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CONJUGACY CRITERIA FOR THE HALF-LINEAR SECOND ORDER DIFFERENTIAL EQUATION

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ABSTRACT. In this paper, some conjugacy criteria for the half-linear second order differential equation

$$\Big(|y'(t)|^{p-1}\mathrm{sgn}\, y'(t)\Big)' + c(t)|y(t)|^{p-1}\mathrm{sgn}\, y(t) = 0, \quad p > 1$$

are obtained.

1. Introduction. We are concerned with the zeros of solutions of the half-linear second order differential equation

$$\Big(\phi(y'(t))\Big)' + c(t)\phi(y(t)) = 0 \tag{1.1}$$

where $c(t)$ is a continuous function on $\mathbf{R}$ and $\phi(s)$ is the real function defined by $\phi(s) := |s|^{p-1}\mathrm{sgn}\, s$, with $p > 1$. In the case $p = 2$, equation (1.1) reduces to the linear equation

$$y''(t) + c(t)y(t) = 0. \tag{1.2}$$

The investigation of qualitative properties of (1.1) was initiated by Elbert who proved that the zeros of linearly independent solutions of (1.1) interlace and that the Sturm comparison theorem extends to half-linear equations [**7**]. Moreover, he had attracted the attention of many authors for an expected similarity between the qualitative properties of (1.1) and (1.2).

As in the case of linear equations, equation (1.1) is called *disconjugate* if it has a solution without any zeros. Otherwise, it is called *conjugate*.

Concerning the linear case, conjugacy criteria for (1.2) and the general equation

$$(a(t)y'(t))' + c(t)y(t) = 0 \tag{1.3}$$

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have been investigated in several papers, see [3, 4, 6, 9, 12, 16]. In [**16**], Tipler proved that the linear equation (1.2) is conjugate under the condition $\int_{-\infty}^{\infty} c(t)\,dt > 0$. Pena [**13**] has shown that the same condition is also sufficient for conjugacy in $\mathbf{R}$ for the half-linear equation (1.1). Other conjugacy criteria for (1.1) can be found in [5, 13, 15].

In this paper, we give some conditions on the function $c(t)$ to guarantee the conjugacy of (1.1).

Let $y(t)$ be a solution of the equation (1.1) such that $y(t) \neq 0$; then the function $W(t) = \phi(y')/\phi(y)$ is a solution of the generalized Riccati equation

$$W'(t) + (p-1)|W(t)|^q = -c(t) \tag{1.4}$$

where $q = p/(p-1)$ is the conjugate number of p. The relation between solutions of (1.1) and (1.4) is very useful for the investigation of conjugacy property of the half-linear equations.

2. Main results. For the convenience of the reader, we start with a lemma which is a variant of the generalized Hartman's lemma, cf. [**14**].

Lemma 2.1. *Suppose that the generalized Riccati equation*

$$W'(t) + (p-1)|W(t)|^q = -c(t), \quad t \in \mathbf{R} \tag{2.1}$$

has a solution. If

$$\liminf_{T\to\infty} \frac{1}{T}\int_0^T \left(\int_{-t}^{t} c(s)\,ds\right) dt = k > -\infty, \tag{2.2}$$

then

$$\int_{-\infty}^{\infty} |W(t)|^q\,ds < \infty. \tag{2.3}$$

Proof. From equation (2.1), we get

$$\frac{1}{t}\int_0^t \Big[W(s) - W(-s)\Big]\,ds + \frac{p-1}{t}\int_0^t \int_{-s}^{s} |W(\xi)|^q\,d\xi\,ds = -\frac{1}{t}\int_0^t \int_{-s}^{s} c(\xi)\,d\xi\,ds. \tag{2.4}$$

That is,

$$\frac{1}{t}\int_0^t \Big[W(s)-W(-s)\Big]\,ds+\frac{p-1}{2t}\int_0^t\int_{-s}^{s}|W(\xi)|^q\,d\xi\,ds \\ = -\frac{p-1}{2t}\int_0^t\int_{-s}^{s}|W(\xi)|^q\,d\xi\,ds-\frac{1}{t}\int_0^t\int_{-s}^{s}c(\xi)\,d\xi\,ds. \tag{2.5}$$

Towards contradiction, suppose that (2.3) does not hold. That is, $\int_{-\infty}^{\infty}|W(s)|^q\,ds=\infty$. Then, by assumption (2.2) the righthand side of (2.5) tends to $-\infty$ as $t\to\infty$.

Thus, for large values of t,

$$\frac{1}{t}\int_0^t \Big[W(s)-W(-s)\Big]\,ds+\frac{p-1}{2t}\int_0^t\int_{-s}^{s}|W(\xi)|^q\,d\xi\,ds<0. \tag{2.6}$$

Then, for $t\ge T$, we have

$$\frac{p-1}{2t}\int_0^t\int_{-s}^{s}|W(\xi)|^q\,d\xi\,ds<-\frac{1}{t}\int_0^t\Big[W(s)-W(-s)\Big]\,ds. \tag{2.7}$$

On the other hand, by Holder's inequality, we get

$$\begin{aligned}\frac{1}{t}\int_0^t &\Big[W(s)-W(-s)\Big]\,ds \\ &\le \frac{1}{t}\int_0^t\Big[|W(s)|+|W(-s)|\Big]\,ds \\ &\le \frac{1}{t}\,t^{1/p}\left[\left(\int_0^t|W(s)|^q\,ds\right)^{1/q}+\left(\int_0^t|W(-s)|^q\,ds\right)^{1/q}\right] \\ &\le \frac{1}{t}\,t^{1/p}(1^p+1^p)^{1/p}\left[\int_0^t(|W(s)|^q+|W(-s)|^q)\,ds\right]^{1/q} \\ &= \frac{(2t)^{1/p}}{t}\left(\int_{-t}^{t}|W(s)|^q\,ds\right)^{1/q}.\end{aligned} \tag{2.8}$$

Let $S(t)=\int_0^t\int_{-s}^{s}|W(\xi)|^q\,d\xi\,ds$, which tends to ∞. Then equation (2.8) can be rewritten as

$$\frac{S'}{S^q}>\left(\frac{p-1}{2}\right)^q\left(\frac{1}{2}\right)^{q/p}\frac{1}{t^{q-1}}. \tag{2.9}$$

Integrating this inequality from T to t, $T \leq t$, we get
(2.10)

$$\frac{1}{1-q}\left[\frac{1}{(S(t))^{q-1}} - \frac{1}{(S(T))^{q-1}}\right] > \left(\frac{p-1}{2^{(p+1)/p}}\right)^q \cdot \frac{1}{2-q}\Big[T^{2-q} - t^{2-q}\Big], \quad \text{if} \quad q \neq 2$$

and

$$\frac{1}{S(T)} - \frac{1}{S(t)} > \frac{1}{8}\ln\left(\frac{t}{T}\right), \quad \text{if} \quad q = 2, \tag{2.11}$$

which is a contradiction, since the righthand side tends to ∞ as $t \to \infty$ but the lefthand side is bounded. Then $\int_{-\infty}^{\infty} |W(s)|^q \, ds < +\infty$. □

Theorem 2.2. *Equation* (1.1) *is conjugate on* **R** *if* $c(t) \not\equiv 0$ *and*

$$\liminf_{T\to\infty} \frac{1}{T}\int_0^T \int_{-t}^{t} c(s)\, ds\, dt \geq 0. \tag{2.12}$$

Proof. Assume the converse, that is, there exists a solution of equation (1.1) having no zero in **R**. So, the generalized Riccati equation (2.1) has a solution $W(t) \not\equiv 0$.

Following the same steps in the proof of Lemma 2.1, we arrive at inequality (2.8) which, together with (2.3), leads to

$$\lim_{t\to\infty} \frac{1}{t}\int_0^t \Big[W(s) - W(-s)\Big]\, ds = 0.$$

Now, taking the lower limit, as $t \to \infty$, of both sides of equation (2.4), one gets

$$\liminf_{t\to\infty} \frac{1}{t}\int_0^t \int_{-s}^{s} c(\xi)\, d\xi\, dt = -(p-1)\int_{-\infty}^{\infty} |W(s)|^q\, ds < 0,$$

which contradicts assumption (2.12). □

The following two lemmas will be used in the sequel.

Lemma 2.3. *If $q > 1$, then for any two real numbers X, Y, we get*

$$\left(\frac{|X| + |Y|}{2}\right)^q \leq \frac{|X|^q + |Y|^q}{2}. \tag{2.13}$$

Equality holds if and only if $|X| = |Y|$.

Lemma 2.4. *Suppose that equation* (1.1) *has a positive solution $x(t)$. Then, for every $a \in \mathbf{R}$, the equation*

$$(\phi(y'(t)))' + \frac{c(a+t) + c(a-t)}{2}\,\phi(y(t)) = 0, \quad t \in (-\infty, \infty), \tag{2.14}$$

has a positive solution $y(t)$.

Proof. Let $W(t) := \phi(x'(t))/\phi(x(t))$. Then $W(t)$ is a solution of the generalized Riccati equation

$$W'(t) + (p-1)|W(t)|^q = -c(t). \tag{2.15}$$

Now let $R(t) := (1/2)\Big[W(a+t) + W(a-t)\Big]$. Then, by Lemma 2.3, we have

$$\begin{aligned} &R'(t) + (p-1)|R(t)|^q \\ &= \frac{1}{2}\Big[W'(a+t) + W'(a-t)\Big] + (p-1)\left|\frac{W(a+t) - W(a-t)}{2}\right|^q \\ &\leq \frac{1}{2}\Big[W'(a+t) + W'(a-t)\Big] + \frac{p-1}{2}\Big[|W(a+t)|^q + |W(a-t)|^q\Big] \\ &\leq -\frac{1}{2}\Big[c(a+t) + c(a-t)\Big]. \qquad \square \end{aligned} \tag{2.16}$$

Theorem 2.5. *If equation* (1.1) *is disconjugate on $(-\infty, \infty)$, then, for any $a \in \mathbf{R}$, the equation*

$$\Big(t\phi(z'(t))\Big)' + \left(\frac{1}{2}\int_{a-t}^{a+t} c(s)\,ds\right)\phi(z(t)) = 0 \tag{2.17}$$

is disconjugate on $(0, \infty)$.

Proof. Let $R(t) := (1/2)\Big[W(a+t) - W(a-t)\Big]$ as in Lemma 2.4. Then $R(0) = 0$. From inequality (2.16), by integration from 0 to t, we get

$$\begin{aligned} R(t) + (p-1)\int_0^t |R(s)|^q\,ds &\le -\frac{1}{2}\int_0^t \Big[c(a+s) + c(a-s)\Big]\,ds \\ &= -\frac{1}{2}\int_{a-t}^{a+t} c(s)\,ds. \end{aligned} \tag{2.18}$$

Using Holder's inequality, we find that

$$\left|\int_0^t R(s)\,ds\right| \le \left(\int_0^t ds\right)^{1/p}\left(\int_0^t |R(s)|^q\,ds\right)^{1/q}. \tag{2.19}$$

That is,

$$t^{q/p}\left|\int_0^t R(s)\,ds\right|^q \le \int_0^t |R(s)|^q\,ds. \tag{2.20}$$

From inequalities (2.19) and (2.20), we get

$$R(t) + \frac{p-1}{t^{q-1}}\left|\int_0^t R(s)\,ds\right|^q \le -\frac{1}{2}\int_{a-t}^{a+t} c(s)\,ds. \tag{2.21}$$

Putting $V(t) := \int_0^t R(s)\,ds$, we may rewrite inequality (2.21) as

$$V'(t) + \frac{p-1}{t^{q-1}}|V(t)|^q \le -\frac{1}{2}\int_{a-t}^{a+t} c(s)\,ds.$$

This means that the equation

$$\Big(t\phi(z'(t))\Big)' + \left(\frac{1}{2}\int_{a-t}^{a+t} c(s)\,ds\right)\phi(z(t)) = 0, \tag{2.22}$$

is disconjugate. □

Theorem 2.6. *Equation* (1.1) *is conjugate if* $c(t) \not\equiv 0$ *and there exists an* $a \in \mathbf{R}$ *such that*

$$\liminf_{T\to\infty}\int_0^T\int_{a-t}^{a+t} c(s)\,ds\,dt \ge 0. \tag{2.23}$$

Proof. Suppose, towards contradiction, that equation (1.1) is disconjugate. Let $R(t)$ be as in the proof of Lemma 2.4. Then, $R(0) = 0$ and, by equation (2.16),

$$R'(s) + (p-1)|R(s)|^q = -U^2(s) - \frac{1}{2}\Big[c(a+s) + c(a-s)\Big],$$

where $U(s)$ is some continuous function of s. Integrating from 0 to t, we get

$$\begin{aligned}(2.24)\quad R(t) + (p-1)\int_0^t |R(s)|^q\,ds &= -\int_0^t U^2(s)\,ds - \frac{1}{2}\int_0^t \Big[c(a+s) + c(a-s)\Big]\,ds \\ &= -\int_0^t U^2(s)\,ds - \frac{1}{2}\int_{a-t}^{a+t} c(s)\,ds\end{aligned}$$

Using Holder's inequality, we find that
(2.25)

$$R(t) + (p-1)t^{q/p}\left|\int_0^t R(s)\,ds\right|^q \le -\int_0^t U^2(s)\,ds - \frac{1}{2}\int_{a-t}^{a+t} c(s)\,ds.$$

Now, letting $V(t) := \int_0^t R(s)\,ds$, we may rewrite inequality (2.25) as

$$V'(t) + \frac{p-1}{t^{q-1}}|V(t)|^q \le -\int_0^t U^2(s)\,ds - \frac{1}{2}\int_{a-t}^{a+t} c(s)\,ds.$$

This means that the equation
(2.26)

$$\Big(t\phi(z'(t))\Big)' + \left(\int_0^t U^2(s)\,ds + \frac{1}{2}\int_{a-t}^{a+t} c(s)\,ds\right)\phi(z(t)) = 0, \quad t \in (0,\infty)$$

is disconjugate.

On the other hand, noting that $\int_0^1 (1/t)\,dt = \int_1^\infty (1/t)\,dt = \infty$ and that, in the presence of (2.23),

$$\int_0^t U^2(s)\,ds + \frac{1}{2}\int_{a-t}^{a+t} c(s)\,ds \equiv 0 \quad \text{implies that} \quad c(t) \equiv 0,$$

which contradicts the assumption. Also, if $U \not\equiv 0$ and $1/2 \int_{a-t}^{a+t} c(s)\,ds \equiv -\int_0^t U^2(s)\,ds$, then

$$\lim_{T\to\infty} \int_0^t \int_{a-t}^{a+t} c(s)\,ds\,dt = -\infty \tag{2.27}$$

which also contradicts assumption (2.23). If $U \equiv 0$, then $\int_{a-t}^{a+t} c(s)\,ds \equiv 0$ implies $c(a-s) \equiv c(a+s)$, for all s. □

The contrapositive parts of the following theorem give sufficient conditions for conjugacy of equation (1.1).

Theorem 2.7. *If equation* (1.1) *has a positive solution on* $(-\infty,\infty)$ *and* $a \in \mathbf{R}$, $\int_0^\infty |R(s)|^q\,ds < \infty$ *then for all* $T > 0$,

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T \left(\int_0^t \int_{a-s}^{a+s} c(\xi)\,d\xi\,ds \right) dt \le 0. \tag{2.28}$$

Moreover, if $c(a+t) \not\equiv c(a-t)$, *then*

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T \left(\int_0^t \int_{a-s}^{a+s} c(\xi)\,d\xi\,ds \right) dt = -\infty. \tag{2.29}$$

Proof. Suppose that equation (1.1) has a positive solution $x(t)$ defined on $\mathbf{R}$. Define $W(t) = \phi(x'(t))/\phi(x(t))$, $R(t) = [W(a+t) - W(a-t)]/2$, then

$$R'(\xi) + (p-1)|R(\xi)|^q = -U^2(\xi) - \frac{1}{2}\Big[c(a+\xi) + c(a-\xi)\Big], \tag{2.30}$$

where $U(\xi)$ is some continuous function in ξ. Multiplying both sides of equation (2.30) by $(T-\xi)^2$, we get

$$\begin{aligned}(T-\xi)^2 R'(\xi) &+ (p-1)(T-\xi)^2|R(\xi)|^q \\ &= -(T-\xi)^2U^2(\xi) - \frac{(T-\xi)^2}{2}\left[c(a+\xi) + c(a-\xi)\right].\end{aligned}$$

So

$$\frac{d}{d\xi}\Big((T-\xi)^2 R(\xi)\Big) + 2(T-\xi)R(\xi) + (T-\xi)^2(p-1)|R(\xi)|^q = -(T-\xi)^2\left[\frac{c(a+\xi)+c(a-\xi)}{2} + U^2(\xi)\right].$$

Adding $|R(\xi)|^{2-q}/p-1$ to both sides of the last equation, we arrive at

(2.31)
$$\frac{d}{d\xi}\Big((T-\xi)^2 R(\xi)\Big) + \frac{|R(\xi)|^{2-q}}{p-1} + 2(T-\xi)R(\xi) + (T-\xi)^2(p-1)|R(\xi)|^q = \frac{|R(\xi)|^{2-q}}{p-1} - (T-\xi)^2\left[\frac{c(a+\xi)+c(a-\xi)}{2} + U^2(\xi)\right].$$

That is,

(2.32)
$$\frac{d}{d\xi}\Big((T-\xi)^2 R(\xi)\Big) + \left(\frac{|R(\xi)|^{1-q/2}}{\sqrt{p-1}} + \sqrt{p-1}(T-\xi)(R(\xi))^{q/2}\right)^2 = \frac{|R(\xi)|^{2-q}}{p-1} - (T-\xi)^2\left[\frac{c(a+\xi)+c(a-\xi)}{2} + U^2(\xi)\right].$$

Integrating equation (2.32) and noting that $R(0)=0$, one gets

(2.33)
$$\int_0^T \left[\left(\frac{|R(\xi)|^{1-q/2}}{\sqrt{p-1}} + \sqrt{p-1}(T-\xi)(R(\xi))^{q/2}\right)^2 + (T-\xi)^2 U^2(\xi)\right] d\xi = \int_0^T \frac{|R(\xi)|^{2-q}}{p-1}\, d\xi - \frac{1}{2}\int_0^t (T-\xi)^2\Big[c(a+\xi)+c(a-\xi)\Big]\, d\xi.$$

Since the integrand in the lefthand side of equation (2.33) is nonnegative and cannot be identically zero, we have

$$\begin{aligned}(2.34)\quad \frac{1}{2T}\int_0^T (T-\xi)^2\Big[c(a+\xi)+c(a-\xi)\Big]\,d\xi &\\ &\le \frac{1}{T}\int_0^T \frac{|R(\xi)|^{2-q}}{p-1}\,d\xi\\ &\le \frac{1}{T}\Big(\int_0^T d\xi\Big)^{2/p}\Big(\int_0^T \frac{|R(\xi)|^q}{(p-1)^{q/(q-2)}}\,d\xi\Big)^{(2-q)/q}\\ &\le T^{(2-p)/p}\Big(\int_0^T \frac{|R(\xi)|^q}{(p-1)^{q/(2-q)}}\,d\xi\Big)^{(2-q)/q}\\ &\le \frac{1}{p-1}\Big(\frac{1}{T}\int_0^T |R(\xi)|^q\,d\xi\Big)^{(2-q)/q}.\end{aligned}$$

Now, if $\int_0^\infty |R(\xi)|^q\,d\xi < \infty$, then

$$\frac{1}{T}\int_0^T |R(\xi)|^{2-q}\,d\xi \longrightarrow 0, \quad \text{as} \quad T\to\infty.$$

It follows that

$$\lim_{T\to\infty}\frac{1}{2T}\int_0^T (T-\xi)^2\Big[c(a+\xi)+c(a-\xi)\Big]\,d\xi \le 0$$

So, we get

$$\lim_{T\to\infty}\frac{1}{T}\int_0^T\int_0^t\int_{a-s}^{a+s} c(\xi)\,d\xi\,ds\,dt \le 0.$$

We notice that, if $U\not\equiv 0$, then

$$\lim_{T\to\infty}\int_0^T U^2(\xi)\,d\xi = A, \quad 0<A\le\infty$$

implies that

$$\lim_{T\to\infty}\frac{1}{T}\int_0^T (T-\xi)^2U^2(\xi)\,d\xi = 2\lim_{T\to\infty}\frac{1}{T}\int_0^T (T-t)\int_0^t U^2(\xi)\,d\xi\,dt = \infty$$

which, in view of equality (2.33), gives the relation

$$\lim_{T\to\infty}\frac{1}{T}\int_0^T\Big(\int_0^t\int_{a-s}^{a+s} c(\xi)\,d\xi\,ds\Big)\,dt = -\infty.$$

On the other hand, it is easy to show that if $U \equiv 0$, then $c(a+t) \equiv c(a-t)$. □

Theorem 2.8. *For $p \geq 2$, equation* (1.1) *is conjugate if there is a point $a \in \mathbf{R}$ such that $c(a+t) + c(a-t) \not\equiv 0$ and*

$$\mu\biggl(\Bigl\{ t \in \mathbf{R}^+ : \int_{a-t}^{a+t} c(s)\, ds \geq 0 \Bigr\} \biggr) = \infty, \tag{2.35}$$

where μ is the Lebesgue measure on $\mathbf{R}$.

Proof. Let $C_a(t) := \int_{a-t}^{a+t} c(s)\, ds$, and define the function φ_a to be 1 if $C_a(t) \geq 0$ and 0 otherwise. Suppose that equation (1.1) were disconjugate. Then, by Lemma 2.3 the inequality

$$r'(t) + (p-1)|r(t)|^q = -\frac{1}{2}\Bigl[c(a+t) + c(a-t)\Bigr]$$

would have a solution r satisfying $r(0) = 0$. Integrating, multiplying by φ_a and then integrating once more, we get

$$\int_0^t \varphi_a(s) r(s)\, ds + (p-1) \int_0^t \varphi_a(s) \biggl(\int_0^s |r(\tau)|^q\, d\tau \biggr)\, dt \leq 0. \tag{2.36}$$

Therefore,

$$\begin{aligned}(p-1)\int_0^t \varphi_a(s) \biggl(\int_0^s |r(\tau)|^q\, d\tau \biggr)\, ds &\leq \int_0^t \varphi_a(s) |r(s)|\, ds \\ &\leq \biggl(\int_0^t [\varphi_a(s)]^p\, ds \biggr)^{1/p} \biggl(\int_0^t |r(s)|^q\, ds \biggr)^{1/q}.\end{aligned} \tag{2.37}$$

Since $[\varphi_a]^p \equiv \varphi_a$, inequality (2.37) would lead to

$$\begin{aligned}\biggl((p-1)\int_0^t \varphi_a(s) \biggl(\int_0^s |r(\tau)|^q\, d\tau \biggr)\, ds \biggr)^q &\\ \leq \biggl(\int_0^t \varphi_a(s)\, ds \biggr)^{q-1} &\biggl(\int_0^t |r(s)|^q\, ds \biggr).\end{aligned} \tag{2.38}$$

Hence, for large values of t,

$$(p-1)^q\varphi_a(t)\bigg(\int_0^t \varphi_a(s)\,ds\bigg)^{1-q} \le \frac{\varphi_a(t)\int_0^t |r(s)|^q\,ds}{\Big(\int_0^t \varphi_a(s)(\int_0^s |r(\tau)|^q\,d\tau)\,ds\Big)^q}.$$

This would give

$$\int_{T_o}^T \varphi_a(t)\bigg(\int_0^t \varphi_a(s)\,ds\bigg)^{1-q}\,dt \le \frac{(p-1)^{1-q}}{\Big(\int_0^{T_o} \varphi_a(s)(\int_0^s |r(\tau)|^q\,d\tau)\,ds\Big)^{q-1}}$$
$$< \infty,$$
$$0 < T_0 < T.$$

Noting that $q \le 2$, since $p \ge 2$, and that (2.35) implies that $\int_0^\infty \varphi_a(s)\,ds = \infty$, we have arrived at a contradiction. $\square$

We denote by $C_{00}(\mathbf{R})$ the set of all continuous functions on $\mathbf{R}$ with compact support.

Theorem 2.9. *Equation* (1.1) *is disconjugate on* $\mathbf{R}$ *if and only if the equation*

$$\Big(\phi(y'(t))\Big)' + \Big(f * c(t)\Big)\phi(y(t)) = 0 \tag{2.39}$$

is also disconjugate on $\mathbf{R}$ *for every nonnegative function* $f \in C_{00}(\mathbf{R})$ *with* $\int_{-\infty}^\infty f(s)\,ds \le 1$.

Proof. Suppose that equation (1.1) is disconjugate on $\mathbf{R}$. Let W be a continuous solution of the generalized Riccati equation

$$W'(\xi) + (p-1)|W(\xi)|^q = -c(\xi). \tag{2.40}$$

For each $t \in \mathbf{R}$, define $R(t) := \int_{-\infty}^\infty f(s)W(t-s)\,ds = f * W(t)$. Then,

$$R'(t) = \int_{-\infty}^\infty f(s)W'(t-s)\,ds;$$

and

(2.41)

$$\begin{aligned}|R(t)|^q &= \left|\int_{-\infty}^{\infty}(f(s))^{1/p}(f(s))^{1/q}w(t-s)\,ds\right|^q \\ &\le \left(\int_{-\infty}^{\infty} f(s)\,ds\right)^{q/p}\left(\int_{-\infty}^{\infty} f(s)|W(t-s)|^q\,ds\right) \\ &\le \int_{-\infty}^{\infty} f(s)|W(t-s)|^q\,ds.\end{aligned}$$

It follows that

(2.42)

$$\begin{aligned}&R'(t)+(p-1)|R(t)|^q \\ &\quad\le \int_{-\infty}^{\infty} f(s)W'(t-s)\,ds + (p-1)\int_{-\infty}^{\infty} f(s)|W(t-s)|^q\,ds \\ &\quad= \int_{-\infty}^{\infty} f(s)[W'(t-s)+(p-1)|W(t-s)|^q]\,ds \\ &\quad= -\int_{-\infty}^{\infty} f(s)c(t-s)\,ds \\ &\quad= -f*c(t).\end{aligned}$$

Now suppose that equation (2.39) is disconjugate for every function $f \in C_{00}(\mathbf{R})$ satisfying the condition $\int_{-\infty}^{\infty} f(s)\,ds \le 1$. Define the sequence f_n on $\mathbf{R}$ by

$$f_n(t) = \frac{1}{C_n}\exp\left(-\frac{1}{1-n^2t^2}\right) \quad \text{if} \quad n^2t^2 \le 1$$

and

$$f_n(t) = 0 \quad \text{otherwise},$$

where

$$C_n = \int_{-1/n}^{1/n} \exp\left(-\frac{1}{1-n^2t^2}\right) dt.$$

Then $\{f_n\} \subset C_{00}(\mathbf{R})$, $\int_{-\infty}^{\infty} f_n(s)\,ds = 1$ and $f_n * c \to c$ uniformly on $[-b,b]$ for every $b>0$, see [**10**]. This implies that equation (1.1) is disconjugate on the interval $[-b,b]$ for every $b>0$, cf. [**11**, page 6]. Therefore, it is disconjugate on $\mathbf{R}$. □

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