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LEVELS OF QUATERNION ALGEBRAS

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The level of a ring with identity is the least integer n for which -1 is expressible as a sum of n squares. Pfister [**3**] proved that the level of a field must be a power of two, and later Dai, Lam and Peng [**2**] showed that any positive integer may occur as the level of a commutative ring. There seems to be nothing in the literature about this problem in the non-commutative case. In [**5**] a different notion of level is discussed involving the expression of -1 as a sum of products of squares. In this note we examine the usual notion of level for quaternion division algebras. We show that any power of two may occur and also that $2^k + 1$ occurs for all $k \geq 1$. We have no information on whether other integer values can occur as the level of a quaternion division algebra.

Let D be a quaternion division algebra.

DEFINITION. The *level* $s(D)$ is the least integer n such that $-1 = \sum_{i=1}^n x_i^2$ with each $x_i \in D$. If -1 is not expressible as a sum of squares then we define $s(D)$ to be infinity.

We write $D = \left(\frac{a,b}{F}\right)$, F a field, characteristic $\neq 2, i^2 = a, j^2 = b, ij = -ji$, etc.

We write T_D for the four-dimensional quadratic form $\langle 1, a, b, -ab\rangle$ and T_P for the three-dimensional form $\langle a, b, -ab\rangle$. (Note that, apart from a scalar factor 2, T_D is the usual trace form of D over F, i.e., the map $D \to F$, $\ x \to \mathrm{tr}(x), tr$ denoting the reduced trace.) We use the notation of [**4**] for quadratic forms.

LEMMA 1. *$s(D) \leq 2$ if and only if either $\langle 1,1\rangle \perp T_D$ is isotropic or $\langle 1\rangle \perp 2 \times T_P$ is isotropic.*

PROOF. If $x \in D$, then $x = p + qi + rj + sk$ for p, q, r, s in F and so $x^2 = p^2 + aq^2 + br^2 - abs^2 + 2pqi + 2prj + 2psk$. Now $s(D) \leq 2$ implies

$-1 = x_1^2 + x_2^2$ and hence

$$-1 = \sum_{i=1}^{2} p_i^2 + a\sum_{i=1}^{2} q_i^2 + b\sum_{i=1}^{2} r_i^2 - ab\sum_{i=1}^{2} s_i^2$$

and

$$\sum_{i=1}^{2} p_i q_i = \sum_{i=1}^{2} p_i r_i = \sum_{i=1}^{2} p_i s_i = 0.$$

If each p_i is zero, then $\langle 1\rangle \perp 2 \times T_P$ is isotropic. If the p_i are not both zero, then multiplying the first equation by $\sum_{i=1}^{2} p_i^2$ shows that, because of [**3**, satz 2], the form $\langle 1,1\rangle \perp T_D$ is isotropic.

Conversely, if $\langle 1\rangle \perp 2 \times T_P$ is isotropic, then $2 \times T_P$ represents -1 so that $-1 = a\sum_{i=1}^{2} q_i^2 + b\sum_{i=1}^{2} r_i^2 - ab\sum_{i=1}^{2} s_i^2$ for some $q_i, r_i, s_i, i = 1, 2$, in F. Thus $-1 = x_1^2 + x_2^2$ where $x_i = q_i i + r_i j + s_i k$, $i = 1, 2$. (i.e., -1 is the sum of the squares of two pure quaternions.)

If $\langle 1,1\rangle \perp T_D$ is isotropic, then there exist $\lambda_1, \lambda_2, p, q, r, s$ in F such that $\lambda_1^2 + \lambda_2^2 + p^2 + aq^2 + br^2 - abs^2 = 0$. If λ_1, λ_2 are each zero, then $-1 = x^2$ where $x = 1/p(qi + rj + sk)$, so $s(D) = 1$. (Note that p cannot also be zero, for if it were we would obtain a pure quaternion whose square was zero.) So, assuming λ_1, λ_2 are not both zero, there exist μ_1, μ_2 such that $(\lambda_1^2 + \lambda_2^2)(\mu_1^2 + \mu_2^2) = 1$. This gives $(\mu_1^2 + \mu_2^2)(p^2 + aq^2 + br^2 - abs^2) = -1$, and then $-1 = x_1^2 + x_2^2$ where $x_1 = p\mu_1 + q\mu_2 i + r\mu_2 j + s\mu_2 k$ and $x_2 = p\mu_2 - q\mu_1 i - r\mu_1 j - s\mu_1 k$. □

LEMMA 2. *$s(D) \leq 3$ if and only if $\langle 1\rangle \perp 3 \times T_P$ is isotropic.*

PROOF. $s(D) \leq 3$ implies that

$$-1 = \sum_{i=1}^{3} p_i^2 + a\sum_{i=1}^{3} q_i^2 + b\sum_{i=1}^{3} r_i^2 - ab\sum_{i=1}^{3} s_i^2$$

and

$$\sum_{i=1}^{3} p_i q_i = \sum_{i=1}^{3} p_i r_i = \sum_{i=1}^{3} p_i s_i = 0.$$

Letting $p_4 = 1,\quad q_4 = r_4 = s_4 = 0$ we have

$$0 = \sum_{i=1}^{4} p_i^2 + a\sum_{i=1}^{4} q_i^2 + b\sum_{i=1}^{4} r_i^2 - ab\sum_{i=1}^{4} s_i^2$$

and

$$\sum_{i=1}^{4} p_i q_i = \sum_{i=1}^{4} p_i r_i = \sum_{i=1}^{4} p_i s_i = 0.$$

Multiplying the above equation by $\sum_{i=1}^{4} p_i^2$ yields, by [**3**, satz 2], that $\langle 1\rangle \perp 3 \times T_P$ is isotropic.

Conversely, if $\langle 1\rangle \perp 3 \times T_P$ is isotropic, then -1 is the sum of the squares of three pure quaternions. Thus $s(D) \leq 3$. □

PROPOSITION 1. *There exist quaternion division algebras of level three.*

PROOF. Consider $K = \mathbf{R}(x_1, x_2, x_3)$, the rational function field in three variables x_1, x_2, x_3. Let $a = x_1^2 + x_2^2 + x_3^2$ which, by [**1**], is not a sum of less than three squares. Let F be the Laurent series field $K((t))$. Let $D = \left(\frac{a,t}{F}\right)$ which is a division algebra. We will show that $s(D) = 3$.

Since $T_P = \langle a, t, -at\rangle$ and $a = x_1^2 + x_2^2 + x_3^2$ it is easy to see that $\langle 1\rangle \perp 3 \times T_P$ is isotropic so that $s(D) \leq 3$ by Lemma 2.

If $\langle 1\rangle \perp 2 \times T_P$ is isotropic, then $\langle 1, a, t, -at, a, t, -at\rangle$ is isotropic. Elements of F are Laurent series of the form $\sum_{i=m}^{\infty} y_i t^i$, each $y_i \in K$. The lowest power of t appearing in z^2 for $z \in F$ must be an even power. Equating coefficients of the lowest power of t in an expression for $\langle 1, a, t, -at, a, t, -at\rangle$ being isotropic yields that either $\langle 1, a, a\rangle$ is isotropic over K or $\langle 1, 1, -a, -a\rangle$ is isotropic. The first of these implies that $-a$ is a sum of squares in K while the second implies a is a sum of two squares. Neither of these is possible. If $\langle 1, 1\rangle \perp T_D$ is isotropic then $\langle 1, 1, 1, a, t, -at\rangle$ is isotropic over F. This implies either $\langle 1, 1, 1, a\rangle$ or $\langle 1, -a\rangle$ isotropic over K, and thus either $-a$ is a sum of squares or a is square in F. Again neither of these is possible. Hence $s(D) = 3$. □

LEMMA 3. *$s(D) \leq 2^k$ implies that either $(2^k+1)\langle 1\rangle \perp (2^k-1)\times T_P$ is isotropic or $\langle 1\rangle \perp 2^k \times T_P$ is isotropic.*

The proof is similar to the first half of Lemma 1.

LEMMA 4. *Let n be any positive integer. If $\langle 1\rangle \perp n\times T_P$ is isotropic, then $s(D)\leq n$.*

The proof is similar to the second half of Lemma 2.

COMMENT. It is not clear whether, for $k>1$, the converse of the implication in Lemma 3 is true. The converse of Lemma 4 is true for $n=2^k-1$ but for other values of n we do not know.

PROPOSITION 2. *There exist quaternion division algebras of level 2^k+1 for all $k\geq 1$.*

PROOF. Let $n=2^k+1$ and consider $K=\mathbf{R}(x_1,x_2,\dots,x_n)$, $a=\sum_{i=1}^n x_i^2$, $F=K((t))$ and $D=\left(\frac{a,t}{F}\right)$. Then, similar to Proposition 1, $s(D)=2^k+1$. □

PROPOSITION 3. *There exist quaternion division algebras of level 2^k for all $k\geq 0$.*

PROOF. Let $n=2^k$, $K=\mathbf{R}(x_1,x_2,\dots,x_n)$ and $F=K(y)$, the rational function field over K in one new variable y. Let $a=\sum_{i=1}^n x_i^2$ and let $D=((-y,y-a)/F)$ which can be shown to be a division algebra. We claim that $s(D)=n$.

Firstly, $s(D)\leq n$, since $(i+j)^2=-a$ yields the expression

$$-1=\left(\frac{i+j}{x_1}\right)^2+\left(\frac{x_2}{x_1}\right)^2+\cdots+\left(\frac{x_n}{x_1}\right)^2.$$

Suppose $s(D) \leq n-1$. Then there is an expression

$$\sum_{i=1}^{n} f_i^2 - y\sum_{i=1}^{n} g_i^2 + (y-a)\sum_{i=1}^{n} h_i^2 + y(y-a)\sum_{i=1}^{n} k_i^2 = 0$$

where f_i, g_i, h_i, k_i are in F and $f_1 = 1, g_1 = h_1 = k_1 = 0$. Also $\sum_{i=1}^{n} f_i g_i = \sum_{i=1}^{n} f_i h_i = \sum_{i=1}^{n} f_i k_i = 0$.

After clearing denominators we may assume that f_i, g_i, h_i, k_i are polynomials in y and that the same set of equations hold. We may assume $f_i(0), g_i(0), h_i(0), k_i(0)$ are not all zero for all i. (Divide by a power of y if necessary.) Multiplying the first equation by $\sum_{i=1}^{n} h_i^2$ and putting $y = 0$ gives

$$\left(\sum_{i=1}^{n} f_i(0)^2\right)\left(\sum_{i=1}^{n} h_i(0)^2\right) - a\left(\sum_{i=1}^{n} h_i(0)^2\right)^2 = 0.$$

Since $\sum_{i=1}^{n} f_i(0)h_i(0) = 0$ it follows that a is a sum of $n-1$ squares in K by [**3**, satz 2] unless $h_i(0) = f_i(0) = 0$ for all i. By [**1**], a is a sum of no less than n squares so that $h_i(0) = f_i(0) = 0$ for all i. Hence $f_i = yf_i'$, $h_i = yh_i'$ for new polynomials f_i', h_i', $i = 1, 2, \ldots\ldots, n$.

The first equation now gives

$$0 = y^2\sum_{i=1}^{n} (f_i')^2 - y\sum_{i=1}^{n} g_i^2 + (y-a)y^2\sum_{i=1}^{n} (h_i')^2 + y(y-a)\sum_{i=1}^{n} k_i^2.$$

Dividing by y and then putting $y = 0$ yields $\sum_{i=1}^{n} g_i(0)^2 = -a\left(\sum_{i=1}^{n} k_i(0)^2\right)$. Since $-a$ cannot be a sum of squares in K we have that $g_i(0) = k_i(0) = 0$ for all i, a contradiction. Thus $s(D) = 2^k$. □

COMMENT. Let n be any positive integer, $K = \mathbf{R}(x_1, x_2, \ldots, x_n)$, $F = K((t))$, $a = \sum_{i=1}^{n} x_i^2$, $D = \left(\frac{a,t}{F}\right)$. Then $s(D) \leq n$ by Lemma 4 and also $s(D) > 2^k$ where $2^k < n \leq 2^{k+1}$ by Lemma 3. It is possible that $s(D) = n$ but we do not know how to prove this other than when $n = 2^k + 1$.

Added in proof. J.-P. Tignol and N. Vast have shown in C.R. Acad. Sci. Paris, t305, Série I (1987), 583-586, that for D as in the above comment $S(D)$ can only take the values 2^r or $2^r + 1$ for some natural number r.

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