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GEOMETRIC MODULI FOR KLEIN SURFACES

MIKA SEPPÄLÄ AND TUOMAS SORVALI

1. Introduction and statement of results. The analytic counterpart of a real algebraic curve of genus g is a Klein surface of genus g. That is a compact topological surface Σ^*, which is either non-orientable or has boundary components (or both), together with an analytic structure. This assertion is due to Norman Alling and Newcomb Greenleaf (cf [**1**]). The surface Σ^* is a topological model for the corresponding real curve. Observe that surfaces arising as topological models of real algebraic curves are never classical compact and oriented surfaces without boundary. There are also some other restrictions for the topological type of real curves of genus g. These constraints are not very complicated and we can easily compute the number of different topological types of real algebraic curves of genus g - that number is $(3g+4)/2$ (cf., e.g., [**3**, §2]).

The moduli problem is to give, for an algebraic curve, parameters or moduli which determine its isomorphism class. In the complex case the isomorphism classes of complex algebraic curves of a given genus g form an algebraic variety. The case of real algebraic curves is more complicated. The genus does not yet classify them even topologically. Hence, instead of considering all real algebraic curves of a given genus, we should consider all real algebraic curves of a given topological type Σ^*.

To exclude certain special cases we assume now that all real algebraic curves or Klein surfaces are of genus $g > 1$. That excludes the following Klein surfaces: the disk, the real projective plane, the annulus, the Möbius band and the Klein bottle.

It is now a formal simplification to consider the interior Σ of the surface (Σ^* instead of Σ^* itself. By a real algebraic curve of the topological type Σ we then mean a real algebraic curve of the type Σ^* in the sense of Alling and Greenleaf. These real algebraic curves are

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analytic structures X of Σ such that the (non-compact) Klein surface (Σ, X) is analytically a compact Klein surface from which a certain number of closed disks have been removed. Note, in particular, that the surface Σ also carries analytic structures for which some or all boundary components degenerate to points. Such analytic structures do not come from real algebraic curves of the topological type Σ and we do not consider them here. By the uniformization theorem there is a one-to-one correspondence between the analytic structures of Σ and the complete hyperbolic metrics d of Σ, i.e., complete metrics which have a constant negative curvature -1. Those hyperbolic metrics d for which the boundary curves of Σ^* (if any) are infinitely long correspond then to the analytic structures of Σ arising from real curves. In this way real algebraic curves of topological type Σ are just hyperbolic metrics on Σ.

The moduli problem then leads to the following

PROBLEM. *Find parameters which determine the hyperbolic metric d of Σ up to an isometry homotopic to the identity mapping of Σ.*

This is not exactly a translation of the original moduli problem for real algebraic curves. Two such curves are isomorphic if and only if there is an isometry between the corresponding hyperbolic metrics. In the above geometric version of the moduli problem, we require, in addition, that the isometry is homotopic to the identity mapping of Σ. In other words, we actually want to parametrize the Teichmüller space of real algebraic curves of topological type Σ. This change allows us to give intrinsic geometric parameters or moduli for real algebraic curves of topological type Σ. These geometric moduli are lengths of closed geodesic curves. The connection to the moduli problems of real algebraic geometry motivates the study of Teichmüller spaces $T(\Sigma)$ of non-orientable surfaces Σ with boundary.

We need more notation. Let $\mathcal{M}(\Sigma)$ denote the set of those hyperbolic metrics of Σ which arise from real algebraic curves of topological type Σ. We identify two metrics d_1 and d_2 of Σ if there is an isometry $(\Sigma, d_1) \to (\Sigma, d_2)$ homotopic to the identity mapping of Σ. In this way we obtain the *Teichmüller space* $T(\Sigma)$ of the surface Σ as a quotient space of $\mathcal{M}(\Sigma)$.

Now let λ be a closed curve on Σ. Define the *geodesic length function* $l_\lambda : T(\Sigma) \to \mathbf{R}_+^*$ setting, for a hyperbolic metric d,

$$l_\lambda([d]) = \inf\{d - \text{ length of } \tau \mid \tau \text{ homotopic to } \lambda\}.$$

Here $[d]$ denotes the point of $T(\Sigma)$ defined by the metric d and $\mathbf{R}_+^*$ is the set of positive real numbers.

The problem we want to study is

PROBLEM. *Find a minimal set* $\{\lambda_1, \lambda_2, \dots, \lambda_m\}$ *of closed curves on* Σ *such that the associated geodesic length functions* $l_j = l_{\lambda_j}$, $j = 1, 2, \dots, m$, *give an injective mapping*

$$L : T(\Sigma) \to (\mathbf{R}_+^*)^m, \qquad [d] \mapsto (l_1([d]), \dots, l_m([d])). \tag{1.1}$$

In other words, we want to be able to recognize every real algebraic curve and every Klein surface by measuring how long their handles are, how wide their holes are, and how large their cross caps are, etc..

Assuming that the arithmetic genus of the corresponding real curves is g, or—which is the same thing—that the Euler characteristic $\chi(\Sigma)$ of the surface Σ equals $1 - g$, we can compute the dimension of $T(\Sigma)$. This Teichmüller space is a connected real analytic manifold and $\dim_R T(\Sigma) = 3g - 3$ (cf, e.g., [**4**, Theorem 5.6]). This implies that we need at least 3g-3 closed curves λ_j in order to have an injective mapping (1.1).

THEOREM 1. *Let* Σ *be a topological type of real algebraic curves of genus* g. *There are* $3g - 3$ *closed curves* $\lambda_1, \dots, \lambda_{3g-3}$ *on* Σ *such that the associated geodesic length functions determine an injective mapping* L *by the formula* (1.1).

For oriented topological surfaces Σ with boundary this theorem was proved in [**5**, §7] (see also [**2**] where a particular case of the same problem is studied). For nonorientable surfaces Σ *without boundary* this theorem was shown in [**7**, Theorem 6.1]. The only case that remains to be proved is that of non-orientable surfaces Σ which have a non-empty boundary. We will give the proof in the next section.

By Theorem 1 the Teichmüller spaces $T(\Sigma)$ of real algebraic curves can be parametrized geometrically by $m, m = \dim_{\mathbf{R}} T(\Sigma)$, geodesic length functions. This is, however, not possible for compact and oriented surfaces Σ. In other words, for the geometric parametrization of the Teichmüller space $T(\Sigma)$ of the compact and oriented surface Σ we need more than $m = \dim_{\mathbf{R}} T(\Sigma)$ closed curves. In fact, Scott Wolfert has observed that we cannot even choose a set of m closed curves on Σ such that the associated mapping (1.1) would be *locally injective*. On the other hand, in [**6**] and [**7**] we have constructed, for an oriented compact surface Σ, a set of $m + 2$ closed curves on Σ such that the associated mapping (1.1) is injective. The reason which makes the geometric moduli problem more difficult in the case of compact and oriented surfaces without boundary is that the fundamental group of such surfaces is not freely generated. In the case of non-orientable surfaces without boundary, the fundamental group is not freely generated either, but the relation between generators can be expressed in a rather nice form and it can be used in computations.

2. Proof of Theorem 1. As we have remarked already, it remains to prove Theorem 1 only for non-orientable surfaces Σ which are obtained from compact non-orientable surfaces by deleting a certain number of closed disks. Such a surface Σ can always be built as a connected sum of p tori (i.e., handles) and m cross caps from which we delete n open disks. Then the arithmetic genus of the real algebraic curve of topological type Σ is $g = 2p + m + n - 1$. In the case under consideration, $m > 0$ and $n > 0$. We have assumed furthermore that $g = 2p + m + n - 1 > 1$. Actually one may build all non-orientable surfaces by using two cross caps only. Hence we might suppose that, in our case, $m = 1$ or $m = 2$. This does not, however, simplify considerations at all.

So we think of the surface Σ as one having p handles, m cross caps and n holes. Let Q be a point of Σ. The first fundamental group of Σ at the point $Q, \pi_1(\Sigma, Q)$, has the following generators: $\alpha_1, \beta_1, \ldots, \alpha_p, \beta_p, \sigma_1, \ldots, \sigma_m, \gamma_1, \ldots, \gamma_n$. The generators α_j and β_j correspond to the handles, the generators σ_j go through the cross caps and the generators γ_j go around the holes. The relation is simply

$$\prod_{j=1}^{p}[\alpha_j, \beta_j] \prod_{j=1}^{m} \sigma_j^2 \prod_{j=1}^{m} \gamma_j = 1. \tag{2.1}$$

Here $[\alpha_j, \beta_j] = \alpha_j \beta_j^{-1} \alpha_j^{-1} \beta_j$ is the commutator of α_j and β_j.

Let d be a hyperbolic metric on Σ arising from a real algebraic curve of the topological type Σ. The pair (Σ, d) is a Klein surface which we can represent as the quotient $(\Sigma, d) = U/G$ where $U = \{z \in \mathbf{C} | \mathrm{Im} z > 0\}$ is the hyperbolic upper half-plane and G is a reflection group acting in U. Here G is a properly discontinuous group of hyperbolic Möbius transformations and their complex conjugates. The group G acts freely in U. Let Ω be the domain of discontinuity of G. Then $\Omega \cap \mathbf{R}$ consists of intervals which correspond to the boundary curves of (Σ, d) in the projection $U \to (\Sigma, d) = U/G$.

There is an almost canonical isomorphism $\pi_1(\Sigma, Q) \to G$. It is defined by lifting to U the curves representing points of $\pi_1(\Sigma, Q) \to G$. Some choices are involved and the isomorphism $\pi_1(\Sigma, Q) \to G$ is defined up to an inner automorphism of G. Choose one isomorphism. Let λ be a closed curve on Σ representing a point $[\lambda]$ of $\pi_1(\Sigma, Q)$. If $g \in G$ corresponds to $[\lambda]$ under the isomorphism $\pi_1(\Sigma, Q) \to G$, then $l_\lambda([d]) = \log k(g)$, where $k(g)$ is the multiplier of the transformation g. Hence the values of the geodesic length functions can be computed by the multipliers of the elements of G (for details see e.g., [**5**]).

In order to prove Theorem 1 we have to find $3g - 3$ closed curves on Σ such that the corresponding mapping (1.1) is injective. Consider a set $\{[\lambda_1], \ldots, [\lambda_N]\}$ of elements of $\pi_1(\Sigma, Q)$. Let $g_j \in G$ be the transformation corresponding to $[\lambda_j]$ under the isomorphism $\pi_1(\Sigma, Q) \to G$, $j = 1, \ldots, N$. The following result is a well-known fact. For a proof see, e.g., [**7**, §5 and §6].

PROPOSITION 1. *The mapping L defined by the geodesic length functions associated to the closed curves λ_j, $j = 1, \ldots, N$, is injective if and only if the multipliers of the corresponding transformations $g_j \in G$, $j = 1, \ldots, N$, determine the group G up to a conjugation by a Möbius transformation.*

Let a_j, b_j, s_j and g_j correspond to the generators $\alpha_j, \beta_j, \sigma_j$ and γ_j of $\pi_1(\Sigma, Q)$ under the isomorphism $\pi_1(\Sigma, Q) \to G$. Then they also generate the reflection group G and satisfy the relation

$$\prod_{j=1}^{p}[a_j, b_j]\prod_{j=1}^{m} s_j^2 \prod_{j=1}^{n} g_j = \mathrm{Id}. \tag{2.2}$$

Let F be the subgroup of G generated by $a_i, b_i, i = 1, \dots, p, s_j^2, j = 1, \dots, m$, and $g_k, k = 1, \dots, n$. Then F is a Fuchsian group and it is a subgroup of order 2 in G. The group F has $M = 2p + m + n$ generators which satisfy the relation (2.2). But from the relation (2.2) we can solve, for instance, for the transformation g_n. Recall that, in the case under consideration, $n > 0$. Hence the group F is actually freely generated by the $M - 1$ transformations $a_i, b_i, i = 1, \dots, p, s_j^2, j = 1, \dots, m$, and $g_k, k = 1, \dots, n - 1$. The following result is proved in [**5**, §7].

PROPOSITION 2. *There exists a set* $K = \{g_1, \dots, g_{3(M-1)-3}\}$ *of* $3(M - 1) - 3$ *elements of* F *such that the multipliers of the Möbius transformations* $g_j \in K$ *determine* F *up to a conjugation by a Möbius transformation.*

Assume that the multipliers of the elements of K are known. Then the group F is determined up to a conjugation by a Möbius transformation. Normalizing in such a way that the attracting and the repelling fixed-points of g_1 are 0 and ∞ and that the repelling fixed-point of s_1^2 is 1, the group F becomes uniquely determined by the multipliers of the elements of K. The transformations $s_j \in G$ are orientation reversing glide-reflections. Since the group F is now determined, also the hyperbolic Möbius-transformations s_j^2 are determined. On the other hand, glide reflections g_j are already determined if we know what their squares are [**7**, Lemma 6.1]. Consequently, the multipliers of the elements of the set K determine also the group G up to a conjugation by a Möbius-transformation.

For each index $j = 1, \dots, 3(M - 1) - 3$, let λ_j be a closed curve for which $[\lambda_j] \in \pi_1(\Sigma, Q)$ corresponds to $g_j \in K$ under the isomorphism $\pi_1(\Sigma, Q) \to G$. By Proposition 1, the mapping L defined by the geodesic length functions associated to the curves λ_j is injective. This proves Theorem 1, since $3g-3=6p+3m+3n-6=3(M-1)-3$ for $g=2P+m+n-1$ and $M=2p+m+n$.

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DEPARTMENT OF MATHEMATICS, UNIVERSITY OF REGENSBURG, FRG-8400 REGENSBURG, FEDERAL REPUBLIC OF GERMANY

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF JOENSUU, SF-8100 JOENSUU, FINLAND