\|), \\ \operatorname{supp} \lambda^- &\subseteq (f - g_0)^{-1}(-\|f - g_0\|). \end{aligned}$$

If $f \in C([0,1]) \setminus W_n$, then a measure $\lambda \in \mathcal{M}([0,1])$, which satisfies condition I(b) will be called a *separating measure* for f and W_n. Let $S(f, W_n)$ denote the set of separating measures for f and W_n. Note that if λ is a separating measure for f and W_n, then, by condition II(iii), $\|\lambda\| = 1$ if and only if $\|\lambda\| \leq 1$ and $\lambda(f - g_0) = \|f - g_0\|$. It follows that the set $\{\lambda \in S(f, W_n) : \|\lambda\| = 1\}$ is a weak*-compact subset of $C([0,1])^\star \cong \mathcal{M}([0,1])$.

If $\lambda \in S(f, W_n)$, then the function $w_n(\lambda)$ will be called an *associated function* of f and W_n or the associated function of λ.

The next proposition, which is a straightforward consequence of Theorem 2.1, gives conditions which are sufficient to ensure that a regular Borel measure λ is an element of $S(f, W_n)$.

Proposition 2.2. *Suppose that* $f \in C([0,1]) \setminus W_n$ *and* $\lambda_0 \in S(f, W_n)$. *If* $\lambda \in \mathcal{M}([0,1]) \cap P_{n-1}^{\perp}$,

$$\text{(1)} \qquad \operatorname{supp} \lambda^+ \subseteq \operatorname{supp} \lambda_0^+, \qquad \operatorname{supp} \lambda^- \subseteq \operatorname{supp} \lambda_0^-,$$

$$\text{(2)} \qquad w_n(\lambda_0)^{-1}(0) \subseteq w_n(\lambda)^{-1}(0)$$

and

$$\text{(3)} \qquad w_n(\lambda)(y)\, w_n(\lambda_0)(y) \geq 0 \quad \textit{for all} \quad y \in [0,1],$$

then λ *is also an element of* $S(f, W_n)$.

Proof. If $\lambda_0 \in S(f, W_n)$, then λ_0 satisfies conditions II(i)–(iii) of Theorem 2.1. It follows from the conditions (1)–(3) that λ also satisfies conditions II(i)–(iii). So the conclusion follows by Theorem 2.1. □

In this part it is first established that there exist separating measures with minimal supports and associated functions with maximal zero sets.

Theorem 2.3. *Suppose that* $f \in C([0,1]) \setminus W_n$. *If* λ_1 *is a separating measure for* f *and* W_n, *then there exists a separating measure* λ_0 *such that* $\operatorname{supp} \lambda_0 \subseteq \operatorname{supp} \lambda_1$ *and if* λ *is also a separating measure and* $\operatorname{supp} \lambda \subseteq \operatorname{supp} \lambda_0$, *then* $\operatorname{supp} \lambda = \operatorname{supp} \lambda_0$.

Proof. Suppose that $\lambda \in S(f, W_n)$. Let

$$L(\lambda) = \{\lambda' \in S(f, W_n) : \operatorname{supp} \lambda' \subseteq \operatorname{supp} \lambda,\ \|\lambda'\| = 1\},$$

which is a weak*-compact subset of $\mathcal{M}([0,1])$, and let $\mathcal{L} = \{\operatorname{supp} \lambda' : \lambda' \in L(\lambda)\}$. If L' is a chain in $\mathcal{L}$, then $A = (\lambda_\alpha : \alpha \in L')$ is a net of measures (for $\alpha \in L'$ choose a measure in $L(\lambda)$, λ_α say, such that $\alpha = \operatorname{supp} \lambda_\alpha$). We want to find a lower bound in $\mathcal{L}$ for the chain L'. Then by Zorn's lemma, $\mathcal{L}$ contains a minimal element.

Since $L(\lambda)$ is weak*-compact then the net A has a cluster point, λ_0 say, that is, the net A has a convergent subnet (convergent to λ_0). If there is an open set $V \subset [0,1]$ such that $\lambda(f) = 0$ whenever $\operatorname{supp} f \subset V$

(that is, $f|_{[0,1]\setminus V} \equiv 0$) for λ in convergent subnet of A, then $\lambda_0(f) = 0$; that is, for each open set $V \subset [0,1]$,

$$V \subset [0,1] \setminus \operatorname{supp} \lambda \longrightarrow V \subset [0,1] \setminus \operatorname{supp} \lambda_0.$$

So $\operatorname{supp} \lambda_0 \subseteq \operatorname{supp} \lambda_\alpha$, for each $\alpha \in L'$. That is, $\operatorname{supp} \lambda_0$ is a lower bound in $\mathcal{L}$ for chain L'. □

Theorem 2.4. *Suppose $n \in \mathbf{N}$. Let $f \in C([0,1]) \setminus W_n$, and let λ_1 be a separating measure for f and W_n with minimal support. Then there exists $\lambda_0 \in S(f, W_n)$ such that*

(i) $\operatorname{supp} \lambda_0 = \operatorname{supp} \lambda_1$,

(ii) $w_n(\lambda)^{-1}(0) = w_n(\lambda_0)^{-1}(0)$ *wherever* $\lambda \in S(f, W_n)$, $\operatorname{supp} \lambda = \operatorname{supp} \lambda_0$ *and* $w_n(\lambda)^{-1}(0) \supseteq w_n(\lambda_0)^{-1}(0)$.

Proof. For each $\lambda \in S(f, W_n)$, let $I(\lambda)$ denote the set of $\lambda' \in S(f, W_n)$ such that $\|\lambda'\| = 1$, $\operatorname{supp} \lambda' \subseteq \operatorname{supp} \lambda$ and $w_n(\lambda')^{-1}(0) \supseteq w_n(\lambda)^{-1}(0)$. Each of the sets $I(\lambda)$ is a nonempty and compact subset of $\mathcal{M}([0,1])$. The completion of the proof now follows that of the previous theorem. □

The next lemma is the nonperiodic variant of [**4**, Lemma 3.1.6]. The proof of the two lemmas are essentially the same.

Lemma 2.5. *Let $n \in \mathbf{N}$. Let $f \in C([0,1]) \setminus W_n$, and let $\lambda_0 \in S(f, W_n)$ be such that λ_0 has minimal support and $w_n(\lambda_0)$ has maximal zero set, that is, satisfies condition* (ii) *of Theorem* 2.4. *If $0 \le a < b \le 1$ and $w_n(\lambda_0)^{-1}(0) \cap [a,b] = \varnothing$, then $|\operatorname{supp} \lambda_0 \cap [a,b]| \le 3n+1$.*

Proof. The lemma will be proved by contradiction. Suppose that $|\operatorname{supp} \lambda_0 \cap [a,b]| > 3n+1$. Let $B_1, \ldots, B_{3n+2}$ be disjoint closed subintervals of $[a,b]$ such that $\lambda_0|_{B_j} \neq 0$, for $j = 1, \ldots, 3n+2$. Let $\lambda_j \in \mathcal{M}([0,1])$, for $j = 1, \ldots, 3n+1$, be such that $\lambda_j(A) = \lambda_0(A \cap B_j)$ for each Borel subset A of $[0,1]$.

Let $y_1, \ldots, y_n$ be distinct points of $(0,a)$, and let $y_{n+1}, \ldots, y_{2n}$ be distinct points of $(b,1)$. (If $a = 0$ or $b = 1$ and $(a,b) \neq (0,1)$, then a part of this step will be removed and the upper bound for $|\operatorname{supp} \lambda_0 \cap [a,b]|$

is $2n+1$.) Then there exists a nonzero $(a_1, \dots, a_{3n+1}) \in \mathbf{R}^{3n+1}$ such that

$$(a_1\lambda_1 + \cdots + a_{3n+1}\lambda_{3n+1})(p_i) = 0,$$
$$\text{where} \quad p_i(x) = x^{i-1}, \quad \text{for} \quad i = 1, \dots, n,$$

and

$$w_n(a_1\lambda_1 + \cdots + a_{3n+1}\lambda_{3n+1})(y_k) = 0 \quad \text{for} \quad k = 1, \dots, 2n.$$

Let $\lambda = a_1\lambda_1 + \cdots + a_{3n+1}\lambda_{3n+1}$. So $\lambda \neq 0$ and $\lambda \in P_{n-1}^{\perp}$. Now $\operatorname{supp} \lambda \cap (b, 1) = \varnothing$ (and also $\operatorname{supp} \lambda \cap (0, a) = \varnothing$). Thus, Proposition 1.2 implies that the restriction of $w_n(\lambda)$ to each of $(b, 1)$ and $(0, a)$ is a polynomial of degree less than or equal to $n-1$ with n zeros in $(0, a)$ and n zeros in $(b, 1)$. So $[0, a)$ and $(b, 1]$ are zero intervals of $w_n(\lambda)$.

Let $\varepsilon = \operatorname{sgn} w_n(\lambda_0)(y)$, for all $y \in [a, b]$. Let J be the set of $t \in \mathbf{R}$ such that

$$ta_j > -1 \quad \text{for} \quad j = 1, \dots, 3n+1,$$

and

$$\varepsilon w_n(\lambda_0 + t\lambda)(y) > 0 \quad \text{for all} \quad y \in [a, b].$$

Then $0 \in J \neq \mathbf{R}$ and J is an open subinterval of $\mathbf{R}$.

Suppose $t \in \overline{J}$. Then $\operatorname{supp}(\lambda_0 + t\lambda) \supseteq \operatorname{supp} \lambda_0 \cap B_{3n+2} \neq \varnothing$ so that $\lambda_0 + t\lambda \neq 0$; $\lambda_0 + t\lambda \in P_{n-1}^{\perp}$ and

$$\operatorname{supp}(\lambda_0 + t\lambda)^+ \subseteq \operatorname{supp} \lambda_0^+,$$
$$\operatorname{supp}(\lambda_0 + t\lambda)^- \subseteq \operatorname{supp} \lambda_0^-.$$

Furthermore,

$$w_n(\lambda_0 + t\lambda) = w_n(\lambda_0) + t\, w_n(\lambda),$$

so it follows that

$$w_n(\lambda_0 + t\lambda)^{-1}(0) \supseteq w_n(\lambda_0)^{-1}(0)$$

and

$$w_n(\lambda_0 + t\lambda)(y) \quad w_n(\lambda_0)(y) \geq 0 \quad \text{for all} \quad y \in [0, 1].$$

Therefore, by Proposition 2.2, $\lambda_0 + t\lambda \in S(f, W_n)$.

Now let t be a point of the nonempty boundary of J. Then either $ta_j = -1$ for some $j \in \{1, \dots, 3n+1\}$, in which case $\operatorname{supp}(\lambda_0 + t\,\lambda) \cap$

$B_j = \varnothing$ and $\operatorname{supp}(\lambda_0 + t\lambda) \neq \operatorname{supp}\lambda_0$, in contradiction to the fact that λ_0 is a separating measure of minimal support, or $w_n(\lambda_0 + t\lambda)(y) = 0$ for some $y \in [a, b]$, in contradiction to the fact that $w_n(\lambda_0)$ has a maximal zero set. The proof of the lemma is complete. □

The *finite support theorem* now follows easily.

Theorem 2.6. *Let $n \in \mathbf{N}$. If λ is a separating measure of minimal support for some $f \in C([0, 1]) \setminus W_n$, then* $\operatorname{supp}\lambda$ *is finite.*

Proof. By Theorem 2.4 it may be supposed, after replacing λ by another measure with the same support if necessary, that the associated function $w_n(\lambda)$ has maximal zero set. It follows from (ii) of Proposition 1.2 that $w_n(\lambda)$ is a nonzero piecewise monotonic function. Therefore, $w_n(\lambda)^{-1}(0)$ is a union of a finite family of zero intervals and a finite set of isolated points. If I is a zero interval of $w_n(\lambda)$, then Proposition 1.2 implies that $\operatorname{supp}\lambda \cap \operatorname{int} I$ is empty. If J is an open interval of $[0, 1]$ disjoint from $w_n(\lambda)^{-1}(0)$, then in Lemma 2.5 it follows that $|\operatorname{supp}\lambda \cap J| < 3n + 2$. This proves the theorem. □

3. On the separating measures for W_n. The main result of the previous section (Theorem 2.6) is the fact that if $n \in \mathbf{N}$ and λ is a separating measure of minimal support for some $f \in C([0, 1]) \setminus W_n$, then $\operatorname{supp}\lambda$ is finite. If $\mathcal{M}$ is a finite-dimensional subspace of $C([0, 1])$, then there exists a separating measure λ for $f \in C([0, 1]) \setminus \mathcal{M}$ and $\mathcal{M}, |\operatorname{supp}\lambda| \leq \dim \mathcal{M}, +1$, [**5**]. The main result of this section, Theorem 3.3, shows that no such result holds for best approximation by W_n. In fact, it is established that for any $m \in \mathbf{N}$, there exists $f \in C([0, 1]) \setminus W_n$ such that $|\operatorname{supp}\lambda| \geq m$ for any $\lambda \in S(f, W_n)$, that is, there is no m such that $|\operatorname{supp}\lambda| \leq m$ for all $\lambda \in S(f, W_n)$ and $f \in C([0, 1]) \setminus W_n$.

Let ψ_{n-1} be the set of spline functions defined on $\mathbf{R}$ which are of degree $n - 1$ and have a finite set of simple knots. If w is a continuous function, then it will be said that $[a, b]$ is a zero interval of w if $a < b$ and $w(y) = 0$ for all $y \in [a, b]$. If $w \in \psi_{n-1}$ and $w(y) = 0$, then $Z_n(w, y)$ will be the multiplicity of zero y of w as defined in [**2**]. That is, if $1 \le \alpha \le n-2$ and

$$w(y) = w^{(1)}(y) = \cdots = w^{(\alpha-1)}(y) = 0,\ w^{(\alpha)}(y) \neq 0,$$

then $Z_n(w,\, y) = \alpha$. If

$$w(y) = w^{(1)}(y) = \cdots = w^{(n-2)}(y) = 0,$$

then $Z_n(w,\, y)$ is either $n-1$ if w changes sign at y or n if w does not change sign at y. It follows that if y is a point of a zero interval of w then $Z_n(w,\, y) = n$. Distinct zeros y and y' of w are said to be separated zeros of w if the interval with endpoints y, and y' is not a zero interval of w. If I is an interval of $\mathbf{R}$, then $Z_n(w,\, I)$ will denote the maximal number of separated zeros of w on I, each zero being counted according to its multiplicity.

If $\alpha_1, \ldots, \alpha_k \in \mathbf{R}$, then $\operatorname{Scc}(\alpha_1, \ldots, \alpha_k)$ denotes the number of strict sign changes in the sequence $\alpha_1, \ldots, \alpha_k$. Let $\lambda \in \mathcal{M}([a,b])$ and $\operatorname{supp} \lambda \cap [a,b]$ be finite. If I is an interval and $\operatorname{supp} \lambda \cap I = \{x_1, \ldots, x_m\}$, define

$$\operatorname{Scc}(\lambda,\, I) = \operatorname{Scc}(\lambda(x_1), \ldots, \lambda(x_m)).$$

The following proposition is a result of [**2**, Corollary 1.7].

Proposition 3.1. *Let $n > 1$ and $a < b$. Then $Z_n(w_n(\lambda), [a,b]) \le \operatorname{Scc}(\lambda, [a,b]) + n$.*

The following lemma [**3**, Lemma 4.7.7] is required.

Lemma 3.2. *Let $n \in \mathbf{N}$, $q \ge 2$, $m = q + n - 1$, $x_1 < \cdots < x_m$, $z_1 < \cdots < z_q$, $x_1 = z_1$, $x_m = z_q$ and $Z = \{z_1, \ldots, z_q\}$. The following two sets of conditions on $x_1, \ldots, x_m, z_1, \ldots, z_q$ and Z are equivalent.*

(i) *$x_i < z_i < x_{i+n-1}$ for all $i = 2, \ldots, q-1$,*

(ii) *$|Z \cap (x_j, x_k)| > k - j - n$ whenever $1 \le j \le k \le m$ and $(j,k) \neq (1,m)$.*

Proof. If $j \in \{2, \ldots, m-n\}$, then

$$x_j < z_j \quad \text{if and only if} \quad |Z \cap (x_j, x_m)| > m - j - n. \tag{4}$$

If $k \in \{2, \dots, m-n\}$, then

(5) $$z_k < x_{k-1+n} \quad \text{if and only if} \quad |Z \cap (x_1, x_k)| > k-1-n.$$

If $1 < j < k < m$, then

(6) $$|Z \cap (x_j, x_k)| = |Z \cap (x_1, x_k)| + |Z \cap (x_j, x_m)| - |Z \cap (x_1, x_m)|.$$

The lemma now follows from (4), (5) and (6). □

Theorem 3.3. *Let $n \in \mathbf{N}$. For a given $m \geq n+1$, there exists $f \in C([0,1]) \setminus W_n$ such that if λ is a separating measure for f and W_n then $|\operatorname{supp} \lambda| = m$.*

Proof. At first, we find $f \in C([0,1]) \setminus W_n$ and $g_0 \in W_n$ where g_0 is a best approximation to f from W_n, and then it is established that, for any separating measure λ for f and W_n, $|\operatorname{supp} \lambda \cap [0,1]| = m$. Now let $m \in \mathbf{N}$ and $m \geq n+1$. We choose arbitrary interval $[a,b]$, where $0 < a < b < 1$, an integer $q = m-n+1 > 1$, a sign $\varepsilon \in \{-1, 1\}$, points

$$a = z_1 < \cdots < z_q = b,$$

let $Z = \{z_1, \dots, z_q\}$ and $g_0 \in W_n$ such that

$$g_0^{(n)}(x) = (-1)^{q+i}\varepsilon \quad \text{for all} \quad x \in (z_{i-1}, z_i) \quad \text{and} \quad i \in \{2, \dots, q\},$$

and

(7) $$g_0^{(n)}(x) = 0 \quad \text{for all} \quad x \in [0,1] \setminus [a,b].$$

Next we choose

$$a = x_1 < \cdots < x_m = b$$

such that

(8) $$x_j < z_j < x_{j-1+n} \quad \text{for} \quad j = 2, \dots, m-n.$$

We choose $f \in C([0,1]) \setminus W_n$ and $d \in \mathbf{R}^+$ such that

$$(f - g_0)(x_j) = (-1)^{m+j}\varepsilon d \quad \text{for all} \quad j \in \{1, \dots, m\},$$

and

$$|(f-g_0)(x)| < d \quad \text{for all} \quad x \in [0,1] \setminus \{x_1, \dots, x_m\}.$$

In fact $d = \|f - g_0\|$.

Suppose that $\lambda \in S(f, W_n)$ with minimal support (existence by Theorem 2.3) and associated function $w_n(\lambda)$ satisfies condition II(ii) of Theorem 2.1. Then $\lambda(f - g_0) = \|\lambda\| \|f - g_0\|$ or, equivalently,

$$(f-g_0)(x) = \|f-g_0\| \quad \text{for all} \quad x \in \operatorname{supp} \lambda^+$$

and

$$(f-g_0)(x) = -\|f-g_0\| \quad \text{for all} \quad x \in \operatorname{supp} \lambda^-.$$

So $\operatorname{supp} \lambda \subseteq \{x_1, \dots, x_m\} \subseteq [a,b]$ and $\operatorname{supp} \lambda \cap ([0,1] \setminus [a,b]) = \varnothing$. Thus, the restriction of $w_n(\lambda)$ to each of $[0,a]$ and $[b,1]$ is a polynomial of degree less than or equal to $n-1$, Proposition 1.2.

On the other hand, Theorem 2.1 II(ii) and (7) imply that $w_n(\lambda)$ is zero almost everywhere on $[0,1] \setminus [a,b]$; but, it is a polynomial on each of $[0,a]$ and $[b,1]$, so

$$w_n(\lambda)(x) = 0 \quad \text{for all} \quad x \in [0,1] \setminus [a,b],$$

i.e., $[0,1] \setminus [a,b] \subseteq w_n(\lambda)^{-1}(0)$. Now we claim that $w_n(\lambda)$ has no zero interval in $[a,b]$, for if $[x_j, x_k] \subset [x_1, x_m]$ for some $(j,k) \neq (1,m)$ is a maximal interval in $[x_1, x_m]$ such that $w_n(\lambda)$ has no zero interval in $[x_j, x_k]$, then Proposition 3.1 implies that

$$Z_n(w_n(\lambda),\, (x_j, x_k)) \leq \operatorname{Scc}(\lambda,\, [x_j, x_k]) - n.$$

So Lemma 3.2 and (8) imply that

$$\begin{aligned} k - j - n &< |Z \cap (x_j, x_k)| \leq |w_n(\lambda)^{-1}(0) \cap (x_j, x_k)| \\ &\leq Z_n(w_n(\lambda),\, (x_j, x_k)) \leq \operatorname{Scc}(\lambda,\, [x_j, x_k]) - n \leq k - j - n, \end{aligned}$$

which is impossible. Therefore, $w_n(\lambda)$ has no zero interval in $[a,b]$. Now Proposition 3.1 implies that

$$\begin{aligned} q - 2 = |Z \cap (x_1, x_m)| &\leq |w_n(\lambda)^{-1}(0) \cap (x_1, x_m)| \\ &\leq Z_n(w_n(\lambda),\, (x_1, x_m)) \leq \operatorname{Scc}(\lambda,\, [x_1, x_m]) - n. \end{aligned}$$

Now if $\operatorname{supp}\lambda \neq \{x_1, \dots, x_m\}$, then $\operatorname{Scc}(\lambda, [x_1, x_m]) \leq m - 2$ and so

$$q - 2 \leq m - n - 2,$$

but $m = q + n - 1$, which is absurd. So $\operatorname{supp}\lambda = \{x_1, \dots, x_m\}$, and so the proof is complete. □

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