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A DISTORTION THEOREM FOR THE CLASS OF MÖBIUS TRANSFORMATIONS OF CONVEX MAPPINGS

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ABSTRACT. For the class of Möbius transformations of convex mappings, this article examines the problems of maximum and minimum modulus.

1. Introduction. Let S denote the class of analytic univalent functions f defined in the unit disk $D = \{z : |z| < 1\}$ and normalized so that $f(0) = f'(0) - 1 = 0$.

If $f \in S$ and $w \notin f(D)$, then the function

$$\hat{f} = wf/(w-f)$$

belongs again to S. This omitted - value transformation is important in the analysis of the class S and other related classes, where it has been utilized in the proofs of properties of these classes.

If F is a subset of S, let

$$\hat{F} = \{\hat{f} : f \in F,\ w \in C^* \backslash f(D)\}.$$

Here C^* is the extended complex plane which is $C \cup \{\infty\}$. Since we allow $w = \infty$, it is clear that $F \subset \hat{F} \subset S$, and since the composition of normalized Möbius transformations is again a normalized Möbius transformation, it follows that $\hat{\hat{F}} = \hat{F}$.

Some elementary properties of F are immediately inherited by $\hat{F}$. If F is compact in the topology of locally uniform convergence, then so is $\hat{F}$. If F is rotationally invariant, that is, $f_\alpha(z) = e^{-i\alpha} f(e^{i\alpha} z)$ belongs to F whenever f does, then $\hat{F}$ is also rotationally invariant.

We shall consider the subclass K of S consisting of those functions f in S which map the unit disk D conformally onto convex domains. We can easily construct examples to show that K is a proper subset of

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$\hat{K}$. Indeed, let f be a parallel strip mapping and w be a finite point of the boundary $\partial f(D)$. Then the boundary of the range $\hat{f} = wf/(w-f)$ consists of a straight line and a full circle which intersect only at the point w. Thus $\hat{f}$ is not a convex function. In view of the proper containment, some interesting properties of K are not preserved under the omitted - value transformation. For example, it is well-known that the second coefficient of functions in K is bounded by 1. However, Barnard and Schober [**1**] proved that the second coefficient of functions in $\hat{K}$ is bounded by $1.327\cdots$, and that there exists an extremal function whose second coefficient attains the given bound.

In a recent note, Hall [**5**] proved that the coefficients of functions in $\hat{K}$ have a uniform bound. However, the question of the best uniform bound remains open as well as the individual coefficient problems, except as noted earlier, the second coefficient.

The purpose of this note is to examine the problems of maximum and minimum modulus for the class $\hat{K}$. As an application, we obtained the Koebe disk for $\hat{K}$.

We will apply a result obtained by Barnard and Schober [**1**], and, for the sake of completeness, we state this result in

THEOREM A. *If $\lambda : \hat{K} \to R$ is an admissible continuous functional, then λ assumes its maximum over $\hat{K}$ at a function $\hat{f} = wf/(w-f)$ where either $\hat{f}$ is a half-plane mapping or else f is a strip mapping and w is a finite point of $\partial f(D)$.*

Barnard and Schober [**1**] also observed the following application of Theorem A:

Let λ be defined by

$$\lambda(\hat{f}) = \mathrm{Re}\{\phi(\log[\hat{f}(z)/z])\},$$

where ϕ is a nonconstant entire function, and $z \in D\backslash\{0\}$ is fixed. By a result of Kirwan [**6**], λ is a continuous admissible functional as defined in [**1**]. Choosing $\phi(w) = \pm w$, Theorem A implies that an extremal function to the problems of maximum and minimum modulus is either a half-plane mapping or is generated by a strip mapping. Notice that the domain of the extremal strip mapping need not be symmetric about the origin. Although the half-plane mapping is extremal in K for these problems, we will show that an extremal function for the class $\hat{K}$ is in fact generated by a non-degenerate strip mapping.

2. A distortion theorem. For each convex function f in K, it is well-known that

$$(2.1) \qquad r(1+r)^{-1} \le |f(z)| \le r(1-r)^{-1}, \quad |z| = r < 1,$$

with equality occurring only for functions which are half-plane mappings, that is, functions of the form $f(z) = z(1-e^{i\theta}z)^{-1}$, $\theta \in \mathbf{R}$.

We derive sharp upper and lower bounds for $|f(z)|$, $f \in \hat{K}$. The result is as follows:

THEOREM 2.1. *Let r, $0 < r < 1$, be fixed. For x in $(0,\pi)$, let*

$$h(x,r) = \frac{[x - 2\arg(1+re^{ix})]\sin x}{x\arg(1+re^{ix})}$$

and

$$H(x,r) = \frac{[x + 2\arg(1-re^{-ix})]\sin x}{x\arg(1-re^{-ix})}.$$

Then, for each $f \in \hat{K}$,

$$m(r) \le |f(z)| \le M(r), \quad |z| = r < 1,$$

where

$$[M(r)]^{-1} = \min\{h(x,r) : 0 < x < \pi\} < (1-r)r^{-1}$$

and

$$[m(r)]^{-1} = \max\{H(x,r) : 0 < x < \pi\} > (1+r)r^{-1}.$$

For the functions $e^{-i\alpha}f(e^{i\alpha}z)$, $\alpha \in R$, $|f(z)| = m(r)$ occurs where $f(z) = g(1)g(z)/[g(1)-g(z)]$ and g is the vertical strip mapping defined by

$$g(z) = \frac{1}{2i\sin x_1}\log\frac{1+e^{ix_1}z}{1+e^{-ix_1}z},$$

where H attains its maximum at x_1. Similarly, $|f(z)| = M(r)$ occurs for the functions of the form given above, except that

$$g(z) = \frac{1}{2i \sin x_2} \log \frac{1 + e^{ix_2} z}{1 + e^{-ix_2} z},$$

where h attains its minimum at x_2.

The proof is rather lengthy, so we will break it into several parts. Notice that, from Theorem A, it suffices to extremize $|f(z)|$ where f is either a half-plane mapping or else f is the transform of a strip mapping. Although this is clearly a major step in determining the extremal values, as occasionally happens in these type of problems, determining the explicit values still requires some work.

The bounds for the modulus of half-plane mappings are given by (2.1). Since $\hat{K}$ is rotationally invariant, we may assume that f is generated by a vertical strip mapping. Thus f has the form

$$f(z) = g(e^{i\psi})g(z)/[g(e^{i\psi}) - g(z)], \tag{2.2}$$

where $g(e^{i\psi}) \neq \infty$ and

$$g(z) = \frac{1}{2i \sin x} \log \frac{1 + e^{ix} z}{1 + e^{-ix} z} \tag{2.3}$$

for some x in $(0, \pi)$. We first establish the following lemma.

LEMMA 2.2. *Let g be given as in (2.3). Then, for each fixed x,*

$$\arg\left(1 - g(z)\frac{\sin x}{x}\right) < \arg\left(\frac{zg'(z)}{g(z)}\right) < \arg\left(1 + g(z)\frac{\sin x}{\pi - x}\right)$$

whenever $\operatorname{Im}\{z\} > 0$ in D.

(Here it is understood that the argument function vanishes at $z = 0$.)

PROOF. We will prove the left assertion; the right assertion is proved analogously.

Specifically, we shall show that

$$\begin{aligned} -\infty &< \limsup_{z \to \zeta} \arg(1 - g(z)x^{-1} \sin x) \\ &\leq \liminf_{z \to \zeta} \arg(zg'(z)/g(z)) < \infty \end{aligned} \tag{2.4}$$

for each point ζ on the boundary ∂D^+, where D^+ is the upper half-disk. An application of the generalized maximum principle for harmonic functions [**3**, p. 254] will then complete the proof.

For each fixed x in $(0,\pi)$, $\operatorname{Re} g < x/(2\sin x)$ in D, so $\operatorname{Re}\{1 - gx^{-1}\sin x\} > 1/2$. Since g is a convex function, $\operatorname{Re}\{zg'/g\} > 1/2$ [**4**, p. 73]. Thus the harmonic function $\arg(1 - gx^{-1}\sin x)$ is uniformly bounded by $\pi/2$ and continuous in $\overline{D}$, while $\arg(zg'/g)$ is bounded by $\pi/2$ and continuous in $\overline{D}$ except at $z = e^{i(\pi\pm x)}$. Therefore it is sufficient to compare radial limits almost everywhere in (2.4).

If ζ is real in ∂D^+, then $\zeta g'(\zeta)/g(\zeta)$ and $1 - g(\zeta)x^{-1}\sin x$ are both real, so (2.4) holds.

If $\zeta = e^{i(\pi-x)}$, then $\lim_{r\to 1}\arg(1 - g(re^{i(\pi-x)})x^{-1}\sin x) = -\pi/2$. Since $\arg(zg'/g) > -\pi/2$ in D, (2.4) again holds.

It remains to show the validity of (2.4) for nonreal ζ in ∂D^+ with $\zeta \neq e^{i(\pi-x)}$. In this case, if

$$\log\frac{1+e^{ix}z}{1+e^{-ix}z} = u(r,\theta) + iv(r,\theta),$$

then, for $z = re^{i\theta}$,

$$\lim_{r\to 1}\arg(zg'(z)/g(z)) = \arctan\frac{u(1,\theta)}{v(1,\theta)}, \tag{2.5}$$

and

$$\lim_{r\to 1}\arg(1 - g(z)x^{-1}\sin x) = \arctan\frac{u(1,\theta)}{2x - v(1,\theta)}. \tag{2.6}$$

The identity

$$\frac{1+e^{i(\theta+x)}}{1+e^{i(\theta-x)}} = \frac{\cos((x+\theta)/2)}{\cos((x-\theta)/2)}e^{ix}$$

yields

$$v(1,\theta) = \begin{cases} x, & \theta\in(0,\pi-x)\\ -(\pi-x), & \theta\in(\pi-x,\pi)\end{cases} \tag{2.7.1}$$

$$u(1,\theta) < 0, \qquad \theta\in(0,\pi). \tag{2.7.2}$$

So, from (2.5) and (2.6), the validity of (2.4) for $\zeta = e^{i\theta}$, $\theta \in (0,\pi)$ and $\theta \neq \pi - x$, is equivalent to

$$\frac{[2x - 2v(1,\theta)]u(1,\theta)}{[2x - v(1,\theta)]v(1,\theta)} \geq 0.$$

But (2.7) implies the above inequality, and completes the proof of the lemma. □

We will now use the above lemma to prove

LEMMA 2.3. *Let g be defined as in* (2.3), *and fix r, $0 < r < 1$. Then, for $z = re^{i\theta}$, $0 \leq \theta \leq \pi$,*

$$\frac{1}{g(r)} - \frac{\sin x}{x} \leq \left|\frac{1}{g(z)} - \frac{\sin x}{x}\right| \leq \frac{\sin x}{x} - \frac{1}{g(-r)}$$

and

$$-\frac{1}{g(-r)} - \frac{\sin x}{\pi - x} \leq \left|\frac{1}{g(z)} + \frac{\sin x}{\pi - x}\right| \leq \frac{1}{g(r)} + \frac{\sin x}{\pi - x}.$$

PROOF. As before, we will only prove the first assertion since the other assertion follows similarly.

For $z = re^{i\theta}$, $0 < r < 1$,

$$\frac{\partial}{\partial\theta}|h|^2 = 2\mathrm{Re}\{iz\bar{h}h'\}, \text{ where } h(z) = \frac{1}{g(z)} - \frac{\sin x}{x}.$$

Differentiation yields

$$\begin{aligned}\frac{\partial}{\partial\theta}|h|^2 &= |g|^{-2}\mathrm{Im}\{zg'(z)/g(z)\} - \mathrm{Im}\{zg'(z)/g^2(z)\}x^{-1}\sin x \\ &= |g|^{-2}l(\theta)\end{aligned}$$

where

$$\begin{aligned}l(\theta) = {} & [1 - (x^{-1}\sin x)\mathrm{Re}\ g(z)]\mathrm{Im}\{zg'(z)/g(z)\} \\ & + x^{-1}\sin x(\mathrm{Im}\ g(z))\mathrm{Re}\{zg'(z)/g(z)\}.\end{aligned}$$

Since $\operatorname{Re}\{zg'/g\}$ and $1-(x^{-1}\sin x)\operatorname{Re} g$ are positive, $l(\theta)>0$ is equivalent to

$$\frac{\operatorname{Im}\{zg'(z)/g(z)\}}{\operatorname{Re}\{zg'(z)/g(z)\}} > \frac{-(x^{-1}\sin x)\operatorname{Im} g(z)}{1-(x^{-1}\sin x)\operatorname{Re} g(z)} = \frac{\operatorname{Im}\{1-g(z)x^{-1}\sin x\}}{\operatorname{Re}\{1-g(z)x^{-1}\sin x\}}.$$

This inequality is equivalent to

$$\arg(zg'(z)/g(z)) > \arg(1-g(z)x^{-1}\sin x).$$

Applying Lemma 2.2, we conclude that $l(\theta)>0$ in $(0,\pi)$. □

We now proceed with the proof of the theorem.

PROOF OF THEOREM 2.1. Let

$$\delta(r)=\inf\{|f(z)| : f\in \hat{K},\ |z|=r\},$$

and

$$\Delta(r)=\sup\{|f(z)| : f\in \hat{K},\ |z|=r\}.$$

As observed earlier, it suffices to consider f where either $f(z)=z(1-z)^{-1}$ or else f is given by (2.2).

If $f(z)=z(1-z)^{-1}$, then f attains its maximum at $z=r$ and minimum at $z=-r$. Thus

$$r(1+r)^{-1}\le |f(z)|\le r(1-r)^{-1}.$$

Next let f be given by (2.2). We write the associated function g in (2.3) as $g(z)=g(z,x)$ to emphasize its dependence on x. Let

$$M(r)=\sup\{|f(z)| : |z|=r\},$$

and

$$m(r)=\inf\{|f(z)| : |z|=r\}.$$

Then, by considering the reciprocal of f, we deduce that

$$[M(r)]^{-1}\ge \inf_{0<x<\pi}\min_{\theta}\min_{\psi}\left\{\left|\frac{1}{g(re^{i\theta},x)}-\frac{1}{g(e^{i\psi},x)}\right|\right\} \tag{2.8}$$

and

$$(2.9)\qquad [m(r)]^{-1} \leq \sup_{0<x<\pi} \max_{\theta} \max_{\psi} \left\{ \left| \frac{1}{g(re^{i\theta},x)} - \frac{1}{g(e^{i\psi},x)} \right| \right\}.$$

In what follows, we shall show that equality is obtained in both (2.8) and (2.9) and that

$$\delta(r) = \min\{r(1+r)^{-1}, m(r)\} = m(r),$$

$$\Delta(r) = \max\{r(1-r)^{-1}, M(r)\} = M(r).$$

The boundary of the range of $1/g$ consists of two circles C_1, C_2 with centers at $x^{-1}\sin x, -(\pi - x)^{-1}\sin x$, and of radii $x^{-1}\sin x$, $(\pi - x)^{-1}\sin x$, respectively. Moreover, these two circles are symmetric with respect to the real axis.

Notice that the range of $1/g$ is also symmetric with respect to the real axis. Since $1/g(z,x)$ is real if and only if z is real, it suffices to consider $z = re^{i\theta}$ for θ in $[0,\pi]$. For each fixed $\theta, 1/g(re^{i\theta},x)$ lies outside the circles C_1 and C_2. Thus the minimum distance between $1/g(re^{i\theta},x)$ and C_1 is given by $d_1(\theta,x)$, where

$$d_1(\theta,x) = |[g(re^{i\theta},x)]^{-1} - x^{-1}\sin x| - x^{-1}\sin x$$

while the minimum distance between $1/g(re^{i\theta},x)$ and C_2 is $d_2(\theta,x)$, where

$$d_2(\theta,x) = |[g(re^{i\theta},x)]^{-1} + (\pi - x)^{-1}\sin x| - (\pi - x)^{-1}\sin x.$$

So, for a fixed θ and x,

$$\min\{|[1/g(re^{i\theta},x)] - [1/g(e^{i\psi},x)]| : 0 \leq \psi \leq 2\pi\} = \min\{d_1(\theta,x), d_2(\theta,x)\}.$$

Similarly, for a fixed θ and x,

$$\max\{|[1/g(re^{i\theta},x)] - [1/g(e^{i\psi},x)]| : 0 \leq \psi \leq 2\pi\} = \max\{D_1(\theta,x), D_2(\theta,x)\},$$

where

$$D_1(\theta,x) = |[g(re^{i\theta},x)]^{-1} - x^{-1}\sin x| + x^{-1}\sin x$$

and

$$D_2(\theta, x) = |[g(re^{i\theta}, x)]^{-1} + (\pi - x)^{-1} \sin x| + (\pi - x)^{-1} \sin x.$$

Applying Lemma 2.3, it follows that, for a fixed x,

$$\min\{d_1(\theta, x) : 0 \le \theta \le \pi\} = d_1(0, x) = [1/g(r, x)] - [1/g(1, x)]$$

and

$$\min\{d_2(\theta, x) : 0 \le \theta \le \pi\} = d_2(\pi, x) = [1/g(-1, x)] - [1/g(-r, x)].$$

Also, for a fixed x,

$$\max\{D_1(\theta, x) : 0 \le \theta \le \pi\} = D_1(\pi, x) = [1/g(1, x)] - [1/g(-r, x)],$$

and

$$\max\{D_2(\theta, x) : 0 \le \theta \le \pi\} = D_2(0, x) = [1/g(r, x)] - [1/g(-1, x)].$$

Now $g(r, x) = \arg(1 + re^{ix})/\sin x$, $g(1, x) = x/2\sin x$, $g(-r, x) = -\arg(1 - re^{-ix})/\sin x$, and $g(-1, x) = -(\pi - x)/2\sin x$. Since $g(r, \pi - x) = -g(-r, x)$, we see that

$$d_1(0, \pi - x) = d_2(\pi, x),$$

and

$$D_2(0, \pi - x) = D_1(\pi, x).$$

Thus it suffices to minimize $d_1(0, x)$ and to maximize $D_1(\pi, x)$. Specifically, $d_1(0, x) = h(x, r)$ and $D_1(\pi, x) = H(x, r)$, where

$$h(x, r) = [x - 2\arg(1 + re^{ix})](\sin x)/[x\arg(1 + re^{ix})]$$

and

$$H(x, r) = [x + 2\arg(1 - re^{-ix})](\sin x)/[x\arg(1 - re^{-ix})].$$

From (2.8) and (2.9),

$$[M(r)]^{-1} \ge \inf\{h(x, r) : 0 < x < \pi\} \tag{2.10}$$

and

$$[m(r)]^{-1} \leq \sup\{H(x,r) : 0 < x < \pi\}. \tag{2.11}$$

A straightforward calculus argument shows that

$$h(\pi/2, r) = \frac{1}{\arctan r} - \frac{4}{\pi}$$

is less than $h(0,r) = (1-r)r^{-1} = h(\pi, r)$; hence h attains its minimum value in $(0, \pi)$.

Suppose $h(x_2) = \alpha$ is the minimum value. The proof thus far shows that the reciprocal of the function $f(z) = g(1,x_2)g(z,x_2)/[g(1,x_2) - g(z,x_2)]$ assumes the value α at $z = r$. Combining this with (2.10), we conclude that

$$[M(r)]^{-1} = \min\{h(x,r) : 0 < x < \pi\}.$$

Similarly, $H(0,r) = (1+r)r^{-1} = H(\pi, r)$, and since

$$H(\pi/2, r) = \frac{4}{\pi} + \frac{1}{\arctan r} > \frac{1+r}{r},$$

H assumes its maximum in $(0, \pi)$. Proceeding analogously as before, we conclude that

$$[m(r)]^{-1} = \max\{H(x,r) : 0 < x < \pi\}.$$

Finally, it is clear that we obtain sharpness of our result for those f as given in the statement of the theorem. □

Let us take a closer examination of the function h as given in Theorem 2.1. It is difficult to determine explicitly the point(s) in $(0, \pi)$ at which h assumes its minimum value. So we would want to ascertain the number of zeros of $\partial h/\partial x$. Numerical evidence seems to suggest that $\partial h/\partial x$ and $\partial H/\partial x$, where H is also given in Theorem 2.1, vanish exactly once in $(0, \pi)$. Under this assumption, we give below the approximate extremal values of h and H. Note that x_1 and x_2 denote the approximate zero to $\partial H/\partial x$ and $\partial h/\partial x$, respectively.

The Extremal Values of H and h

r	x_1	x_2	$H(x_1,r)$	$h(x_2,r)$
0.1	2.024425	2.140862	11.351998	8.698759
0.2	1.969090	2.202705	6.376196	3.725457
0.4	1.862916	2.336489	3.922480	1.281990
0.6	1.761633	2.490874	3.132843	0.510468
0.8	1.664398	2.686461	2.757472	0.163417
0.9	1.617154	2.821787	2.638368	0.064163

COROLLARY 2.4. *For each* $f \in \hat{K}$,

$$\frac{r}{(1-r^2)M(r)} \leq \left|\frac{zf'(z)}{f(z)}\right| \leq \frac{r}{(1-r^2)m(r)}, \quad |z| = r < 1,$$

where $M(r)$ *and* $m(r)$ *are defined in Theorem* 2.1. *This result is sharp.*

PROOF. Let $f \in \hat{K}$ and ζ in D be fixed. Let F be the Marty transformation of f, that is, $F(z) = [f((z+\zeta)/(1+\overline{\zeta}z)) - f(\zeta)]/f'(\zeta)$

$(1-|\zeta|^2)$. Then $F \in \hat{K}$ and so, from Theorem 2.1,

$$m(r) \leq |F(-\zeta)| \leq M(r), \quad |\zeta| = r.$$

Thus

$$m(r)(1-r^2)r^{-1} \leq |f(\zeta)/[\zeta f'(\zeta)]| \leq M(r)(1-r^2)r^{-1}.$$

The Marty transformation of the extremal functions in Theorem 2.1 yields the sharpness of our result. □

The theorem below has appeared in [**2**]. As a final application of Theorem 2.1, we provide an alternative proof of the result.

THEOREM 2.5. *The range of every function $f \in \hat{K}$ contains the disk $\{w : |w| < \pi/8\}$. The radius $\pi/8$ is best possible.*

PROOF. From Theorem 2.1, if $f \in \hat{K}$, then $\lim_{r\to 1} \inf |f(z)| \geq \lim_{r\to 1} m(r), \quad |z| = r$. But

$$\begin{aligned} 1/m(1) &= \max_{0<x<\pi} H(x,1) \\ &= \max_{0<x<\pi} 2\pi q(x) \end{aligned}$$

where $q(x) = (\sin x)/[x(\pi - x)]$.

Now q' vanishes only at $x = \pi/2$ in $(0,\pi)$. Since $q(\pi/2) = 4/\pi^2$ is greater than $q(0) = 1/\pi = q(\pi)$, we deduce that q attains its absolute maximum at $x = \pi/2$. Thus $m(1) = \pi/8$.

On the other hand, if g is the symmetric vertical strip mapping in (2.2), that is,

$$g(z) = (1/2i)\log[(1+iz)/(1-iz)],$$

then $g(1) = \pi/4$ and $g(-1) = -\pi/4$. So $f(z) = g(1)g(z)/[g(1) - g(z)]$ omits the point $-\pi/8$ at $z = -1$. Thus $\lim_{r\to 1} \inf |f(z)| = \pi/8$, $|z| = r$.
□

REFERENCES

1. R.W. Barnard and G. Schober, *Möbius transformations of convex mappings*, Complex Variables Theory Appl. **3** (1984), 55-69.

2. ——— and ———, *Möbius transformations of convex mappings* II, to appear in Complex Variables Theory Appl. **7** (1986), 205-214.

3. J.B. Conway, *Functions of one complex variable*, 2nd Ed., Springer-Verlag, New York, 1978.

4. P.L. Duren, *Univalent Functions*, Springer-Verlag, New York, 1983.

5. R.R. Hall, *On a conjecture of Clunie and Sheil-Small*, Bull. London Math. Soc. **12** (1980), 25-28.

6. W.E. Kirwan, *A note on extremal problems for certain classes of analytic functions*, Proc. Amer. Math. Soc. **17** (1966), 1028-1030.

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