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INTEGRAL AND NUCLEAR OPERATORS ON THE SPACE $C(\Omega, c_0)$

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ABSTRACT. We give necessary and sufficient conditions for a linear and continuous operator on the space $C(\Omega, c_0)$ to be integral and nuclear. Based on this result some examples are given.

1. Introduction. Let Ω be a compact Hausdorff space, let X be a Banach space with dual space X^*, and let $C(\Omega, X)$ stand for the Banach space of continuous X-valued functions on Ω under the uniform norm and denoted by $C(\Omega)$ when X is the scalar field. It is well known that if Y is a Banach space, then any linear and continuous operator $U : C(\Omega, X) \to Y$ has associated with it a finitely additive vector measure $G : \Sigma_\Omega \to L(X, Y^{**})$, where Σ_Ω is the σ-field of Borel subsets of Ω, such that

$$y^* U(f) = \int_\Omega f dG_{y^*}, \quad f \in C(\Omega, X), \quad y^* \in Y^*,$$

see [**3, 4, 8, 9**] for more details. The measure G is called the representing measure of U.

Also, for a linear and continuous operator $U : C(\Omega, X) \to Y$, we can associate in a natural way two linear and continuous operators

$$U^{\#} : C(\Omega) \longrightarrow L(X, Y) \quad \text{and} \quad U_{\#} : X \longrightarrow L(C(\Omega), Y)$$

defined by

$$\left(U^{\#}\varphi\right)(x) = U(\varphi \otimes x) \quad \text{and} \quad (U_{\#}x)(\varphi) = U(\varphi \otimes x)$$

where for $\varphi \in C(\Omega)$, $x \in X$, we define $(\varphi \otimes x)(\omega) = \varphi(\omega) x$, for $\omega \in \Omega$.

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The operator $U^{\#}$ occurs also in [**9**, Theorem 1, page 377], where it is denoted by U'.

Swartz in [**19**] characterized absolutely summing operators U on the space $C(\Omega, X)$ in terms of the representing measure and the operator $U^{\#}$.

Saab in [**17**] characterized integral operators U on the space $C(\Omega, X)$ in terms of the representing measure, and later Montgomery-Smith and Saab in [**10**] presented a characterization of integral operators U on an injective tensor product in terms of the operator $U^{\#}$.

Partial characterizations of nuclear operators U on the space $C(\Omega, X)$ in terms of the representing measure and the operator $U^{\#}$ are given in [**1, 12, 17, 18**], and a partial characterization of Pietsch integral operators U on the space $C(\Omega, X)$ in terms of the representing measure and the operator $U^{\#}$ is given in [**13**].

The space $C(\Omega, X)$ is an injective tensor product [**5, 8**]. In [**10**], for p-absolutely summing operators, and in [**14**], for (r,p)-absolutely summing operators, necessary conditions are given for an operator U on an injective tensor product to be p-absolutely summing, respectively (r,p)-absolutely summing, in terms of the operator $U^{\#}$. By symmetry, these necessary conditions are also true for the operator $U_{\#}$. We will use in our proofs this corresponding fact for $U_{\#}$.

We denote by $(As, \| \quad \|_{as})$, $(I, \| \quad \|_{\text{int}})$, $(\mathcal{N}, \| \quad \|_{\text{nuc}})$ the normed ideal of all absolutely summing, (Grothendieck) integral operators and nuclear operators, respectively. We refer the reader to [**5, 7, 8, 11**] for details.

If X is a Banach space, a series $\sum_{n=1}^{\infty} x_n$ in X is called a weak Cauchy series if and only if for every $x^* \in X^*$ the series $\sum_{n=1}^{\infty} |x^*(x_n)|$ is convergent.

We denote by $(e_n)_{n\in\mathbf{N}}$ the canonical basis in the Banach space c_0.

For a vector measure $G : \Sigma \to X$, where Σ is a σ-field of sets, we denote by $|G|$ the variation measure of G, see [**8, 9**].

Also, for $\Omega = [0,1]$ we denote $\mu : \Sigma_\Omega \to [0,1]$ the Lebesgue measure, and by $(r_n)_{n\in\mathbf{N}}$ the sequence of Rademacher functions. If (S, Σ, ν) is a finite measure space, X is a Banach space and $f : S \to X$ is a ν-Bochner integrable function, then we denote by $B - \int_{(\ )} f \, d\nu$ the indefinite Bochner integral. It is always a σ-additive vector measure with finite variation.

All notations and notions used and not defined in this paper can be found in [**7, 8**].

In the sequel we will use the following well-known result, [**5, 7**].

Fact. *If X is a Banach space, then $As(c_0, X) = I(c_0, X) = \mathcal{N}(c_0, X)$.*

More precisely, $T \in As(c_0, X)$ if and only if $\sum_{n=1}^{\infty} \|T(e_n)\| < \infty$, and in this case

$$\|T\|_{\mathrm{as}} = \|T\|_{\mathrm{int}} = \|T\|_{\mathrm{nuc}} = \sum_{n=1}^{\infty} \|T(e_n)\| .$$

The main result and examples. In the following theorem, which is the main result of our paper, we present a characterization of both integral and nuclear operators on the space $C(\Omega, c_0)$.

Theorem 1. *Let Ω be a compact Hausdorff space, X a Banach space and $U : C(\Omega, c_0) \to X$ a linear and continuous operator. Suppose that the representing measure G takes its values in $L(c_0, X) \subseteq L(c_0, X^{**})$, and let $G_{e_n} : \Sigma_\Omega \to X$ be defined by $G_{e_n}(E) = G(E)(e_n)$, for $n \in \mathbf{N}$.*

(a) *The following assertions are equivalent*:

(i) *U is absolutely summing.*

(ii) *U is integral.*

(iii) *For each $n \in \mathbf{N}$, we have*

$$U_{\#}(e_n) \in As(C(\Omega), X)$$

and

$$\sum_{n=1}^{\infty} \|U_{\#}(e_n)\|_{\mathrm{as}} < \infty.$$

(iv) *For each $n \in \mathbf{N}$, the set function G_{e_n} has bounded variation and $\sum_{n=1}^{\infty} |G_{e_n}|(\Omega) < \infty$.*

(v) $G : \Sigma \to \mathcal{N}(c_0, X)$ *has bounded variation with respect to the nuclear norm.*

In addition,

$$\|U\|_{\mathrm{int}} = \|U\|_{\mathrm{as}} = \sum_{n=1}^{\infty} \|U_{\#}(e_n)\|_{\mathrm{as}} = \sum_{n=1}^{\infty} |G_{e_n}|(\Omega) = |G|_{\mathrm{nuc}}(\Omega).$$

(b) *The following assertions are equivalent*:

(i) U *is nuclear.*

(ii) U *is integral and* $U_{\#}(e_n) \in \mathcal{N}(C(\Omega), X)$ *for every* $n \in \mathcal{N}$.

(iii) U *is integral and* $U_{\#}(\xi) \in \mathcal{N}(C(\Omega), X)$ *for every* $\xi \in c_0$.

In addition, $\|U\|_{\mathrm{nuc}} = \sum_{n=1}^{\infty} \|U_{\#}(e_n)\|_{\mathrm{nuc}} = |G|_{\mathrm{nuc}}(\Omega)$.

Proof. (a) (i) $\Rightarrow$ (iii). If U is absolutely summing, then by [**10**, Theorem 3.1] or [**14**, Theorem 1], $U_{\#} : c_0 \to As(C(\Omega), X)$ is absolutely summing and $\|U_{\#}\|_{\mathrm{as}} \le \|U\|_{\mathrm{as}}$. This implies, in particular, that $U_{\#}(e_n) \in As(C(\Omega), X)$ for each $n \in \mathbf{N}$ and by the Fact, $\sum_{n=1}^{\infty} \|U_{\#}(e_n)\|_{\mathrm{as}} \le \|U_{\#}\|_{\mathrm{as}}$, i.e., (iii) holds.

(iii) $\Rightarrow$ (iv). For each $n \in \mathbf{N}$, the representing measure of $U_{\#}(e_n)$ is G_{e_n} and $\|U_{\#}(e_n)\|_{\mathrm{as}} = |G_{e_n}|(\Omega)$, [**8**, Theorem 3, page 162]. Hence, (iv) follows from (iii).

(iv) $\Rightarrow$ (i). For $E \in \Sigma_{\Omega}$, we have

$$\begin{aligned}\sum_{n=1}^{\infty} \|G(E)(e_n)\| = \sum_{n=1}^{\infty} \|G_{e_n}(E)\| &\le \sum_{n=1}^{\infty} |G_{e_n}|(E) \\ &\le \sum_{n=1}^{\infty} |G_{e_n}|(\Omega) < \infty.\end{aligned}$$

By the above Fact, $G(E) \in As(c_0, X)$ and

$$\|G(E)\|_{\mathrm{nuc}} = \|G(E)\|_{\mathrm{as}} = \sum_{n=1}^{\infty} \|G_{e_n}(E)\|, \ E \in \Sigma_{\Omega}.$$

From this and (iv) we obtain that $G : \Sigma_{\Omega} \to As(c_0, X)$ has bounded variation and $|G|_{\mathrm{nuc}}(E) = |G|_{\mathrm{as}}(E) \le \sum_{n=1}^{\infty} |G_{e_n}|(E)$ for any $E \in \Sigma_{\Omega}$.

Now, by Swartz's theorem [**19**] or by a more general result [**15**, Theorem 2], it follows that U is absolutely summing and, moreover,

$$\|U\|_{\mathrm{as}} = |G|_{\mathrm{as}}(\Omega) \le \sum_{n=1}^{\infty} |G_{e_n}|(\Omega).$$

(i) $\Rightarrow$ (v). It follows from Swartz's theorem [**19**] and the above Fact.

(v) $\Rightarrow$ (ii). Apply Saab's theorem [**17**, Theorem 3].

(ii) $\Rightarrow$ (i). This is true in general and is known [**5, 7, 11**].

We also have the equality from the statement.

We remark that $|G|_{\mathrm{nuc}}(E) = \sum_{n=1}^{\infty} |G_{e_n}|(E)$ for all $E \in \Sigma_\Omega$. This follows since (iv) $\Rightarrow$ (i) yields

$$|G|_{\mathrm{nuc}}(\Omega \setminus E) \le \sum_{n=1}^{\infty} |G_{e_n}|(\Omega \setminus E), E \in \Sigma_\Omega,$$

and then use $|G|_{\mathrm{nuc}}(\Omega) = \sum_{n=1}^{\infty} |G_{e_n}|(\Omega)$.

(b) (i) $\Rightarrow$ (iii). Follows from the ideal property of nuclear operators and the obvious relation that for any $\xi \in c_0$ we have $U_{\#}(\xi) = U\sigma_\xi$, where $\sigma_\xi : C(\Omega) \to C(\Omega, c_0)$ is defined by $\sigma_\xi(\varphi) = \varphi \otimes \xi$.

(iii) $\Rightarrow$ (ii). This is trivial.

(ii) $\Rightarrow$ (i). We will prove that in this case there is a $|G|_{\mathrm{nuc}}$-Bochner integrable function $h : \Omega \to \mathcal{N}(c_0, X)$ such that

$$G(E) = B - \int_E h \, d|G|_{\mathrm{nuc}}, \quad E \in \Sigma_\Omega,$$

which then assures, by [**1**, Theorem III. 4], as it is cited in [**18**] or [**12**, Theorem 1] or [**18**, Theorem 5], that U is nuclear.

Indeed, since for each $n \in \mathbf{N}$ we suppose that $U_{\#}(e_n) \in \mathcal{N}(C(\Omega), X)$, it follows that there is a $|G_{e_n}|$-Bochner integrable function $\varphi_n : \Omega \to X$ such that $G_{e_n}(E) = B - \int_E \varphi_n \, d|G_{e_n}|$, for $E \in \Sigma_\Omega$ [**8**, Theorem 4, page 173]. Because U is integral, (a) implies that $|G_{e_n}| \ll |G|_{\mathrm{nuc}}$ and, hence, there is a $|G_{e_n}|$-integrable function $h_n : \Omega \to [0, \infty)$ such that $|G_{e_n}|(E) = \int_E h_n \, d|G|_{\mathrm{nuc}}$ for $E \in \Sigma_\Omega$. Then $G_{e_n}(E) =$

$B-\int_E h_n\varphi_n \, d|G|_{\text{nuc}}$ for $E\in\Sigma_\Omega$ and $|G_{e_n}|(E)=\int_E \|\varphi_n\| h_n \, d|G|_{\text{nuc}}$ for all $n\in\mathbf{N}$ and $E\in\Sigma_\Omega$. Hence, by (a), it follows that

$$|G|_{\text{nuc}}(E)=\sum_{n=1}^{\infty}|G_{e_n}|(E)=\int_E\left(\sum_{n=1}^{\infty}\|\varphi_n\|\,h_n\right)d\,|G|_{\text{nuc}}\,,\quad E\in\Sigma_\Omega.$$

Then $\sum_{n=1}^{\infty}\|\varphi_n\|h_n=1$ for $|G|_{\text{nuc}}$-almost everywhere $\omega\in\Omega$, which implies that the function $h:\Omega\to L(c_0,X)$ defined by

$$h(\omega)(\xi)=\sum_{n=1}^{\infty}\langle\xi,e_n\rangle\,\varphi_n(\omega)\,h_n(\omega)\,,\quad \omega\in\Omega,\quad \xi\in c_0,$$

takes $|G|_{\text{nuc}}$-almost everywhere value in $L(c_0,X)$; without loss of generality, we can suppose that h takes all its values in $L(c_0,X)$.

Then, for all $\omega\in\Omega$ and $n\in\mathbf{N}$, we have $h(\omega)(e_n)=\varphi_n(\omega)h_n(\omega)$, which by the above Fact, implies that h takes its values in $\mathcal{N}(c_0,X)$ and

$$\|h(\omega)\|_{\text{nuc}}=\sum_{n=1}^{\infty}\|\varphi_n(\omega)\|\,h_n(\omega)\,.$$

Thus,

$$\begin{aligned}\int_\Omega\|h(\omega)\|_{\text{nuc}}\;d\,|G|_{\text{nuc}}(\omega)&=\int_\Omega\left(\sum_{n=1}^{\infty}\|\varphi_n\|\,h_n\right)\;d\,|G|_{\text{nuc}}\\&=|G|_{\text{nuc}}(\Omega)<\infty.\end{aligned}$$

Since $h:\Omega\to\mathcal{N}(c_0,X)$ is obviously $|G|_{\text{nuc}}$-Bochner measurable, it follows that h is $|G|_{\text{nuc}}$-Bochner integrable.

Now, if $F(E)=B-\int_E h\,d|G|_{\text{nuc}}$, then by the Hille theorem [**8**, page 47], for any $\xi=(\xi_n)_{n\in\mathbf{N}}\in c_0$ and $E\in\Sigma_\Omega$, we have

$$F(E)(\xi)=B-\int_E h(\omega)(\xi)\;d\,|G|_{\text{nuc}}(\omega)$$

and

$$\begin{aligned}F(E)(\xi)&=\sum_{n=1}^{\infty}\xi_n\left(B-\int_E\varphi_n(\omega)\,h_n(\omega)\,d\,|G|_{\text{nuc}}(\omega)\right)\\&=\sum_{n=1}^{\infty}\xi_nG_{e_n}(E)=\sum_{n=1}^{\infty}\xi_nG(E)(e_n)=G(E)(\xi)\,.\end{aligned}$$

Thus, $G(E) = B - \int_E h\, d|G|_{\text{nuc}}$ for all $E \in \Sigma_\Omega$, and the proof is finished. □

If X^* and Y have the Radon-Nikodym property and Y is complemented in its bidual, then the space $\mathcal{N}(X,Y)$ has the Radon-Nikodym property, (see [**13**, Corollary 5] for this result or [**2**, Theorem 7] for the more general normed ideal of operators, but in [**2**], under some approximation hypotheses). In the examples which we present the space $\mathcal{N}(X,Y)$ (in our situation $\mathcal{N}(c_0,Y)$) does not a priori have the Radon-Nikodym property.

As applications of Theorem 1, we now present some relevant examples.

Example 2. (i) Let $a = (a_n)_{n\in\mathbf{N}} \in l_\infty$, and $U : C([0,1], c_0) \to c_0(C[0,1])$ be the operator defined by

$$(Uf)(x) = \left(a_n \int\limits_0^x \langle f(t), e_n \rangle \, dt \right)_{n\in\mathbf{N}}, \quad x \in [0,1].$$

Then U is integral if and only if $a \in l_1$, while U is nuclear if and only if $a = 0$.

(ii) Let $a = (a_n)_{n\in\mathbf{N}} \in l_\infty$ and $U : C([0,1], c_0) \to c_0(L_1[0,1])$ be the operator defined by

$$(Uf)(x) = \left(a_n \int\limits_0^x \langle f(t), e_n \rangle \, dt \right)_{n\in\mathbf{N}}, \quad x \in [0,1].$$

Then U is integral if and only if U is nuclear if and only if $a \in l_1$.

(iii) Let $\sum_{n=1}^\infty x_n$ be a weak Cauchy series in a Banach space X and $U : C[0,1], c_0) \to X$ the operator defined by

$$U(f) = \sum_{n=1}^{\infty} \left(\int_0^1 r_n(t) \langle f(t), e_n \rangle \, dt \right) x_n.$$

Then U is integral if and only if it is nuclear if and only if the series $\sum_{n=1}^\infty x_n$ is absolutely convergent.

Proof. (i) Let $\Omega = [0,1]$. The representing measure of U is

$$G(E)(\xi)(t) = \left(a_n \langle \xi, e_n\rangle \mu\left(E \bigcap [0,t]\right)\right)_{n\in\mathbf{N}},$$
$$\xi \in c_0, \quad E \in \Sigma_\Omega, \quad t \in [0,1].$$

Then

$$\|G_{e_n}(E)\| = \sup_{t\in[0,1]} |a_n| \mu\left(E \bigcap [0,t]\right) = |a_n| \mu(E), \quad E \in \Sigma_\Omega,$$

and thus

$$\sum_{n=1}^{\infty} |G_{e_n}|([0,1]) = \sum_{n=1}^{\infty} |a_n|.$$

The statement follows from Theorem 1 (a).

If U is nuclear then, by Theorem 1 (b), for each $n \in \mathbf{N}$ the operator $U_{\#}(e_n) : C[0,1] \to c_0(C[0,1])$ is nuclear and, by the ideal property of nuclear operators, it follows that $p_n U_{\#}(e_n) : C[0,1] \to C[0,1]$ is nuclear (where p_n are the canonical projections in $c_0(C[0,1])$). Observe, for each $n \in \mathbf{N}$, that

$$(p_n U_{\#}(e_n))(\varphi)(x) = a_n \int_0^x \varphi(t)\, dt, \quad \varphi \in C[0,1], \quad x \in [0,1].$$

Since the Volterra operator $V : C[0,1] \to C[0,1]$ defined by $(V\varphi)(x) = \int_0^x \varphi(t)\, dt$ is not nuclear [**8**, page 73], it follows that $a_n = 0$.

(ii) Argue as in (i) and use the fact that the Volterra operator $V : C[0,1] \to L_1[0,1]$ defined by $(V\varphi)(x) = \int_0^x \varphi(t)\, dt$ is nuclear [**8**, page 78].

(iii) Let $f \in C([0,1], c_0)$. Then $\langle f(t), e_n\rangle \to 0$ and $|\langle f(t), e_n\rangle| \le \|f\|$ for $t \in [0,1]$. From the Lebesgue dominated convergence theorem it follows that $\int_0^1 r_n(t)\langle f(t), e_n\rangle\, dt \to 0$ and, since $\sum_{n=1}^{\infty} x_n$ is a weak Cauchy series in X, it follows that U is well defined [**6**, Theorem 6, page 44]. The representing measure of U is

$$G(E)(\xi) = \sum_{n=1}^{\infty} \langle \xi, e_n\rangle \left(\int_E r_n(t)\, dt\right) x_n, \quad E \in \Sigma_\Omega.$$

Then $G_{e_n}(E) = (\int_E r_n(t)\,dt)x_n$ and $|G_{e_n}|([0,1]) = \|x_n\| \int_0^1 |r_n(t)|\,dt$. We now apply Theorem 1 (a) and (b) to obtain the statement.

Example 3. (i) Let $\Omega = [0,1]$, $(\varphi_n)_{n\in\mathbf{N}} \subset L_1([0,1], l_1)$ be such that $\sup_{n\in\mathbf{N}} \|\varphi_n(t)\|_{l_1} < \infty$ for all $t \in [0,1]$ and $\int_E \langle \xi, \varphi_n(t)\rangle\,dt \to 0$ for $E \in \Sigma_\Omega$ and $\xi \in c_0$. Let $U : C([0,1], c_0) \to c_0$ be the operator defined by

$$U(f) = \left(\int_0^1 \langle f(t), \varphi_n(t)\rangle \, dt \right)_{n\in\mathbf{N}}.$$

Then U is nuclear if and only if $\varphi_n(t) \to 0$ weak* for μ-almost everywhere $t \in [0,1]$ and the function $h = (h_k)_{k\in\mathbf{N}}$ belongs to $L_1([0,1], l_1)$, where $h_k(t) = \sup_{n\in\mathbf{N}} |\langle e_k, \varphi_n(t)\rangle|$.

(ii) Let $\mathcal{M} = (\alpha_{nk})_{n,k\in\mathbf{N}}$ be a regular method of summability and $U : C([0,1], c_0) \to c_0$ the operator defined by

$$U(f) = \left(\sum_{k=1}^{\infty} \alpha_{nk} \int_0^1 \langle f(t), e_k\rangle \, r_k(t)\, dt \right)_{n\in\mathbf{N}}.$$

Then U is integral if and only if U is nuclear if and only if the series $\sum_{k=1}^{\infty} (\sup_{n\in\mathbf{N}} |\alpha_{nk}|)$ is convergent.

Proof. (i) By hypothesis, it is clear that U is well defined and that the representing measure of U is

$$G(E)(\xi) = \left(\int_E \langle \xi, \varphi_k(t)\rangle \, dt \right)_{k\in\mathbf{N}}, \quad E \in \Sigma_\Omega, \quad \xi \in c_0.$$

Suppose that U is nuclear. Then, by Theorem 1 (b), for any $\xi \in c_0$ the operator $U_\#(\xi) : C[0,1] \to c_0$ defined by $U_\#(\xi)(f) = (\int_0^1 \langle \xi, \varphi_n(t)\rangle f(t)\,dt)_{n\in\mathbf{N}}$ is nuclear. Using [**16**, Proposition 3 (iv)], it follows that $\langle \xi, \varphi_n(t)\rangle \to 0$ for μ-almost everywhere $t \in [0,1]$ and

$$\|U_\#(\xi)\|_{\text{nuc}} = \int_0^1 h_\xi(t)\, dt, \quad \text{where} \quad h_\xi(t) = \sup_{n\in\mathbf{N}} |\langle \xi, \varphi_n(t)\rangle|.$$

We observe that the exceptional set typically depends on $\xi \in c_0$ but, since c_0 is separable, it follows that the exceptional set is independent of ξ, i.e., there is an $A \in \Sigma_\Omega$ such that $\mu([0,1] \setminus A) = 0$ and $\varphi_n(t) \to 0$ weak* for any $t \in A$.

In particular,

$$\|U_\#(e_k)\|_{\text{nuc}} = \int_0^1 h_k(t)\, dt \quad \text{with} \quad h_k(t) = \sup_{n \in \mathbf{N}} |\langle e_k, \varphi_n(t) \rangle| .$$

Since U is nuclear, by Theorem 1 (a) we have

$$\sum_{k=1}^{\infty} \|U_\#(e_k)\|_{\text{nuc}} = \|U\|_{nuc} \quad \text{i.e.} \quad \sum_{k=1}^{\infty} \int_0^1 h_k(t)\, dt = \|U\|_{\text{nuc}} .$$

Hence, $h = (h_k)_{k \in \mathbf{N}} \in L_1([0,1], l_1)$.

For the converse we observe that, for each $k \in \mathbf{N}$, we have $\langle e_k, \varphi_n(t) \rangle \to 0$ for μ-almost everywhere $t \in [0,1]$ and the function h_k is integrable. Thus, by [**16**, Proposition 3(iv)], it follows that the operator $U_\#(e_k)$ is nuclear and, in addition, $\|U_\#(e_k)\|_{\text{nuc}} = \int_0^1 h_k(t)\, dt$. Using now the fact that $h = (h_k)_{k \in \mathbf{N}} \in L_1([0,1], l_1)$ the proof is completed by Theorem 1 (b).

(ii) Recall that if $\mathcal{M} = (\alpha_{nk})$ is an infinite real matrix and

$$(x_k)_{k \in \mathbf{N}} \longrightarrow \left(\sum_{k=1}^{\infty} \alpha_{nk} x_k \right)_{n \in \mathbf{N}}$$

is its formal action on the space of all sequences of scalars, then $\mathcal{M} = (\alpha_{nk})$ is called a regular method of summability if its action on convergent sequences produces convergent sequences with preservation of limits. As is well known, a matrix $\mathcal{M} = (\alpha_{nk})$ is a regular method of summability if and only if

a) $\sup_{n \in \mathbf{N}} \sum_{k=1}^{\infty} |\alpha_{nk}| < \infty$,

b) for each $k \in \mathbf{N}$, $\lim_{n \to \infty} \alpha_{nk} = 0$,

c) $\lim_{n \to \infty} \sum_{k=1}^{\infty} \alpha_{nk} = 1$.

Let $\varphi_n : [0,1] \to c_0^* = l_1$ be defined by

$$\varphi_n(t)(\xi) = \sum_{k=1}^{\infty} \alpha_{nk} \langle \xi, e_k \rangle r_k(t)$$

and observe that

$$U(f)=\left(\int_0^1 \langle f(t),\varphi_n(t)\rangle\, dt\right)_{n\in\mathbf{N}}.$$

Then $U_{\#}(e_k)=\lambda_k\otimes\alpha_k$ is a rank one operator, where $\lambda_k(\varphi)=\int_0^1\varphi(t)r_k(t)\,dt$ for $\varphi\in C[0,1]$ and $\alpha_k=(\alpha_{nk})_{n\in\mathbf{N}}$.

Also $\langle e_k,\varphi_n(t)\rangle=\alpha_{nk}r_k(t)$. Hence, $h_k(t)=\sup_{n\in\mathbf{N}}|\langle e_k,\varphi_n(t)\rangle|=\sup_{n\in\mathbf{N}}|\alpha_{nk}|$ and by Theorem 1 (a) and part (i) the statement follows.

Example 4. Let $\Omega=[0,1]$ and $(h_n)_{n\in\mathbf{N}}\subset L_\infty[0,1]$ be such that

$$M=\sup_{n\in\mathbf{N}}\|h_n\|_\infty<\infty \quad\text{and}\quad \int_E h_n(t)\,dt\to 0 \quad\text{for}\quad E\in\Sigma_\Omega.$$

Let $(x_n^*)_{n\in\mathbf{N}}\subset c_0^*=l_1$ be a bounded sequence such that, for some $x\in c_0$, we have $\liminf_{n\to\infty}|x_n^*(x)|>0$ and $T:c_0\to l_\infty$ defined by $T(x)=(x_n^*(x))_{n\in\mathbf{N}}$ is nuclear.

Let $U:C([0,1],c_0)\to c_0$ be the operator defined by

$$U(f)=\left(\int_0^1 x_n^*(f(t))\,h_n(t)\,dt\right)_{n\in\mathbf{N}}.$$

Then

(a) U is integral.

(b) U is nuclear if and only if $h_n(t)\to 0$ for μ-almost everywhere $t\in[0,1]$.

Proof. (a) For $\varphi_n:[0,1]\to l_1=c_0^*$ defined by

$$\varphi_n(t)(x)=x_n^*(x)\,h_n(t),\quad t\in[0,1],\quad x\in c_0,$$

we observe that the conditions in Example 3 (i) are satisfied.

The representing measure of U is $G(E)(x)=(x_n^*(x)\int_E h_n(t)\,dt)_{n\in\mathbf{N}}$.

We show that $G:\Sigma\to\mathcal{N}(c_0,c_0)$ has bounded variation with respect to the nuclear norm. By Saab's theorem [**17**, Theorem 3], or Theorem 1(a), U will then be integral.

Indeed, for each $E \in \Sigma$ we have $G(E) = S_E T$, where $S_E : l_\infty \to c_0$ is the multiplication operator defined by $S_E((\alpha_n)_{n\in\mathbf{N}}) = (\alpha_n \int_E h_n(t)\,dt)_{n\in\mathbf{N}}$.

Since $T : c_0 \to l_\infty$ is nuclear, it follows that $G(E) \in \mathcal{N}(c_0, c_0)$ and

$$\|G(E)\|_{\text{nuc}} \leq \|T\|_{\text{nuc}} \|S_E\|,$$

i.e.,

$$\|G(E)\|_{\text{nuc}} \leq \|T\|_{\text{nuc}} \|F(E)\|,$$

where $F(E) = \sup_{n\in\mathbf{N}} |\int_E h_n(t)\,dt|$, for $E \in \Sigma_\Omega$.

Because $\|F(E)\| \leq M\mu(E)$ for $E \in \Sigma_\Omega$, it follows that $G : \Sigma \to \mathcal{N}(c_0, c_0)$ has bounded variation with respect to the nuclear norm.

(b) If U is a nuclear operator, then, by Example 3 (i), it follows that for μ-almost everywhere $t \in [0,1]$ we have $\varphi_n(t) \to 0$ weak*. Let $A \in \Sigma$ be such that $\mu([0,1] \setminus A) = 0$ and for all $t \in A$ and any $x \in c_0$ we have $x_n^*(x) h_n(t) \to 0$. Since, by hypothesis, there is an $x \in c_0$ such that $\liminf_{n\to\infty} |x_n^*(x)| > 0$, we deduce that $\limsup_{n\to\infty} |h_n(t)| = 0$.

For the converse, observe that from $h_n \to 0$, μ-almost everywhere, it follows that $\varphi_n(t) \to 0$ weak* for μ-almost everywhere $t \in [0,1]$. Also

$$h_k(t) = \sup_{n\in\mathbf{N}} |\langle e_k, \varphi_n(t)\rangle| \leq M \|T(e_k)\|, \quad \text{for} \quad t \in [0,1], \quad k \in \mathbf{N}.$$

Since T is nuclear (by the Fact), it follows that $(h_k)_{k\in\mathbf{N}} \in L_1([0,1], l_1)$, and we can use Example 3 (i) to deduce that U is nuclear.

A concrete situation for Example 4 is the following one.

For a nonzero $(\lambda_n)_{n\in\mathbf{N}} \in l_1$, let $x_n^* \in c_0^*$ be defined by

$$x_n^*(x_1, x_2, \dots) = \lambda_1 x_1 + \lambda_2 x_2 + \dots + \lambda_n x_n, \quad (x_1, x_2, \dots) \in c_0.$$

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