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CLASSES OF QUATERNION ALGEBRAS IN THE BRAUER GROUP

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Elman and Lam investigated fields L such that the classes of quaternion algebras over L form a subgroup in the Brauer group $\mathrm{Br}(L)$ of the field L [**4**]. They made the following list of examples: L is a finite field, a local field, a global field, a field of transcendence degree ≤ 2 over $\mathbf{C}$, a field of transcendence degree 1 over $\mathbf{R}$, $\mathbf{C}((t_1))((t_2))((t_3))$, where $K((t))$ means the field of formal power series in t over the field K.

Elman and Lam found in their paper [**4**] that if L is a nonformally real field and the classes of quaternion algebras form a subgroup in the Brauer group $\mathrm{Br}(L)$, then

$$u(L) \in \{1, 2, 4, 8\}.$$

Here u means the so-called u-invariant of the field. (See [**4**], or [**6**; Chapter 11, Theorem 4.10]

DEFINITION 1. A field K is called *linked* if and only if the classes of quaternion algebras form a subgroup of the Brauer group $\mathrm{Br}(K)$. (See also Definition 4.3 in [**2**].)

In [**3**] it is proved that a formally real Pythagorean field F is linked if and only if F is SAP.

Our goal is to characterize all linked fields $L = F(\sqrt{-1})$, where F is formally real Pythagorean with finite chain length. This will generalize the sixth example above. We shall use the possibility to attach, to each order space X of finite chain length, some graded ring $R(X)$, which will be described explicitly in Definition 4. The main motivation for the introduction of $R(X)$ is given by Theorem 5 below.

We use standard notation such as can be found in [**5, 6**, and **8**]. For the reader's convenience, we shall recall just a bit of it.

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By (X, D) we mean an order space, where D is a 2-elementary group and X is a closed subgroup of the group of characters $x(D)$ of D. (See [**8, 9**].) The elements of X are called orderings. Sometimes instead of (X, D) we shall write only X.

$F(2)$ is the maximal 2-extension of the field F,

$G_F = \mathrm{Gal}(F(2)|F)$,

$h_i(G) = \dim_{\mathbf{Z}/2\mathbf{Z}} H^i(G, 2)$

$H^*(G, 2)$ is the graded cohomology ring of the pro-2-group G with coefficients in $\mathbf{Z}/2\mathbf{Z}$,

$\mathrm{cd}\, G$ is the cohomological dimension of G.

One of the basic notions in the theory of order spaces is the chain length. It was introduced by Marshall in the paper [**10**, Definition 1.1].

DEFINITION 2. (MARSHALL). The *chain length* of the order space (X, D) is the largest integer $k \geq 1$ such that there exists elements $a_0, a_1, \ldots, a_k \in D$ such that, for each $i \in \{1, \ldots, k\}$, we have

$$\{x \in X | x(a_{i-1}) = 1\} \subsetneq \{x \in X | x(a_i) = 1\}.$$

If no such k exists we define the chain length to be infinity.

Unless otherwise stated we always assume $\mathrm{cl}\,(X) < \infty$.

Craven seems to have been the first to recognize the importance of order spaces of finite chain length in the field case [**1**]. In [**10**] Marshall proved the following theorem.

THEOREM 3. (MARSHALL). *Let X be an order space and suppose that $\mathrm{cl}(X) < \infty$. Then X can be obtained from one element order spaces by using a finite number of direct sums and group extensions.*

We now define $R(X)$ by induction on $\mathrm{cl}(X)$. By $R^i(X)$, $i \in \mathbf{N} \cup \{0\}$, we denote the subgroup of $R(X)$ consisting of elements of degree i.

DEFINITION 4. (A) If $\mathrm{cl}(X) = 1$, then we define $R(X) = R^0(X) = \mathbf{Z}/2\mathbf{Z}$.

(B) Suppose that

$$(X, D) \simeq (X_1, D_1) \oplus \cdots \oplus (X_s, D_s)$$

is the decomposition of (X, D) into connected components. Then

I. $R^0(X) \simeq \mathbf{Z}/2\mathbf{Z}$.

II. $R^1(X) = R^1(X_1) \oplus \cdots \oplus R^1(X_s) \oplus S(X)$, where $S(X)$ is an abelian group of rank $s-1$ over $\mathbf{Z}/2\mathbf{Z}$.

III. $R^i(X) = R^i(X_1) \oplus \cdots \oplus R^i(X_s)$, for $i \geq 2$.

To define multiplication we view each $R(X_i)$, $i = 1, \dots, s$, as naturally imbedded in $R(X)$ and we set

$$\begin{aligned} &a \in R(X_j), \quad b \in R(X_i) \ \text{ with } \ i \neq j \Rightarrow ab = 0 \\ &c \in S(X), \quad d \in R(X) \Rightarrow \mathrm{cd} = 0. \end{aligned}$$

(C) Suppose that

$$(X, D) = (Y, E) \times H,$$

where H is a 2-elementary abelian group and (Y, E) is a decomposable order space. We define $R(X)$ as follows.

Let $h_i, i \in I$, be a basis of the vector H over $\mathbf{Z}/2\mathbf{Z}$. We set

$$\begin{aligned} R^0(X) &= \{0, 1\} \\ R^1(X) &= R^1(Y) \oplus H \\ R^i(X) &= \oplus_{g_j} g_j R^{i-j}(Y), \quad i \geq 2, \end{aligned}$$

where J means a set consisting of j different elements of $\{h_i | i \in I\}$, $0 \leq j \leq i$, g_j means the formal product of elements of J, and if $j = 0$, then $J = \phi$ and $g_\phi = 1$. $g_j R^{i-j}(Y)$ and $R^{i-j}(Y)$ are isomorphic as abelian groups.

Then multiplication is defined by the formulas

$$g_R a \cdot g_T b = g_R g_T ab,$$

where g_R, g_T are products of r and t different elements of $\{h_i | i \in I\}$, respectively, $a \in R^m(X)$, $b \in R^n(X)$ for some $m, n \in N$. If there exists $h_i, i \in I$, such that h_i divides both g_R and g_T then we put $g_R g_T = 0$. The product ab is defined inductively in $R(Y)$.

EXAMPLES. (1) Suppose that

$$X = X_1 \oplus \cdots \oplus X_s, \quad 2 \leq s,$$

where $|X_1| = \cdots = |X_s| = 1$. Then

$$\begin{aligned} &R^0(X) \simeq \mathbf{Z}/2\mathbf{Z}, \\ &R^1(X) \simeq (\mathbf{Z}/2\mathbf{Z})^{(s-1)}, \\ &R^i(X) = \{0\} \text{ for } i \geq 2. \end{aligned}$$

(2) Suppose that

$$X = Y \times H,$$

where $|Y| = 2$ and $|H| = 4$. Let $\{a, b\}$ be a basis of H over the field $\mathbf{Z}/2\mathbf{Z}$. From example (1) we see that $R^0(Y) \simeq \mathbf{Z}/2\mathbf{Z} = \{0, 1, \}$ and $R^1(Y) \simeq \mathbf{Z}/2\mathbf{Z} = \{0, c\}$. Thus, from Definition 5, we see that

$$\begin{aligned} &R^0(X) = \{0, 1\} \\ &R^1(X) = \{a, b, c, a+b, a+c, b+c, a+b+c, 0\} \\ &R^2(X) = \{ac, bc, ab, ac+bc, bc+ab, ac+ab, ac+bc+ab, 0\} \\ &R^3(X) = \{abc, 0\} \\ &R^i(X) = \{0\} \quad \text{for} \quad i \geq 4. \end{aligned}$$

Thus we see that $R(X)$ is isomorphic to $\mathbf{Z}/2\mathbf{Z}[A, B, C]/(A^2, B^2, C^2)$, where $\mathbf{Z}/2\mathbf{Z}[A, B, C]$ means a polynomial ring over the field $\mathbf{Z}/2\mathbf{Z}$ with indeterminates A, B, C and (A^2, B^2, C^2) is the ideal generated by A^2, B^2, C^2.

THEOREM 5. (See [**5**, **14**, **16**].) *Let F be a Pythagorean field with order space X_F of finite chain length. Then*

$$H^*(G_{F(\sqrt{-1})}, 2) \simeq R(X), \tag{1}$$

where, by isomorphism, we mean isomorphism of graded rings.

REMARK. Craven proved that, for each order space X of finite chain length, there exists a Pythagorean field F such that $X_F \simeq X$ [**1**].

On the other hand it was observed in **[12]** that if X is an order space of finite chain length and F_1 and F_2 are formally real Pythagorean fields with $X_{F_1} \simeq X \simeq X_{F_2}$, then $G_{F_1} \simeq G_{F_2}$ and $G_{F_1(\sqrt{-1})} \simeq G_{F_2(\sqrt{-1})}$. Thus also $H^*(G_{F_1(\sqrt{-1})}, 2) \simeq H^*(G_{F_2(\sqrt{-1})}, 2)$. Hence we see that $R(X)$ is well defined by formula (1) in Theorem 5. Our Definition 4 tells us how to compute $R(X)$ directly from X.

Finally we can write our theorem.

THEOREM 6. *Let $L = F(\sqrt{-1})$, where F is a formally real Pythagorean field with $\mathrm{cl}\,(F) < \infty$.*

Then L is linked if and only if X_F can be written as a finite sum of order spaces Y of the following type:

(1) $\mathrm{st}\,(Y) \leq 2$

(2) $\mathrm{st}\,(Y) = 3$ *and* $|Y| = 8$.

PROOF. It is well known that $\mathrm{Br}\,(L)_2$ can be identified with $H^2(G_L, 2)$, by sending $[(\frac{a,b}{L})] \in \mathrm{Br}_2(Z)$ to $(a) \cup (b) \in H^2(G_L, 2)$. Here (a), (b) are elements of $H^1(G_L, 2)$ which correspond to $a, b \in \dot{L}$ respectively. $\cup$ means cup product.

From Theorem 5 we see that $H^2(G_L, 2)$ is additively generated by cup products of elements of the ring $H^1(G_L, 2)$. (From Merkurjev's Theorem we know that this is true for every field M with char $M \neq 2$.) Thus L is linked if and only if each element of $H^2(G_L, 2)$ is a cup product of two elements of $H^1(G_L, 2)$. Since $H^*(G_L, 2) \simeq R(X_F)$, it is enough to investigate when each element of $R^2(X_F)$ is the product of two elements of $R^1(X_F)$.

(1) Suppose first that $\mathrm{st}(X_F) \leq 1$. If $\mathrm{st}(X_F) = 0$, then $|X_F| = 1$. Then, from Definition 5, we see that

$$R(X_F) \simeq \mathbf{Z}/2\mathbf{Z} = R^0(X_F).$$

In particular $R^2(X_F) = \{0\}$.

If $\mathrm{st}\,(X_F) = 1$, then from Example 1, we see that $R^2(X_F) = \{0\}$. Therefore $L = F(\sqrt{-1})$ is linked.

(2) Suppose that $\mathrm{st}\,(X_F) = 2$ and

$$X_F = X_1 \cup \cdots \cup X_s,$$

is the decomposition of X_F into connected components. We may assume that there exists $c \in N, c \leq s$, such that

$$\begin{aligned} \mathrm{st}(X_i) &= 2 \quad \text{for } 1 \leq i \leq c \\ \mathrm{st}(X_i) &= 0 \quad \text{for } c+1 \leq i \leq s (\text{ if } c < s). \end{aligned}$$

(Recall that if $\mathrm{st}(Y) = 1$, then Y is a sum of one element order spaces $Y_j, 1 \leq j \leq m$. Thus $\mathrm{st}(Y_j) = 0$ for each $j \in \{1, \ldots, m\}$.)

By definition, part (1), we see that

$$R^2(X_F) = \oplus_{i=1}^{c} R^2(X_i).$$

Let $1 \leq i \leq c$. Then X_i is an indecomposable order space. Thus

$$X_i \simeq Z_i \times H_i,$$

where $\mathrm{st}\,(Z_i) = 1$ and $|H_i| = 2$. From Definition 4, we see that

$$\begin{aligned} R^1(X_i) &= R^1(Z_i) \oplus H_i \\ R^2(X_i) &= A_i R^1(Z_i), \end{aligned}$$

where A_i is the non-zero element of H_i. From Defintion 4 we see that $A_i^2 = 0$. Thus we can write

$$R^2(X_i) = \{A_i B_i | B_i \in R^1(X_i)\}.$$

Moreover, from the definition of multiplication in Definition 4, we see that, for any $B_i \in R^1(X_i), i = 1, \ldots, c$, we have

(2) $$A_1 B_1 + \cdots + A_c B_c = (A_1 + \cdots + A_c)(B_1 + \cdots + B_c).$$

Thus we see that every element of $R^2(X_F)$ is a product of two elements of $R^1(X_F)$.

(3) Suppose that $\mathrm{st}(X_F) = 3$ and $|X_F| = 8$. Then there exist elements $A, B, C \in R^1(X_F)$ such that

$$R(X_F) = \mathbf{Z}/2\mathbf{Z}\,[A, B, C]/(A^2, B^2, C^2),$$

where $\mathbf{Z}/2\mathbf{Z}[A,B,C]$ means the polynomial ring in A,B,C over $\mathbf{Z}/2\mathbf{Z}$ and (A^2,B^2,C^2) means the ideal generated by A^2, B^2 and C^2 (see Example 2). Hence

$$R^2(X_F) \\ = \{AB, AC, BC, A(B+C), (A+B)C, B(A+C), (A+B)(B+C), 0\}.$$

Thus again we see that every element of $R^2(X_F)$ is a product of two elements of $R^1(X_F)$.

(4) Suppose that X_F is a finite sum $\oplus_{i=1}^d Y_i$ of order spaces $Y_i, i = 1,\dots,d$ of type (1) or (2) described in our theorem. Then

$$R^2(X_F) = \oplus_{i=1}^d R^2(Y_i).$$

On the other hand we see from (1), (2), and (3) that, for each $i \in \{1,\dots,d\}$, every element C_i of $R^2(Y_i)$ can be written in the form A_iB_i where $A_i, B_i \in R^1(Y_i)$. Thus any element of $R^2(X_F)$ can be written as

$$A_1B_1 + \cdots + A_nB_n = (A_1 + \cdots + A_n)(B_1 + \cdots + B_n),$$

where $A_i, B_i \in R^1(X_i)$ for each $i \in \{1,\dots,d\}$. Thus we see that if X_F can be written as a finite sum of order spaces Y of the type (1) $\mathrm{st}(Y) \le 2$ or (2) $\mathrm{st}(Y) = 3$ and $|Y| = 8$, then the field $L = F(\sqrt{-1})$ is linked.

Suppose now that $L = F(\sqrt{-1})$ is linked. Then, from Elman and Lam [**4**], we see that

$$u(L) \le 8.$$

Hence, from a theorem in [**13**], which asserts that $u(L) = s^{\mathrm{st}(F)}$ if $\mathrm{st}(F) < \infty$ and $u(L) = \infty$ if and only if $\mathrm{st}(F) = \infty$, we see that

$$\mathrm{st}(F) \le 3.$$

Since we already know that if $\mathrm{st}(X_F) \le 2$, then $L = F(\sqrt{-1})$ is always linked, we will assume that $\mathrm{st}(F) = 3$.

First we show that X_F cannot be indecomposable unless $|X_F| = 8$. Suppose that, contrary to our assumption, X_F is indecomposable and $|X_F| > 8$. Then we can write

$$X_F = Y \times H,$$

where $H \neq \{1\}$. Since $\mathrm{st}(X_F) = \mathrm{st}(Y) + \log_2 |H|$ and $\mathrm{st}(X_F) = 3$ we see that $\log_2 |H| \in \{1,2,3\}$. If $\log_2 |H| = 3$, then $\mathrm{st}(Y) = 0$. Hence $|X| = 8$. Thus we may assume that $|H| = 2$ or 4.

Case 1. $|H| = 2$. We may assume that $R(Y) \subset R(X_F)$ and that

$$Y = Y_1 \cup \cdots \cup Y_s, \quad 2 \leq s,$$

is the decomposition of Y into its connected components. Since $\mathrm{st}(Y) = 2$, there exists $i \in \{1, \ldots, s\}$ such that $\mathrm{st}(Y_i) = 2$.

We may assume that $i = 1$. Since $\mathrm{st}(Y_1) = 2$ and Y_1 is indecomposable we can write $Y_1 = Z_1 \times H_1$, where H_1 is a 2-elementary group of rank 1 or rank 2. We may and will assume moreover that the rank of H_1 is 1, since if the rank H_1 is 2 then $\mathrm{st}(Z_1) = 0$, and we can write

$$Y_1 = Z_1' \times H_1',$$

where $|Z_1'| = 2$ and $|H_1'| = 2$.

From Definition 5 we see that

$$\begin{aligned} R^1(X_F) &= R^1(Y) \oplus H \\ R^1(Y) &= M \oplus \left(\oplus_{i=1}^{s} R^1(Y_i)\right) \\ R^2(X_F) &= dR^1(Y) \oplus R^2(Y) \\ R^2(Y) &= \oplus_{i=1}^{s} R^2(Y_i) \\ R^2(Y_1) &= a \cdot R^1(Z_1), \end{aligned} \tag{3}$$

where M is a 2-elementary abelian group of rank $s-1$, a is the non-zero element of H_1 and d is the non-zero element of H.

Let, furthermore, $b \in R^1(Z_1) - \{0\}$ and $c \in R^1(Y_2)$. Then, from (3) and Definition 4, we see that

$$\begin{aligned} \{0, d\} &= \{z \in R^1(X_F) | dz = 0\} \\ 0 &\neq ab \\ bc &= 0 = ac \end{aligned}$$

and the elements a, b, c, d are linearly independent over $\mathbf{Z}/2\mathbf{Z}$.

We claim that the element $ab + cd$ cannot be expressed as a product of two elements of $R^1(X_F)$. Indeed, suppose that we are wrong and that there exist elements $v, w \in R^1(X_F)$ such that

$$ab + cd = vw. \tag{4}$$

To show that (4) is impossible we shall construct a basis U of the vector space $R^1(X_F)$ such that $\{a, b, c, d\} \subset U$ and the set $\{U_1U_2 | U_1, U_2 \in U\} - \{0\}$ is a basis of the vector space $R^2(X_F)$ over $\mathbf{Z}/2\mathbf{Z}$. Indeed, from (3), we see that we can find U inductively as follows:

$$U = \{d\} \cup T,$$

where T is a basis of the vector space $R^1(Y)$,

$$\begin{aligned} T &= T_M \cup \textstyle\bigcup_{i=1}^{s} T_i, \\ T_i &= T \cap R^1(Y_i), \\ T_M &= T \cap M. \end{aligned}$$

We shall assume that each element of $R^1(X_F)$ is written in the basis U. Then we say that an element $u \in U$ enters the expression of $Z \in R^1(X_F)$ if and only if

$$Z = u + \sum_{i=1}^{m} u_i,$$

where $u_i \in U$ and $u_i \neq u$ for each $i \in \{1, \dots, m\}$. Otherwise we say that an element u does not enter Z.

From relation (4) we see that d enters the expression of either v or w. Suppose for example that

$$v = d + A, \quad A \in R^1(X_F),$$

and d does not enter the expression of A. Then we have

$$w = c + B, \quad B \in R^1(X_F),$$

and c does not enter the expression of B. From the equality $vw = ab + cd$ we get

$$ab = dB + cA + AB.$$

Thus

$$dB = AB + ab + cA.$$

If d does not enter the expression of B we see that d does not enter expression of the element $AB + ab + cA$. Thus $dB = 0$ and $B = 0$, too. Hence

$$ab = cA.$$

Since c does not enter the expression of ab and $ab \neq 0$ we see that equality $ab = cA$ is impossible.

Suppose now that d enters the expression of B. Then we can write

$$B = d + C,$$

where d does not enter the expression of C. Then we find $ab = dC + cA + Ad + AC$.

Hence

$$d(C + A) = ab + cA + AC.$$

As before we find that

$$C + A = 0.$$

($C + A$ cannot be d, since d does not enter the expressions of C and A.) Thus

$$ab = cA,$$

which is impossible.

This proves that element $ab + cd$ cannot be written as $u \cdot w$ with $u, w \in R^1(X_F)$.

Case 2. $X_F = Z \times H$, $|H| = 4$ and $\mathrm{st}(Z) = 1$. Since we have already investigated the case $|X_F| = 8$ and $\mathrm{st}(X_F) = 3$, we shall assume that $|X_F| \neq 8$. This means that $|Z| \geq 3$.

From Definition 5 we see that

$$R^1(X_F) = R^1(Z) \oplus H \tag{5}$$

$$R^2(X_F) = cR^1(Z) \oplus dR^1(Z) \oplus cdR^0(Z), \tag{6}$$

where $\{c, d\}$ is the vector basis of H.

Since $|Z| \geq 3$, there exist elements $a, b \in R^1(Z)$ linearly independent over $\mathbf{Z}/2\mathbf{Z}$.

By a calculation completely analogous to the calculation in Case 1, we see that the element

$$ac + bd \in R^2(X_F)$$

cannot be written in the form $v \cdot w$, where v an w are elements of $R^1(X_F)$.

Thus we see that, in both Cases 1 and 2, the classes of quaternion algebras do not form a subgroup of the Brauer group $\mathrm{Br}(L)$.

We now claim that if L is a linked field and Y is any connected component of the order space X_F with $\mathrm{st}(Y) = 3$, then $|Y| = 8$.

Indeed, if this were not true, there would exist a connected component Y of the order space X_F such that $\mathrm{st}(Y) = 3$ and $|Y| > 8$. According to the considerations above we know that there exists an element $f \in R^2(Y)$ such that f cannot be expressed as a product of two elements of the group $R^1(Y)$. On the other hand, from the way $R(X)$ is constructed from $R(Z)$, where Z runs over all connected components of X,we see that any element of the group $R^2(Y)$ which is a product of two elements of the group $R^1(X_F)$ is actually a product of two elements of the group $R^1(Y)$. Thus we see that the element $f \in R^2(Y)$ cannot be expressed as a product of two elements of the group $R^1(X)$. Since the additive group generated by products of two elements of the group $R^1(X_F)$ is the group $R^2(X_F)$ we see that the set of products of two elements of the group $R^1(X_F)$ does not form a group, a contradiction to the definition of linked field.

This proves that if the field $L = F(\sqrt{-1})$ is a linked field, then X_F is a finite sum of order spaces Y such that $\mathrm{st}(Y) \leq 2$ or $\mathrm{st}(Y) = 3$ and $|Y| = 8$.

Since we have already proved that if X_F is a sum of order spaces Y as above, then L is a linked field; our proof is finished. □

REMARK. It would be interesting to characterize all Pythagorean fields F such that $F(\sqrt{-1})$ is linked.

Note that if $\mathrm{st}(F) \leq 1$, then $H^2(G_{F(\sqrt{-1})}, 2) = \{0\}$ and therefore

$F(\sqrt{-1})$ is linked. Also if $\mathrm{st}(F) \geq 4$, then $2I^3F \neq I^4F$ and

$$I^4F(\sqrt{-1}) \simeq I^4F/2I^3F \neq \{0\}.$$

Thus $\mathrm{st}(F(\sqrt{-1})) \geq 4$ and $u(F(\sqrt{-1}) \geq 16$. Therefore, from [**4**], we see that $F(\sqrt{-1})$ is not linked.

It remains to investigate the cases $\mathrm{st}(F) \in \{2,3\}$. As far as I know this is still an open question.

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