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MULTIPLICITY OF POSITIVE SOLUTIONS FOR A MIXED BOUNDARY ELLIPTIC SYSTEM

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ABSTRACT. In this paper we are concerned with the existence and multiplicity of positive solutions for the following class of elliptic system

$$\begin{cases} -\varepsilon^2 \Delta u + u = Q_u(u,v), & -\varepsilon^2 \Delta v + v = Q_v(u,v) \quad \text{in } \Omega \\ u = v = 0 \quad \text{on} \quad \Gamma \quad \text{and} & \partial u/\partial \eta = \partial v/\partial \eta = 0 \quad \text{on } \Sigma \end{cases}$$

where Ω is a bounded domain in $\mathbf{R}^N$, Q is a p-homogeneous function with $2 < p < 2N/(N-2)$ for $N \geq 3$. The main tool used in this paper is the variational method combined with the Ljusternick-Schnirelman category of Σ in itself.

1. Introduction. In this paper we are concerned with the existence of positive solutions for the following class of elliptic system

$$\text{(S)} \qquad \begin{cases} -\varepsilon^2 \Delta u + u = Q_u(u,v) & \text{in } \Omega \\ -\varepsilon^2 \Delta v + v = Q_v(u,v) & \text{in } \Omega \\ u = v = 0 & \text{on } \Gamma \\ \partial u/\partial \eta = \partial v/\partial \eta = 0 & \text{on } \Sigma \end{cases}$$

where Ω is a bounded domain in $\mathbf{R}^N$ with smooth boundary $\partial\Omega = \overline{\Gamma} \cup \overline{\Sigma}$, where Γ and Σ are smooth $(N-1)$-dimensional submanifolds of $\partial\Omega$ with positive measures such that $\Gamma \cap \Sigma = \varnothing$, $Q \in C^1(\Theta, \mathbf{R})$ is a homogeneous function of degree p, with $2 < p < 2N/(N-2)$ and $\Theta = [0, +\infty) \times [0, +\infty)$. Let us state the hypotheses on the nonlinearity Q:

(Q_1) There exists a $C > 0$ such that

$$\begin{cases} |Q_u(u,v)| \leq C(u^{p-1} + v^{p-1}) & \text{for all } (u,v) \in \Theta \\ |Q_v(u,v)| \leq C(u^{p-1} + v^{p-1}) & \text{for all } (u,v) \in \Theta. \end{cases}$$

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(Q_2) $Q_u(0,1) = Q_v(1,0) = 0$;

(Q_3) $Q_u(1,0) = Q_v(0,1) = 0$;

(Q_4) $Q(u,v) > 0$ for all $u, v > 0$;

(Q_5) $Q_u(u,v), Q_v(u,v) \geq 0$ for all $u, v \geq 0$.

Since Q is a C^2 homogeneous function of degree $p > 2$, then

1. $pQ(u,v) = uQ_u(u,v) + vQ_v(u,v)$;
2. ∇Q is a homogeneous function of degree $p-1$.

Some examples of this type of homogeneous function can be found in [**9, 12**].

Following a well-known device used to obtain a solution of (S), let us extend function Q to the whole plane as a C^1-function as

$$Q(s,t) = \begin{cases} Q(s,t) & s,t \geq 0 \\ 0 & \text{otherwise.} \end{cases}$$

In the last years, many papers have considered the scalar equation

$$(P_\varepsilon) \qquad -\varepsilon^2 \Delta u + u = |u|^{p-2}u, \ \Omega$$

with Dirichlet or Neumann boundary conditions. The main points considered by these papers were the following:

- Existence and multiplicity of solutions.
- The concentration of the maximum points of the solutions, which is strongly related to the boundary conditions considered in the problem.
- The relation between the geometry of domain with the multiplicity of solution using the Ljusternick-Schnirelman category of Ω or $\partial\Omega$ in itself.

An important point when we are working with the problem (P_ε) is the properties of the limit problem, which in general involves the following equation

$$(P_\infty) \qquad -\Delta u + u = |u|^{p-2}u$$

in $\mathbf{R}^N$ or $\mathbf{R}^N_+$. We cite the works [**2, 6, 13, 16**–**22** and references therein] to the reader interested in getting more information about problem (P_ε). For elliptic systems of the gradient or Hamiltonian type, we cite the papers [**4, 5, 7**].

In relation to the mixed boundary condition, there are a lot of interesting questions to study; we cite the papers [**3, 10, 11, 15, 22, 23**] and references therein.

Motivated by the works [**4, 10, 11**], we use the Ljusternick-Schnirelman category of Σ in itself to obtain multiplicity results for system (S). The main difficulties found in this paper were to make a careful study among the minimizing sequences associated to the system when we are considering the domains Ω, $\mathbf{R}^N$ and $\mathbf{R}^N_+$ and to get some relations involving the limits of these sequences. Adapting some arguments found in [**14, 19, 22**] for the scalar case, we prove that similar results of those proved in [**10**] also hold for system (S).

Our main result is the following

Theorem 1. *There exists $\varepsilon^* > 0$ such that for any $\varepsilon \in (0, \varepsilon^*)$, system* (S) *has at least* $\operatorname{cat}(\Sigma)$ *positive solutions. Moreover, if Σ is not contractible in itself, then* (S) *admits at least* $\operatorname{cat}(\Sigma) + 1$ *positive solutions.*

We recall that $\operatorname{cat}(\Sigma)$ is the Ljusternick-Schnirelman category of Σ in itself, that is, the least number of closed and contractible sets in Σ which cover Σ.

This paper is divided in the following way: In Section 2, we make some definitions and prove some technical results and in Section 3, we prove Theorem 1.

2. Notations and preliminary results. In this section, we fix some notations and show technical results. Hereafter, Let us denote by $\mathbf{R}^N_+$ the half-space,

$$\mathbf{R}^N_+ = \left\{ x = (x^1, \dots, x^N) \in \mathbf{R}^N; \quad x^N > 0 \right\}$$

and by $m(\mathbf{R}^N)$ and $m(\mathbf{R}^N_+)$ the following numbers

$$m(\mathbf{R}^N) = \inf_{\substack{(u,v)\in H_\infty \\ \int_\Omega Q(u,v)\,dx \neq 0}} \frac{\int_{\mathbf{R}^N} (|\nabla u|^2 + |\nabla v|^2 + u^2 + v^2)\,dx}{(\int_{\mathbf{R}^N} Q(u,v)\,dx)^{2/p}}$$

and

$$m(\mathbf{R}^N_+) = \inf_{(u,v)\in H_{\infty,+}} \int_\Omega Q(u,v)\,dx \neq 0 \frac{\int_{\mathbf{R}^N_+}(|\nabla u|^2+|\nabla v|^2+u^2+v^2)\,dx}{(\int_{\mathbf{R}^N_+} Q(u,v)\,dx)^{2/p}},$$

where $H_\infty = H^1(\mathbf{R}^N)\times H^1(\mathbf{R}^N)$ and $H_{\infty,+} = H^1(\mathbf{R}^N_+)\times H^1(\mathbf{R}^N_+)$.

Using standard arguments, more precisely a result by Lions [**14**], we can prove that the numbers $m(\mathbf{R}^N)$ and $m(\mathbf{R}^N_+)$ are reached with

$$m(\mathbf{R}^N_+) = 2^{(2/p)-1} m(\mathbf{R}^N). \tag{2.1}$$

An important point is the fact that $m(\mathbf{R}^N)$ is reached by a vector $(w_1, w_2) \in H_\infty$ such that both w_1, w_2 are positive radially symmetric functions at the origin. Moreover, the vector $(\widetilde{w_1}, \widetilde{w_2}) = (2^{(1/p)}w_1, 2^{1/p}w_2) \in H_{\infty,+}$ reaches the number $m(\mathbf{R}^N_+)$.

Now we remark upon the solutions of system (S) when Ω is a smooth bounded domain.

In what follows, let us denote by $m(\varepsilon,\Omega)$ and $m(1,\Omega_\varepsilon)$ the following numbers

$$m(\varepsilon,\Omega) = \inf_{\substack{(u,v)\in H \\ \int_\Omega Q(u,v)\,dx\neq 0}} \frac{\int_\Omega(\varepsilon^2[|\nabla u|^2+|\nabla v|^2]+u^2+v^2)\,dx}{(\int_\Omega Q(u,v)\,dx)^{2/p}}$$

and

$$m(1,\Omega_\varepsilon) = \inf_{\substack{(u,v)\in H_\varepsilon \\ \int_{\Omega_{\varepsilon_\varepsilon}} Q(u,v)\,dx\neq 0}} \frac{\int_{\Omega_\varepsilon}(|\nabla u|^2+|\nabla v|^2+u^2+v^2)\,dx}{(\int_{\Omega_\varepsilon} Q(u,v)\,dx)^{2/p}}$$

where $\Omega_\varepsilon = \{x \in \mathbf{R}^N; \varepsilon x \in \Omega\}$, $H = E(\Omega)\times E(\Omega), H_\varepsilon = E(\Omega_\varepsilon)\times E(\Omega_\varepsilon)$,

$$E(\Omega) = \{u \in H^1(\Omega);\ u = 0 \quad \text{on} \quad \Gamma\}$$

and

$$E(\Omega_\varepsilon) = \{u \in H^1(\Omega_\varepsilon);\ u = 0 \quad \text{on} \quad \Gamma_\varepsilon\}$$

where $\Gamma_\varepsilon = \Gamma/\varepsilon$.

From Sobolev imbeddings, it is easy to prove that the numbers $m(\varepsilon,\Omega)$ and $m(1,\Omega_\varepsilon)$ are reached. Moreover, for example, if $m(\varepsilon,\Omega)$ is reached by (u_o,v_o), the functions $u_1 = m(\varepsilon,\Omega)^{1/(p-2)}u_0$ and $v_1 = m(\varepsilon,\Omega)^{1/(p-2)}v_0$ are solutions of (S).

Another important result involving the numbers $m(\varepsilon,\Omega)$ and $m(1,\Omega_\varepsilon)$ is the following identity

$$\varepsilon^{-2\alpha}m(\varepsilon,\Omega) = m(1,\Omega_\varepsilon), \quad \alpha = N\left(\frac{1}{2}-\frac{1}{p}\right).$$

Other notations that we will use in this paper are the following:

$$J_{\varepsilon,\Omega}(u,v) = \int_\Omega (\varepsilon^2[|\nabla u|^2+|\nabla v|^2]+u^2+v^2)\,dx, \quad \text{for all} \quad (u,v)\in V(\Omega)$$

where

$$V(\Omega) = \left\{(u,v)\in H; \int_\Omega Q(u,v)\,dx = 1\right\}.$$

Let $r>\rho>0$ be such that both the sets

$$\Sigma^- = \{x\in\Sigma; \text{dist}\,(x,\Gamma)\geq r\} \quad \text{and} \quad \Sigma^+ = \{x\in\mathbf{R}^N; \text{dist}\,(x,\Sigma)<r\}$$

are homotopically equivalent to Σ. Moreover, without loss generality, we assume that $0\in\Sigma^-$.

Let $\eta\in C^\infty([0,\infty),\mathbf{R})$ verifying:

$$0\leq\eta(t)\leq 1, \eta(t)=1$$
$$\text{if} \quad 0\leq t\leq\frac{1}{2}, \ \eta(t)=0 \quad \text{if} \quad t\geq 1, \quad \text{and} \quad |\eta'|\leq C$$

for some positive constant C. For any $y\in\Sigma^-$ and for $x\in\Omega$, we set $\phi_\varepsilon(y)(x) = (\phi^1_\varepsilon(y)(x),\phi^2_\varepsilon(y)(x))$, where

$$\phi^i_\varepsilon(y)(x) = \eta\left(\frac{|x-y|}{\rho}\right)w_i\left(\frac{x-y}{\varepsilon}\right), \quad i=1,2.$$

Moreover, we define $\Phi_\varepsilon(y)(x) = (\Phi^1_\varepsilon(y)(x),\Phi^2_\varepsilon(y)(x))$ with

$$\Phi^i_\varepsilon(y)(x) = \frac{\phi^i_\varepsilon(y)(x)}{(\int_\Omega Q(\phi^1_\varepsilon,\phi^2_\varepsilon)\,dx)^{1/p}}, \quad i=1,2.$$

By construction, Φ_ε is a continuous map from Σ^- to H.

Lemma 1. *For any $y \in \Sigma^-$, we have $\Phi_\varepsilon(y) \in V(\Omega)$. Moreover,*

$$J_{\varepsilon,\Omega}(\Phi_\varepsilon(y)) = \varepsilon^{2\alpha}[m(\mathbf{R}^N_+) + o(1)] \quad as \quad \varepsilon \to 0 \tag{2.2}$$

uniformly for $y \in \Sigma^-$.

Proof. It is easy to see that $\Phi^i_\varepsilon(y) \geq 0$ for all $y \in \Sigma^-$ and $\int_\Omega Q(\Phi^1_\varepsilon, \Phi^2_\varepsilon)\, dx = 1$. Moreover, since $\operatorname{dist}(B_\rho(y), \Gamma) > 0$, then $spt(\Phi^i_\varepsilon) \subset\subset \Omega \cup \Sigma$ for $i = 1, 2$.

Using the fact that Ω is a smooth bounded domain, for each $y \in \Sigma^-$, there exists a $\delta > 0$, an open neighborhood $\mathcal{N}$ of y and a diffeomorphism $\Psi : B_\delta(y) \to \mathcal{N}$ which has the Jacobian determinant at y verifying $\Psi'(y) = I$ and $\Psi(B^+_\delta) = \mathcal{N} \cap \Omega$ where $B^+_\delta = B_\delta \cap \mathbf{R}^N_+$, see [**1**] for more details.

For each n, let us choose a unitary matrix T_n such that $\widetilde{\Omega}_n = T_n(\Omega_n - y_n)$ has y^N as the inner normal vector of $\partial\widetilde{\Omega}_n$ at the origin. Using the same arguments explored in [**19, 22**], for any $R > 0$, if $\{y_n\} \subset \Sigma^-$ is a convergent sequence to $y \in \Sigma^-$, we have

$$\left| B^+_R(0) \setminus \frac{1}{\varepsilon_n}(\widetilde{\Omega}_n) \cap \{x : |x| \leq R\} \right| \to 0 \quad \text{as} \quad n \to \infty$$

where $\varepsilon_n \to 0$ as $n \to \infty$ and $|A|$ denotes the Lebesgue measure of a mensurable set $A \subset \mathbf{R}^N$. The last limit together with the change of variable theorem and Lebesgue theorem imply that

$$\varepsilon_n^{-2\alpha} J_{\varepsilon_n,\Omega}(\Phi_{\varepsilon_n}(y_n)) = m(\mathbf{R}^N_+) + o_n(1). \tag{2.3}$$

The uniform estimate mentioned in (2.2) follows from (2.3). □

Hereafter, let us denote by Y_ε the Banach space $Y_\varepsilon = H^1(\Omega) \times H^1(\Omega)$ endowed with the norm

$$\|(u, v)\|_\varepsilon = \left(\int_\Omega (\varepsilon^2[|\nabla u|^2 + |\nabla v|^2] + u^2 + v^2)\, dx \right)^{1/2}$$

and by $m^*(\varepsilon,\Omega)$ the following number

$$m^*(\varepsilon,\Omega) = \inf_{\substack{(u,v)\in Y_\varepsilon \\ \int_\Omega Q(u,v)\,dx\neq 0}} \frac{\|(u,v)\|_\varepsilon^2}{(\int_\Omega Q(u,v)\,dx)^{2/p}}.$$

Corollary 1. *The numbers $m^*(\varepsilon,\Omega)$ and $m(\mathbf{R}^N_+)$ satisfy the following equality*

$$\varepsilon^{-2\alpha} m^*(\varepsilon,\Omega) = m(\mathbf{R}^N_+) + o_\varepsilon(1).$$

Proof. From the definitions of $m^*(\varepsilon,\Omega)$ and $m(\varepsilon,\Omega)$, we have

$$m^*(\varepsilon,\Omega) \leq m(\varepsilon,\Omega) \leq J_{\varepsilon,\Omega}(\Phi_\varepsilon(0));$$

consequently,

$$\varepsilon^{-2\alpha} m^*(\varepsilon,\Omega) \leq \varepsilon^{-2\alpha} m(\varepsilon,\Omega) \leq \varepsilon^{-2\alpha} J_{\varepsilon,\Omega}(\Phi_\varepsilon(0)).$$

On the other hand, from Lemma 1,

$$\varepsilon^{-2\alpha} J_{\varepsilon,\Omega}(\Phi_\varepsilon(0)) = m(\mathbf{R}^N_+) + o_\varepsilon(1);$$

then,

$$m^*(1,\Omega_\varepsilon) \leq m(1,\Omega_\varepsilon) \leq m(\mathbf{R}^N_+) + o_\varepsilon(1). \tag{2.4}$$

Now, we will prove the following claim.

Claim 1. *Denoting $\Omega_n = \Omega_{\varepsilon_n}$, we have*

$$\lim_{n\to\infty} m^*(1,\Omega_n) = m(\mathbf{R}^N_+) \quad \textit{as} \quad \varepsilon_n \to 0.$$

In fact, first of all, if $\{(u_n,v_n)\}$ satisfies

$$m^*(1,\Omega_n) = \int_{\Omega_n} (|\nabla u_n|^2 + |\nabla v_n|^2 + |u_n|^2 + |v_n|^2)\,dx,$$

using the fact that Ω_n verifies the uniform cone condition for all $n \in \mathbf{N}$, there exists $c > 0$ satisfying

$$|u|_{L^p(\Omega_n)} \leq c\|u\|_{W^{1,2}(\Omega_n)} \quad \text{for all} \quad n \in \mathbf{N} \text{ (see [22])}.$$

The last inequality together with Lions [14] imply that there exist R, $\tau > 0$ and $\{y_n\} \subset \partial\Omega_n$ such that

$$\lim_{n\to\infty} \int_{B_R(y_n)\cap\Omega_n} Q(u_n, v_n)\, dx \geq \tau. \tag{2.5}$$

For each n, denoting again by T_n a unitary matrix such that $\widetilde{\Omega}_n = T_n(\Omega_n - y_n)$ has y^N as inner normal vector of $\partial\widetilde{\Omega}_n$ at origin, we get by a direct calculus

$$\chi_{\widetilde{\Omega}_n} \to \chi_{\mathbf{R}^N_+} \quad \text{a.e in} \quad \mathbf{R}^N,$$

where $\chi_{\widetilde{\Omega}_n}$ and $\chi_{\mathbf{R}^N_+}$ are the characteristic functions of $\widetilde{\Omega}_n$ and $\mathbf{R}^N_+$, respectively. Defining the sequences $\widehat{u}_n(x) = u_n(T_n^{-1}(x) + y_n)$ and $\widehat{v}_n(x) = v_n(T_n^{-1}(x) + y_n)$, it is easy to check that

$$m^*(1, \Omega_n) = m^*(1, \widetilde{\Omega}_n)$$

with $(\widehat{u}_n, \widehat{v}_n)$ satisfying the equality

$$m^*(1, \widetilde{\Omega_n}) = \int_{\widetilde{\Omega}_n} (|\nabla\widehat{u}_n|^2 + |\nabla\widehat{v}_n|^2 + |\widehat{u}_n|^2 + |\widehat{v}_n|^2)\, dx$$

and $\int_{\widetilde{\Omega}_n} Q(\widehat{u}_n, \widehat{v}_n)\, dx = 1$. Moreover, there exist nonnegative functions $u, v \in H^1_{\text{loc}}(\mathbf{R}^N_+) \setminus \{0\}$ such that $\widehat{u}_n \to u$ and $\widehat{v}_n \to v$ in $H^1_{\text{loc}}(\mathbf{R}^N_+)$. Thus, if $w_n = \widehat{u}_n - u$ and $z_n = \widehat{v}_n - u$, we have

$$\int_{\widetilde{\Omega}_n} |\nabla w_n|^2\, dx + \int_{I\!R^N_+} |\nabla u|^2 dx = \int_{\widetilde{\Omega}_n} |\nabla\widehat{u}_n|^2\, dx + o_n(1)$$

and

$$\int_{\widetilde{\Omega}_n} |\nabla z_n|^2\, dx + \int_{\mathbf{R}^N_+} |\nabla v|^2\, dx = \int_{\widetilde{\Omega}_n} |\nabla\widehat{v}_n|^2\, dx + o_n(1);$$

hence,

$$m^*(1,\widetilde{\Omega}_n) = \int_{\mathbf{R}^N_+} (|\nabla u|^2 + |\nabla v|^2 + |u|^2 + |v|^2)\, dx \\ + \int_{\widetilde{\Omega}_n} (|\nabla w_n|^2 + |\nabla z_n|^2 + |w_n|^2 + |z_n|^2)\, dx + o_n(1).$$

Denoting $\lambda = \int_{\mathbf{R}^N_+} Q(u,v)\, dx$, it follows that

$$m^*(1,\widetilde{\Omega}_n) \geq m(\mathbf{R}^N_+)\lambda^{2/p} + (1-\lambda)^{2/p} m^*(1,\widetilde{\Omega}_n) + o_n(1)$$

and, from (2.4),

$$m^*(1,\widetilde{\Omega}_n) \geq (\lambda^{2/p} + (1-\lambda)^{2/p}) m^*(1,\widetilde{\Omega}_n) + o_n(1).$$

Since there exists $\delta > 0$ such that

$$m^*(1,\widetilde{\Omega}_n) \geq \delta \quad \text{for all} \quad n \in \mathbf{N},$$

we have

$$1 \geq \lambda^{2/p} + (1-\lambda)^{2/p} + o_n(1);$$

then, passing the limit $n \to \infty$, it follows that

$$1 \geq \lambda^{2/p} + (1-\lambda)^{2/p}.$$

If $\lambda \in (0,1)$, we have

$$\lambda^{\frac{2}{p}} + (1-\lambda)^{\frac{2}{p}} > 1$$

which is absurd with the above inequality, and we can conclude that $\lambda \in \{0,1\}$. Once that $\lambda \neq 0$ by (2.5), we have $\lambda = 1$ and so

$$\int_{\mathbf{R}^N_+} Q(u,v)\, dx = 1.$$

On the other hand,

$$\int_{\mathbf{R}^N_+} (|\nabla u|^2 + |\nabla v|^2 + |u|^2 + |v|^2)\, dx \\ \leq \liminf_{n\to\infty} \int_{\widetilde{\Omega}_n} (|\nabla \widehat{u}_n|^2 + |\nabla \widehat{v}_n|^2 + |\widehat{u}_n|^2 + |\widehat{v}_n|^2)\, dx,$$

hence

$$m(\mathbf{R}^N_+) \leq \int_{\mathbf{R}^N_+} (|\nabla u|^2 + |\nabla v|^2 + |u|^2 + |v|^2)\, dx \leq \liminf_{n\to\infty} m^*(1, \widetilde{\Omega}_n).$$

By (2.4),

$$\limsup_{n\to\infty} m^*(1, \widetilde{\Omega}_n) \leq m(\mathbf{R}^N_+)$$

so, from the last two inequalities, it follows

$$\lim_{n\to\infty} m^*(1, \widetilde{\Omega}_n) = m(\mathbf{R}^N_+),$$

consequently

$$\lim_{n\to\infty} m^*(1, \Omega_n) = m(\mathbf{R}^N_+).$$

The last limit implies that

$$\lim_{\varepsilon\to 0} \varepsilon^{-2\alpha} m^*(\varepsilon, \Omega) = m(\mathbf{R}^N_+). \qquad \square$$

Lemma 2. *Let (u_n, v_n) be a sequence satisfying*

$$\int_{\Omega_n} [|\nabla u_n|^2 + |\nabla v_n|^2 + |u_n|^2 + |v_n|^2]\, dx = m(1, \Omega_n) + o_n(1)$$

and

$$(u_n, v_n) \in V(\Omega_n)$$

where $\Omega_n = \Omega_{\varepsilon_n}$ and $\varepsilon_n \to 0$ as $n \to \infty$. Then, for some subsequence, there exists $y_n \in \partial\Omega$ such that: For each $\varepsilon > 0$ there is an $R > 0$ with the property that

$$\lim_{n\to\infty} \int_{B_R(y_n)\cap\Omega_n} Q(u_n, v_n)\, dx \geq 1 - \varepsilon.$$

Proof. In what follows, we will show that sequence $\{\chi_n Q(u_n, v_n)\}$ satisfies the condition of compactness mentioned in the concentration-compactness lemma due to Lions [**14**], where χ_n is the characteristic function of Ω_n. To this end, we will divide our arguments in two steps:

Step 1. Vanishing does not hold. Hereafter, let us denote by m_Q and M_Q two positive constants verifying

$$m_Q[|u|^p+|v|^p] \leq Q(u,v) \leq M_Q[|u|^p+|v|^p] \quad \text{for all} \quad (u,v) \in \mathbf{R}^2. \tag{2.6}$$

Assuming by contradiction that

$$\lim_{n\to\infty}\left(\sup_{y\in\mathbf{R}^N}\int_{B_R(y)\cap\Omega_n}\chi_n Q(u_n,v_n)\,dx\right) = 0, \tag{2.7}$$

from (2.6)–(2.7) it follows that

$$\begin{aligned}\lim_{n\to\infty}\left(\sup_{y\in\mathbf{R}^N}\int_{B_R(y)\cap\Omega_n}|u_n|^p\,dx\right) &= \lim_{n\to\infty}\left(\sup_{y\in\mathbf{R}^N}\int_{B_R(y)\cap\Omega_n}|v_n|^p\,dx\right)\\ &= 0,\end{aligned}$$

hence by [**22**, Lemma 2.2], we get

$$\lim_{n\to\infty}\int_{\mathbf{R}^N}\chi_n|u_n|^p\,dx = \lim_{n\to\infty}\int_{\mathbf{R}^N}\chi_n|v_n|^p\,dx = 0.$$

The last limits together with (2.6) imply that

$$\limsup_{n\to\infty}\int_{\Omega_n}Q(u_n,v_n)\,dx = 0$$

which is absurd, because $(u_n,v_n)\in V(\Omega_n)$. Therefore, we can conclude that vanishing does not hold.

Step 2. Dichotomy does not hold. Adapting again the arguments explored in [**22**], there exists $\gamma\in(0,1)$, such that: For each $\varepsilon>0$, there exists $R_0>0$ and $\{z_n\}\subset\mathbf{R}^N$ satisfying

$$\int_{B_{R_0}(z_n)\cap\Omega_n}Q(u_n,v_n)\,dx \geq \gamma - \frac{\varepsilon}{2} \tag{2.8}$$

and

$$\int_{B_{2R_0}(z_n)\cap\Omega_n} Q(u_n, v_n)\,dx \leq \gamma + \frac{\varepsilon}{2} \tag{2.9}$$

for some subsequence, still denoted by (u_n, v_n).

Let η be a smooth nonincreasing function defined on $[0, +\infty)$ such that $\eta(t) = 1$, $0 \leq t \leq 1$; $\eta(t) = 0$, $t \geq 2$ and $|\eta'| \leq 2$. Also define $\xi(t) = 1 - \eta(t)$ which is a nondecreasing function on $[0, \infty)$. Let

$$u_n^1(x) = \chi_n(x)\eta\left(\frac{|x - z_n|}{R_0}\right)u_n(x)$$

and

$$v_n^1(x) = \chi_n(x)\eta\left(\frac{|x - z_n|}{R_0}\right)v_n(x)$$

and

$$u_n^2(x) = \chi_n(x)\xi\left(\frac{|x - z_n|}{2R_0}\right)u_n(x)$$

and

$$v_n^2(x) = \chi_n(x)\xi\left(\frac{|x - z_n|}{2R_0}\right)v_n(x).$$

From the above definitions, we have

$$\left|\int_{\mathbf{R}^N} Q(u_n^1, v_n^1)\,dx - \gamma\right| \leq 2\varepsilon \tag{2.10}$$

$$\left|\int_{\mathbf{R}^N} Q(u_n^2, v_n^2)\,dx - (1 - \gamma)\right| \leq 2\varepsilon \tag{2.11}$$

$$\begin{aligned}\int_{\Omega_n} (|\nabla u_n|^2 + |u_n|^2)\,dx &- \int_{\Omega_n} (|\nabla u_n^1|^2 + |u_n^1|^2)\,dx \\ &- \int_{\Omega_n} (|\nabla u_n^2|^2 + |u_n^2|^2)\,dx \geq -2\varepsilon\end{aligned} \tag{2.12}$$

and

$$\begin{aligned}\int_{\Omega_n} (|\nabla v_n|^2 + |v_n|^2)\,dx &- \int_{\Omega_n} (|\nabla v_n^1|^2 + |v_n^1|^2)\,dx \\ &- \int_{\Omega_n} (|\nabla v_n^2|^2 + |v_n^2|^2)\,dx \geq -2\varepsilon.\end{aligned} \tag{2.13}$$

From (2.10)–(2.13),

$$
\begin{aligned}
m(1,\Omega_n) + o_n(1) \geq & \left(\int_{\Omega_n} Q(u_n^1, v_n^1)\, dx \right)^{2/p} m(1,\Omega_n) \\
& + \left(\int_{\Omega_n} Q(u_n^2, v_n^2)\, dx \right)^{2/p} m(1,\Omega_n) - 4\varepsilon.
\end{aligned} \tag{2.14}
$$

Using the fact that there exists $\delta_0 > 0$ such that

$$m(1,\Omega_n) \geq \delta_0 \quad \text{for all} \quad n \in \mathbf{N}$$

from (2.14),

$$1 + o_n(1) \geq \left(\int_{\Omega_n} Q(u_n^1, v_n^1)\, dx \right)^{2/p} + \left(\int_{\Omega_n} Q(u_n^2, v_n^2)\, dx \right)^{2/p} - C\varepsilon$$

for some positive constant C. From (2.10)–(2.11),

$$1 \geq (\gamma - 2\varepsilon)^{2/p} + (1 - \gamma - 2\varepsilon)^{2/p} - C\varepsilon;$$

thus, passing to the limit when $\varepsilon \to 0$ in the last inequality and using the fact that $\gamma \in (0,1)$, it follows that

$$1 \geq \gamma^{2/p} + (1-\gamma)^{2/p} > 1$$

which is absurd. Therefore, dichotomy does not hold.

From the above steps, we can conclude that compactness holds, so there exists $\{z_n\} \subset \mathbf{R}^N$ such that for each $\varepsilon > 0$ fixed, there exists $R_1 > 0$ satisfying

$$\lim_{n\to\infty} \int_{B_{R_1}(z_n)\cap\Omega_n} Q(u_n, v_n) \geq 1 - \varepsilon.$$

Claim 2. *There exists* $\overline{C} > 0$ *such that*

$$\operatorname{dist}(z_n, \partial\Omega_n) \leq \overline{C}.$$

If the claim is not true, then for some subsequence, still denoted by $\{z_n\}$, we have

$$\operatorname{dist}(z_n, \partial\Omega_n) \to +\infty \quad \text{as} \quad n \to \infty.$$

Thus, we can conclude that there exists $R_n \in (0, +\infty)$ with $R_n \to \infty$ and $B_{R_n}(z_n) \subset \Omega_n$.

Defining

$$w_n^1(x) = \eta\left(\frac{|x - z_n|}{R_n}\right) u_n(x) \quad \text{and} \quad w_n^2(x) = \eta\left(\frac{|x - z_n|}{R_n}\right) v_n(x),$$

it follows

$$\begin{aligned}\int_{\mathbf{R}^N} Q(w_n^1, w_n^2)\,dx &= \int_{B_{2R_n}(z_n)} Q(w_n^1, w_n^2)\,dx \\ &\geq \int_{B_{R_1}(z_n)} Q(w_n^1, w_n^2)\,dx,\end{aligned}$$

which implies

$$\int_{\mathbf{R}^N} Q(w_n^1, w_n^2)\,dx \geq 1 - \varepsilon \quad \text{for all} \quad n \in \mathbf{N}. \tag{2.15}$$

Once that

$$\begin{aligned}&\int_{\Omega_n} |\nabla u_n|^2 + |\nabla v_n|^2 + |u_n|^2 + |v_n|^2 \\ &\qquad - \int_{\Omega_n} |\nabla w_n^1|^2 + |\nabla w_n^2|^2 + |w_n^1|^2 + |w_n^2|^2 \geq -2\varepsilon,\end{aligned}$$

we have

$$m(1, \Omega_n) + o_n(1) \geq \left(\int_{\Omega_n} Q(w_n^1, w_n^2)\,dx\right)^{2/p} m(\mathbf{R}^N) - 2\varepsilon,$$

and by (2.15),

$$m(1, \Omega_n) + o_n(1) \geq (1 - \varepsilon)^{2/p} m(\mathbf{R}^N) - 2\varepsilon. \tag{2.16}$$

Recalling that

$$\limsup_{n\to\infty} m(1,\Omega_n) \leq m(\mathbf{R}^N_+)$$

from (2.16), it follows that

$$m(\mathbf{R}^N_+) \geq (1-\varepsilon)^{2/p} m(\mathbf{R}^N) - 2\varepsilon,$$

and taking the limit of $\varepsilon \to 0$, we get

$$m(\mathbf{R}^N_+) \geq m(\mathbf{R}^N),$$

which is a contradiction with (2.1), and the claim is proved.

From Claim 2, there exists $y_n \in \partial\Omega_n$ verifying the inequality $|z_n - y_n| \leq \overline{C}$, which implies $B_{R_1}(z_n) \subset B_R(y_n)$ where $R = R_1 + \overline{C}$. Therefore,

$$\int_{B_R(y_n)\cap\Omega_n} Q(u_n, v_n)\,dx \geq \int_{B_{R_1}(z_n)\cap\Omega_n} Q(u_n, v_n)\,dx \geq 1-\varepsilon. \qquad \square$$

Lemma 3. *Let* $\varepsilon_n \downarrow 0$ *and* (u_n, v_n) *satisfy the hypotheses of Lemma* 2*. Then, there exists* $C > 0$ *such that*

$$\operatorname{dist}(y_n, \Sigma_n) \leq C,$$

where $\{y_n\} \in \partial\Omega_n$ *is the sequence given in Lemma* 2*.*

Proof. Assume by contradiction that

$$\lim_{n\to\infty} \operatorname{dist}(y_n, \Sigma_n) = +\infty.$$

Then, there exists $\{R_n\} \subset (0, +\infty)$ with $R_n \to \infty$ such that

$$\operatorname{dist}(y_n, \Sigma_n) \geq 2R_n.$$

Defining

$$w_n^1(x) = \eta\left(\frac{|x-y_n|}{2R_n}\right) u_n(x) \quad \text{and} \quad w_n^2(x) = \eta\left(\frac{|x-y_n|}{2R_n}\right) v_n(x),$$
$$x \in \Omega_n,$$

where η was given in the above lemma and repeating the same type of arguments used in that lemma, we may obtain again

$$m(\mathbf{R}^N_+) \geq m(\mathbf{R}^N),$$

which is absurd. □

In what follows, let us denote by $\beta : V(\Omega) \to \mathbf{R}^N$ the continuous map

$$\beta(u,v) = \int_\Omega Q(u,v)x\,dx.$$

From the definitions of Φ_ε and β, we have that

$$\beta(\Phi_\varepsilon(y)) = y \quad \text{for all} \quad y \in \Sigma^-.$$

Proposition 1. *For each $\theta > 0$ there exists $\varepsilon_1 > 0$ such that for all $\varepsilon \in (0, \varepsilon_1)$ and $(u,v) \in V(\Omega)$, we have*

$$J_{\varepsilon,\Omega}(u,v) \leq m(\varepsilon,\Omega) + \theta\varepsilon^{2\alpha} \quad \Longrightarrow \quad \beta(u,v) \in \Sigma^+.$$

Proof. Assume that there exist $\theta_n, \varepsilon_n \to 0$, $(u_n, v_n) \in V(\Omega)$ such that

$$J_{\varepsilon_n,\Omega}(u_n,v_n) \leq m(\varepsilon_n,\Omega) + \theta_n\varepsilon_n^{2\alpha} \quad \text{and} \quad \beta(u_n,v_n) \notin \Sigma^+;$$

consequently,

$$\varepsilon_n^{-2\alpha} J_{\varepsilon_n,\Omega}(u_n,v_n) \leq m(1,\Omega_n) + \theta_n. \tag{2.17}$$

Defining

$$w_n(x) = \varepsilon_n^{N/p} u_n(\varepsilon_n x) \quad \text{and} \quad z_n(x) = \varepsilon_n^{N/p} v_n(\varepsilon_n x),$$

from (2.17) and Corollary 1, we have

$$(w_n, z_n) \in V(\Omega_n) \quad \text{and} \quad \limsup_{n\to\infty} J_{1,\Omega_{\varepsilon_n}}(w_n,z_n) \leq m(\mathbf{R}^N_+).$$

From Lemmas 2 and 3, there exist $y_n \in \partial\Omega$, $C > 0$, $\varepsilon > 0$ and $R > 0$ such that

$$\lim_{n\to\infty} \int_{B_R(y_n)\cap\Omega_n} Q(w_n, z_n) \geq 1 - \varepsilon \quad \text{and} \quad \operatorname{dist}(y_n, \Sigma_n) \leq C.$$

Thus, there exists $x_n \in \Sigma$ such that

$$\left| y_n - \frac{x_n}{\varepsilon_n} \right| \leq C \quad \text{and} \quad \lim_{n\to\infty} \int_{B_{\varepsilon_n R_1}(x_n)\cap\Omega} Q(u_n, v_n)\,dx \geq 1 - \varepsilon$$

where $R_1 = R + C$. Since Σ is bounded, we can assume that, for some subsequence, $x_n \to x_0 \in \overline{\Sigma}$. Then, for $i = 1, \dots, N$

$$|\beta^i(u_n, v_n) - x_0^i| = \left| \int_\Omega Q(u_n, v_n)(x^i - x_0^i)\,dx \right|,$$

which implies

$$\begin{aligned} |\beta^i(u_n, v_n) - x_0^i| \leq &\int_{\Omega\cap B_{\varepsilon_n R_1}(x_n)} Q(u_n, v_n)|x^i - x_0^i|\,dx \\ &+ \int_{\Omega\setminus B_{\varepsilon_n R_1}(x_n)} Q(u_n, v_n)|x^i - x_0^i|\,dx; \end{aligned}$$

hence,

$$|\beta^i(u_n, v_n) - x_0^i| \leq \varepsilon_n R_1 + |x_n^i - x_0^i| + \varepsilon \max_{x\in\Omega} |x - x_0|,$$

so,

$$\lim_{n\to\infty} |\beta(u_n, v_n) - x_0| = 0,$$

showing that $\beta(u_n, v_n) \in \Sigma^+$ for n sufficiently large, leading to an absurdity. □

3. Proof of main theorem. In this section we will prove Theorem 1. Hereafter, let us denote by θ the number obtained in Proposition 1, $m^*(\varepsilon) = m(\varepsilon, \Omega) + \theta\varepsilon^{2\alpha}$, and by $J_{\varepsilon,\Omega}^{m^*(\varepsilon)}$ the following set

$$J_{\varepsilon,\Omega}^{m^*(\varepsilon)} = \Big\{ (u, v) \in H\,;\; J_{\varepsilon,\Omega}(u, v) \leq m^*(\varepsilon) \Big\}.$$

Lemma 4. *There exists $\varepsilon^* > 0$ such that for any $\varepsilon \in (0, \varepsilon^*)$,*

$$\Phi_\varepsilon(\Sigma^-) \subset V^{m^*(\varepsilon)} \quad \text{and} \quad \beta(V^{m^*(\varepsilon)}) \subset \Sigma^+,$$

where $V^{m^*(\varepsilon)} = J_{\varepsilon,\Omega}^{m^*(\varepsilon)} \cap V(\Omega)$.

Proof. First of all, we have

$$\lim_{\varepsilon\to 0} \varepsilon^{-2\alpha} J_{\varepsilon,\Omega}(\Phi_\varepsilon(y)) = \lim_{\varepsilon\to 0} \varepsilon^{-2\alpha} m(\varepsilon, \Omega) = m(\mathbf{R}_+^N)$$

uniformly for $y \in \Sigma^-$. Thus,

$$\lim_{\varepsilon\to 0} \varepsilon^{-2\alpha} \Big(J_{\varepsilon,\Omega}(\Phi_\varepsilon(y)) - m(\varepsilon, \Omega) \Big) = 0$$

uniformly in Σ^-. Then, there exists an $\varepsilon_2 > 0$ such that

$$\varepsilon^{-2\alpha} \Big(J_{\varepsilon,\Omega}(\Phi_\varepsilon(y)) - m(\varepsilon, \Omega) \Big) \le \theta, \quad \forall \varepsilon \in (0, \varepsilon_2) \quad \text{and} \quad \forall y \in \Sigma^-;$$

that is,

$$J_{\varepsilon,\Omega}(\Phi_\varepsilon(y)) \le m(\varepsilon, \Omega) + \theta\varepsilon^{2\alpha}, \ \forall\, \varepsilon \in (0, \varepsilon_2) \quad \text{and} \quad \forall\, y \in \Sigma^-.$$

Considering $\varepsilon^* = \min\{\varepsilon_1, \varepsilon_2\}$, we have $\Phi_\varepsilon(\Sigma^-) \subset V^{m^*(\varepsilon)}$ and, by Proposition 1,

$$\beta(V^{m^*(\varepsilon)}) \subset \Sigma^+ \quad \text{for all} \quad \varepsilon \in (0, \varepsilon^*). \qquad \square$$

In the proof of the next result, we use similar arguments developed by Benci and Cerami [**8**].

Lemma 5. *Let $\varepsilon^* > 0$ be given by Lemma 4. Then*

$$\operatorname{cat}(V^{m^*(\varepsilon)}) \ge \operatorname{cat}(\Sigma) \quad \textit{for all} \quad \varepsilon \in (0, \varepsilon^*).$$

Proof. Assume that

$$V^{m^*(\varepsilon)} = A_1 \cup \cdots \cup A_n$$

where $A_j, j = 1, \dots, n$ is closed and contractible in $V^{m^*(\varepsilon)}$, that is, there exists

$$h_j \in C([0,1] \times A_j, V^{m^*(\varepsilon)}),$$

such that

$$h_j(0,u) = u, \quad h_j(1,u) = h_j(1,v) \quad \text{for all} \quad u, v \in A_j.$$

Consider $B_j = \Phi_\varepsilon^{-1}(A_j)$, $1 \le j \le n$. The sets B_j are closed and

$$\Sigma^- = B_1 \cup \dots \cup B_n.$$

Using the deformation

$$g_j(t,y) = \beta(h_j(t, \Phi_\varepsilon(y))),$$

we have

$$g_j(0,y) = \beta(h_j(0, \Phi_\varepsilon(y))) = \beta(\Phi_\varepsilon(y)) = y \quad \text{for all} \quad y \in \Sigma^-$$

and

$$g_j(1,y) = \beta(h_j(1, \Phi_\varepsilon(y))) = \beta(z) \quad \text{for all} \quad y \in B_j;$$

thus, B_j is contractible in Σ^+, and

$$\operatorname{cat}(V^{m^*(\varepsilon)}) \ge \operatorname{cat}_{\Sigma^+}(\Sigma^-) = \operatorname{cat}(\Sigma). \qquad \square$$

Proof of Theorem 1. Let ε^* be as in Lemma 4 and $\varepsilon \in (0, \varepsilon^*)$. Using well known Ljusternik-Schirelman arguments, it follows that the existence of at least $\operatorname{cat}(\Sigma)$ distinct critical points of $J_{\varepsilon,\Omega}$ on $V(\Omega)$. For the case where Σ is not contractible in itself, we fix $(u^*, v^*) \in V(\Omega) \setminus \overline{\Phi_\varepsilon(\Sigma^-)}$ and $F : [0,1] \times \overline{\Phi_\varepsilon(\Sigma^-)} \to V(\Omega)$ by setting

$$F(t,u) = \frac{t(u^*, v^*) + (1-t)(u,v)}{\Big(Q((u^*, v^*) + (1-t)(u,v))\Big)^{1/p}}.$$

Repeating the same arguments as found in Candela and Lazzo [**10**], we find one more critical point.

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