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CONTINUED FRACTION TAILS AND IRRATIONALITY

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ABSTRACT. A theorem on irrational numbers using tails is proved. Two examples are given. A comment is made on a classical result in the light of the method used.

1. Introduction. Tails of continued fractions have recently been in focus in continued fraction theory, see [**1**].

Consider the continued fraction

$$\mathbf{K}_{k=1}^{\infty}\frac{a_k}{b_k} \tag{1}$$

where $a_k \neq 0$ and b_k are complex numbers for $k \geq 1$. Then we call the continued fractions

$$T_k = \mathbf{K}_{i=k}^{\infty}\frac{a_i}{b_i} \tag{2}$$

for $K \geq 1$ the tails of (1). In the present treatment we consider only the case where the elements a_k and b_k in (1) are integers. A closer examination of the proof of a well known classical theorem on irrationality shows that the concept of tail can be used with advantage and provides a more transparent access to this irrationality result, see [**2**, p. 56].

A comment at the end of the present treatment will make this explicit and show that this irrationality result is almost trivial in the light of tails. We will illustrate the tail approach by giving a theorem on irrationality below. This theorem contains two possible outcomes precisely like the classical result in [**2**]. This means that (1) is irrational in certain situations and rational in the opposite situations. The theorem gives complete information.

As an illustration, one example of each situation is given. We now formulate the core of the tail method. Choose a number $x_0 \neq 0$. Define

$$x_k = T_k x_{k-1} \tag{3}$$

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for $k \geq 1$ supposing that $T_k \neq 0$ are well defined. Suppose that

$$T_k = \frac{a_k}{b_k + T_{k+1}} \tag{4}$$

is justified for $k \geq 1$. Combining (3) and (4) we obtain the three-term recurrence relation

$$a_{k-1}x_{k-2} = b_{k-1}x_{k-1} + x_k \tag{5}$$

for $k \geq 2$. The idea in the proof is to combine (3) and (5). The equality in (3) gives us the magnitude and sign of $x_k \neq 0$, and the equality in (5) gives us that x_k is an integer if x_0 and x_1 are integers. The aim in the irrationality proof then is to prove the impossibility of such a sequence $\{x_k\}_{k\geq 0}$ in certain situations. In the opposite situations we obtain rationality for trivial reasons. We turn to the actual proof of the theorem on irrationality which shall illustrate explicitly the use of tails.

Tails and irrationality. We intend to prove the following theorem.

THEOREM. *Let the series*

$$\sum_{k=1}^{\infty} \frac{1}{\sqrt{P_{2k-1}}} \tag{6}$$

diverge where P_{2k-1} is a positive integer for $k \geq 1$. Consider the continued fraction

$$\mathbf{K}_{k=1}^{\infty} \frac{a_k}{b_k}, \tag{7}$$

where the elements are positive integers obeying the following two conditions

$$\begin{aligned} a_{2k-1} &= b_{2k-1}P_{2k-1} + 1 \\ b_{2k} &\geq a_{2k}P_{2k-1} - P_{2k+1} \end{aligned} \tag{8}$$

for $k \geq 1$. Then the continued fraction in (7) converges. The value of (7) is irrational if we have strict inequality in the second condition of

(8) *for infinitely many values of* k. *In the opposite case the value of* (7) *is rational.*

PROOF. First, we notice that the tails in (2) are convergent. This follows by appealing to [**2**; Satz 2.11, p. 47]. Here we need only point out that

$$\frac{b_{2k}b_{2k+1}}{a_{2k+1}} \geq \frac{b_{2k+1}}{a_{2k+1}} \geq \frac{1}{2p_{2k+1}} \tag{9}$$

for $k \geq 1$. The inequality (9) provides divergence of the series

$$\sum_{k=1}^{\infty} \sqrt{\frac{a_{2k}b_{2k+1}}{a_{2k+1}}}. \tag{10}$$

Divergence of (10) is sufficient, in the light of the result in [**2**], to insure that the tails converge. The equality, (4), involving tails, is clearly fulfilled with $T_k > 0$ for $k \geq 1$. Second, we estimate the tails T_k for $k \geq 1$. Define

$$T_k(n) = \mathbf{K}_{i=k}^{n} \frac{a_i}{b_i}$$

for $n \geq k \geq 1$. We find, using the backward recurrence algorithm, that

$$T_{2k-1}(2n-1) \geq P_{2k-1} \tag{11}$$

for $n \geq k \geq 1$. For the backward recurrence algorithm see [**3**, p. 26]. We will justify (11) properly. Observe that $0 \leq x \leq 1/p_{2i-1}$ implies

$$p_{2i-1} \leq \frac{a_{2i-1}}{b_{2i-1} + x} \tag{12}$$

for $i \geq 1$. Also, $y \geq p_{2i+1}$ implies

$$0 < \frac{a_{2i}}{b_{2i} + y} \leq \frac{1}{p_{2i-1}} \tag{13}$$

for $i \geq 1$. Repeated application of (12) and (13) gives us (11), remembering that $a_{2i-1}/b_{2i-1} \geq p_{2i-1}$ for $i \geq 1$. Letting $n \to \infty$ in (11) we arrive at

$$T_{2k-1} \geq p_{2k-1} \tag{14}$$

for $k \geq 1$.

We discriminate between two cases in (14). The first case is when there is equality for some $k \geq 1$. The second case is when there is strict inequality for all $k \geq 1$. In the first case we have that (7) is rational for trivial reasons. In the second case we suppose that (7) is a rational number x_1/x_0 where x_0 and x_1 are positive integers. We conclude from (3) that $x_k > 0$ for $k \geq 0$ since $T_k > 0$ for $k \geq 1$. Further we have from (3) that

$$x_{2k-1} > P_{2k-1}x_{2k-2} \tag{15}$$

for $k \geq 1$. Using (5) we see that

$$\begin{aligned} x_{2k} &= a_{2k-1}x_{2k-2} - b_{2k-1}x_{2k-1} \\ &= x_{2k-2} + b_{2k-1}(p_{2k-1}x_{2k-2} - x_{2k-1}) < x_{2k-2} \end{aligned} \tag{16}$$

for $k \geq 1$. In (16) we used (15) and the first condition in (8). Thus we have proved that

$$0 < x_{2k} < x_{2k-2} \tag{17}$$

for $k \geq 1$. The result (17) shows that we have a descending infinite sequence of positive integers which is clearly impossible and (7) is irrational in the second case.

It remains to be shown that the two cases in (14) correspond exactly to the two cases in (8) formulated at the end of the theorem. This is done in the following way. Suppose that $T_{2k-1} = p_{2k-1}$ for some fixed $k \geq 1$. Using (4) we see that this implies $T_{2k} = 1/p_{2k-1}$. Using (4) once more we find that this implies $T_{2k+1} = a_{2k}p_{2k-1} - b_{2k}$. This identity gives us $T_{2k+1} = p_{2k+1}$ and $b_{2k} = a_{2k}p_{2k-1} - p_{2k+1}$ because we have $T_{2k+1} \geq p_{2k+1}$ and $a_{2k}p_{2k-1} - b_{2k} \leq p_{2k+1}$. We can clearly do the same to T_{2k+1} as was already done to T_{2k-1}.

We can continue in this way. This means that $b_{2i} = a_{2i}p_{2i-1} - p_{2i+1}$ for all $i \geq k$. On the other hand, let $b_{2i} = a_{2i}p_{2i-1} - p_{2i+1}$ for $i \geq k$ where $k \geq 1$ is a suitable fixed integer. Using the backward recurrence algorithm we find that

$$T_{2k-1}(2n) \leq p_{2k-1} \tag{18}$$

for $n \geq k \geq 1$. We now justify (18) properly. In the following we consider $i \geq k$. We observe that $x \geq 1/p_{2i-1}$ implies

$$\frac{a_{2i-1}}{b_{2i-1}+x} \leq p_{2i-1} \tag{19}$$

for $i \geq k$. Also, we see that $0 \leq y \leq p_{2i+1}$ implies

$$\frac{a_{2i}}{b_{2i}+y} \geq \frac{1}{p_{2i-1}} \tag{20}$$

for $i \geq k$. Repeated application of (19) and (20) gives us (18). Letting $n \to \infty$ in (18) we arrive at $T_{2k-1} \leq p_{2k-1}$. Since $T_{2k-1} \geq p_{2k-1}$ according to (14), it is clear that $T_{2k-1} = p_{2k-1}$ for this particular k. Finally we have shown that $T_{2k-1} = p_{2k-1}$, for some $k \geq 1$, is equivalent to $b_{2i} = a_{2i}p_{2i-1} - p_{2i+1}$ for i sufficiently large. This means that the two cases in (14) correspond exactly to the two situations described at the end of the theorem and the theorem is proved. □

We will now give two examples illustrating the two different situations in the theorem.

EXAMPLE 1. (IRRATIONALITY). Choosing $p_{2k-1} = p$ for $k \geq 1$ where p is a positive integer, we have that

$$\frac{1}{2}p - \frac{1}{2} + \frac{1}{2}\sqrt{p^2+2p+5} = \frac{p+1}{1}\frac{1}{1+}\frac{p+1}{1\;\;+}\cdots \tag{21}$$

is irrational according to the theorem. This is true by direct control. The formula (21) is easily checked since we, on the right side, have a 2-periodic continued fraction.

EXAMPLE 2. (RATIONALITY). Choosing $p_{2k-1} = p$ for $k \geq 1$ where p is a positive integer we have that

$$p = \frac{p+1}{1\;\;+}\frac{2}{p+}\frac{p+1}{1\;\;+}\cdots \tag{22}$$

is rational according to the theorem. The formula (22) is easily checked since the right side is a 2-periodic continued fraction.

We will finally give a comment on the classical irrationality result in [**2**] previously mentioned.

COMMENT. The result we have in mind is found in [**2**; Satz 2.18, p. 56]. Except for irrelevant differences this theorem states: Suppose that the continued fraction

$$\mathbf{K}_{k=1}^{\infty} \frac{a_k}{b_k} \tag{23}$$

obeys $b_k \geq |a_k|$ and even $b_k \geq |a_k| + 1$ if $a_{k+1} < 0$ where $a_k \neq 0$ and b_k are integers for $k \geq 1$. Then (23) converges. Also (23) is irrational unless $a_k < 0$ and $b_k = |a_k| + 1$ for sufficiently large k. In this exceptional case (23) is rational. The proof of this result in the light of tails will not be given in detail here. But the main steps will be pointed out in the tail terminology. First, we notice that convergence follows from [**2**; Satz 2.14, p. 50]. Using the backward recurrence algorithm analogously we find that

$$0 < |T_k| \leq 1 \tag{24}$$

for $k \geq 1$. If equality in (24) is the case for some $k \geq 1$, clearly (23) is rational. If inequality in (24) is the case for $k \geq 1$, we have, supposing rationality of (23), that

$$0 < |x_k| = |T_k|\ |x_{k-1}| < |x_{k-1}| \tag{25}$$

for $k \geq 1$ where (25) gives us a descending infinite sequence of positive integers which is impossible. Using the backwards recurrence algorithm again analogously we show that the two cases in (24) correspond exactly to the two cases formulated in the classical result in the sense that $|T_k| = 1$ for some $k \geq 1$ is equivalent to $b_i = |a_i| + 1$ and $a_i < 0$ for i sufficiently large. Thus the classical result follows.

Obviously the approach using tails makes this classical result look almost trivial since the basic inequality (24) is evident a priori.

REFERENCES

1. H. Waadeland, *Tales about tails*, Proc. Amer. Math. Soc. **90** (1984), 57-64.

2. O. Perron, *Die Lehre von den Kettenbrüchen*, Vol. II, 3rd edition, Teubner, Stuttgart, 1957.

3. W.B. Jones and W.J. Thron, *Continued fractions, analytic theory and applications*, in *Encyclopedia of Mathematics and Its Applications*, Addison-Wesley, Reading, Massachusetts, 1980.

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