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THE REPRESENTATION THEOREM FOR SPACES OF SIGNATURES

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As shown in [**6**], there is a commutative diagram of form theories and generalization maps as follows:

$$\begin{array}{ccc} \text{HIGHER LEVEL FORMS} & \longrightarrow & \text{SPACES OF SIGNATURES} \\ \uparrow & & \uparrow \\ \text{QUADRATIC FORMS} & \longrightarrow & \text{SPACES OF ORDERINGS} \end{array}$$

On the left we have the reduced theory of quadratic forms over fields, along with Becker and Rosenberg's extension to a reduced higher level form theory [**2**]. The abstract theory of Spaces of Orderings [**5**] facilitates a unified treatment of reduced quadratic forms over fields, skew fields and semi-local rings. The higher level theory can also be carried out over skew fields [**8**], and Spaces of Signatures [**6, 7**] allow for a simultaneous generalization of all of these theories.

Associated to each Space of Signatures (X, G) is a reduced Witt ring $W(X)$, which embeds naturally in a ring of continuous functions $C(X, \mathbf{C})$. The problem of characterizing this subring of $C(X, \mathbf{C})$ was solved in [**7**], under a restrictive 2-power assumption, by extending the representation theorems of [**1, 2, 4**] and [**5**]. In this note we explain how this may be done for any Spaces of Signatures, without the 2-power assumption. Detailed proofs will appear in a forthcoming paper.

Let G be an abelian group of finite (even) exponent, and set $G^* = \mathrm{Hom}_{\mathbf{Z}}(G, \mu)$, with the usual compact-open topology (μ being the complex roots of unity, with the discrete topology). Fix a nonempty subset X of G^*. An m-dimensional form f is an m-tuple $\langle a_1, \ldots, a_m\rangle (a_i \in G)$. The notions of equivalence, isometry, isotropy, represented sets, and sums of forms are defined just as for Spaces of Orderings (see [**6**]).

DEFINITION. (X, G) is a *Space of Signatures* when these axioms hold:

S_0: If $\sigma \in X$, then $\sigma^k \in X$ for all odd k.

S_1: X is closed in G^*.

S_2: There exists $-1 \in G$ such that $\sigma(-1) = -1$ for all $\sigma \in X$.

S_3: If $\sigma(x) = 1$ for all $\sigma \in X$, then $x = 1$.

S_4: If $z \in D(f \oplus g)$, then $z \in D\langle x, y\rangle$ for some $x \in D(f)$ and $y \in D(g)$.

EXAMPLES (i) When $G^2 = 1$, a Space of Signatures is a Space of Orderings in the sense of [**5**]. These arise in connection with fields, semi-local rings, and skew fields.

(ii) Becker and Rosenberg's reduced theory of higher level forms over a field [**2**] gives rise to examples of SOS's. If T is a torsion preorder in a real field K, and we set $G_T = K^{\cdot}/T^{\cdot}$ and $X_T = \{\psi \in \mathrm{Sgn}(K) : \psi(T^{\cdot}) = 1\}$, then $X_T, G_T)$ is a Space of Signatures. Of course (X_T, G_T) is a Space of Orderings if the level of T is 1.

(iii) Powers has shown [**8**] that, under a 2-power assumption, the central notions of [**2**], i.e., higher level preorders and forms, extend to skew fields, giving rise to Spaces of Signatures just as in the last example.

(iv) Let $G = C_{2n}$, the cyclic group of order $2n$, and set $X = \{\sigma \in G^* : \sigma(-1) = -1\}$, where -1 denotes the unique element of order 2 in G. Then (X, G) is an SOS, which we denote by $\mathcal{C}_{2n}$.

THEOREM 1. *Let (X, G) be a Space of Signatures. If $\psi \in G^* \backslash \{1\}$ satisfies $D\langle 1, x\rangle \subset \ker\psi$ for all $x \in \ker\psi$, then $\psi \in X$.*

REMARK. This result shows that axiom S_5 in [**6**] and [**7**] is unnecessary.

Each $a \in G$ induces a continuous function $\hat{a} : X \to \mathbf{C}$ (where $\mathbf{C}$ has the discrete topology), via $\hat{a}(\sigma) = \sigma(a)$ $(\sigma \in X)$.

DEFINITION. The *Witt ring* of (X, G) is the subring $W(X)$ of $C(X, \mathbf{C})$ which is generated by all the $\hat{a}(a \in G)$. Each form $f = \langle a_1, \ldots, a_m\rangle$ thus gives rise to a function $\hat{f} = \sum_{i=1}^m \hat{a}_i$ in $W(X)$.

If T is a preorder in a field K, then $W(X_T)$ is isomorphic to the higher level reduced Witt ring $W_T(K)$ of [**2**].

We are naturally led to consider the *representation problem*:

Given a function $F \in C(X, C)$, how can we tell if $F \in W(X)$?

DEFINITION. (X, G) is a *fan* when it is a group extension (see [**6**]) of C_2, i.e., when $X = \{\psi \in G^* : \psi(-1) = -1\}$.

THEOREM 2. *The following are equivalent:*

(i) *(X, G) is a fan;*

(ii) *Each $x \neq -1$ is rigid, i.e., satisfies $D\langle 1, x\rangle = \{1, x\}$;*

(iii) *Each $x \neq -1$ of order 2 is rigid;*

(iv) *Whenever $\sigma_1, \sigma_2, \sigma_3 \in X$, then $\sigma_1\sigma_2\sigma_3 \in X$.*

THEOREM 3. *The following are equivalent:*

(i) *(X, G) is either a fan or a group extension of a direct sum (see [**6**]) of two fans each of which has cyclic 2-primary part,*

(ii) *For each $x \neq -1$ of order 2, the 2-primary part of the group of elements a satisfying $\langle 1, x\rangle \simeq a\langle 1, x\rangle$ is cyclic.*

(iii) *The 2-primary part of the group generated by -1 and the set of nonrigid elements of G has rank less than or equal to 2.*

DEFINITION. If (X, G) is either a fan or a group extension of $C_{2^s} \oplus C_{2^t}$, for some s, t, (X, G) is said to be a *quasifan*.

THEOREM 4. *The following are equivalent:*

(i) *(X, G) is a quasifan,*

(ii) *For each $x \neq -1$ of order 2, the group of elements b satisfying $\langle 1, x\rangle \simeq b\langle 1, x\rangle$ is 2-primary cyclic.*

(iii) *The group generated by -1 and the set of nonrigid elements of G is 2-primary and has rank less than or equal to 2.*

(iv) *Whenever* $\sigma_1, \sigma_2, \sigma_3 \in X$ *there exists odd* k_1, k_2, *and a permutation* π *of* $\{1,2,3\}$ *such that* $\{\sigma_{\pi(1)}^{k_1}\sigma_{\pi(2)}\sigma_{\pi(3)}, \sigma_{\pi(1)}\sigma_{\pi(2)}^{k_2}\sigma_{\pi(3)}\} \subset X$.

REMARK. Using the signature characterization above it is clear that any subspace of a quasifan is again a quasifan.

DEFINITION. If $Y \subset X$, then $F \in C(X, \mathbf{C})$ is said to be *represented over* Y when $(Y, G/Y^{\perp})$ is a subspace of X (see [**7**]) and there is a form f over (X, G) such that F and $\hat{f}$ agree on Y.

THEOREM 5. *If* $F \in C(X, \mathbf{C})$, *then* F *is represented over* X *precisely when* F *is represented over each finite quasifan* Y *in* X.

In [**7**] necessary and sufficient conditions are given for $F \in C(X, \mathbf{C})$ to be represented over X when (X, G) is a finite quasifan.

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