ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 19, Number 3, Summer 1989

ON THE CLIFFORD-LITTLEWOOD-ECKMANN GROUPS: A NEW LOOK AT PERIODICITY mod 8

T.Y. LAM(*) AND TARA SMITH

1. The Groups $\mathbf{G_{s,t}}$: an historical survey. For any pair of non-negative integers s and t, let $G_{s,t}$ denote the group generated by the symbols ε, $a_1, \ldots, a_s$, $b_1, \ldots, b_t$ with the following relations:

$$(1.1) \qquad \begin{cases} (1) \ \epsilon^2 = 1, \ a_i^2 = \epsilon \ (\forall i), \ b_j^2 = 1, \quad \forall j, \\ (2) \ a_i b_j = \epsilon b_j a_i, \quad \forall i, j, \\ (3) \ a_i a_j = \epsilon a_j a_i, \quad i \neq j, \\ (4) \ b_i b_j = \epsilon b_j b_i, \quad i \neq j, \\ (5) \ \epsilon b_j = b_j \epsilon, \quad \forall j. \end{cases}$$

Here, the relations in (5) are needed only in the special case $(s,t) = (0,1)$. For, as long as $s+t \geq 2$, it is easy to show that these relations follow from the others. In the case $(s,t) = (0,1)$, the inclusion of the relation $\epsilon b_1 = b_1 \epsilon$ ensures that $G_{0,1}$ is the group $\mathbf{Z}_2 \oplus \mathbf{Z}_2$ (and not the free product $\mathbf{Z}_2 * \mathbf{Z}_2$ which is the infinite dihedral group). Thus, in all cases, ϵ is a central element of order 2 in $G_{s,t}$. Intuitively, we think of the element ϵ as "-1", and refer to the relations (2), (3) and (4) above by saying that the elements $\{a_1, \ldots, a_s, \ b_1, \ldots, b_t\}$ "pairwise anticommute". It is easy to see that any element of the group $G_{s,t}$ can be written uniquely in the form $\epsilon^k a_{i_1} \cdots a_{i_m} b_{j_1} \cdots b_{j_n}$, where $1 \leq i_1 < \cdots < i_m \leq s$, $1 \leq j_1 < \cdots < j_n \leq t$ and $k \in \{0,1\}$. Thus, $G_{s,t}$ is a finite group of cardinality 2^{r+1} where $r := s+t$.

The groups $G_{s,t}$ are implicit in Clifford's work on "geometric algebras" [**4**, pp. 398-399]. In fact, if $C(\varphi_{s,t})$ denotes the Clifford algebra of the quadratic form $\varphi_{s,t} := s\langle -1 \rangle \perp t\langle 1 \rangle$ over any field of characteristic not 2, then $G_{s,t}$ appears naturally as a subgroup of the group of units in this Clifford algebra. Some of the groups $G_{s,t}$ are of interest to physicists. In the study of the spin of the electron, the commutation relations between angular momentum operators led to the consideration

Supported by NSF.
Received by the editors on October 8, 1987.

of the triad of Hermitian matrices

$$(1.2) \qquad \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

These "Pauli spin matrices" anticommute in pairs; they have square I_2 and product $i \cdot I_2$, so they generate a group isomorphic to $G_{0,3}$. On the other hand, $i\sigma_1, i\sigma_2$ and $i\sigma_3$ have square $-I_2$ and product I_2, so they generate the quaternion group $G_{2,0}$. In his pioneer work on the quantum theory of the electron [**8**], Dirac found that the "coefficients" α_j, $1 \leq j \leq 4$, in his relativistic wave equation satisfy the commutation rule $[\alpha_j, \alpha_k]_+ = 2\delta_{jk}$, where $[x, y]_+$ denotes the "anticommutator" $xy + yx$. Pointing out that it is not possible to represent the α_j's by 2×2 matrices, he gave a representation of them by the following 4×4 Hermitian matrices:

$$\alpha_j = \begin{pmatrix} 0 & \sigma_j \\ \sigma_j & 0 \end{pmatrix}, \quad 1 \leq j \leq 3, \qquad \alpha_4 = \begin{pmatrix} I_2 & 0 \\ 0 & -I_2 \end{pmatrix}.$$

These four "Dirac matrices" in the unitary group $U(4)$ generate the *Dirac group* (of 32 elements), which is isomorphic to $G_{0,4}$. Haenzel [**19**] brought to bear the group $G_{0,5}$ whereby he related Dirac's relativistic wave equation to the geometry of the icosahedron. More generally, the groups $G_{0,2n}$ arise naturally in quantum field theory. In the theory of Fermion fields, the creation operators x_j and the annihilation operators y_j, $1 \leq j \leq n$, satisfy the commutation relations

$$(1.4) \qquad [x_j, x_k]_+ = 0, \qquad [y_j, y_k]_+ = 0, \qquad [x_j, y_k]_+ = \delta_{jk}$$

for all j, k. Writing $x_j = (q_j + ip_j)/2$ and $y_j = (q_j - ip_j)/2$, the relations above can be transformed into

$$(1.5) \qquad [q_j, q_k]_+ = [p_j, p_k]_+ = 2\delta_{jk}, \qquad [q_j, p_k]_+ = 0$$

for all j, k. Denoting the set $q_1, p_1, \ldots, q_n, p_n$ straight through by $b_1, \ldots, b_{2n}$, (1.5) becomes $[b_j, b_k]_+ = 2\delta_{jk}$, $\forall j, k$, so the b_j's generate the group $G_{0,2n}$. This derivation of the group $G_{0,2n}$ was reported in the "Zusatz bei der Korrektur" of the paper by Jordan and Wigner [**22**] on Pauli's Exclusion Principle. On the last page of this paper, Jordan and Wigner used the Frobenius-Burnside theory of finite group representations to show that, aside from 1-dimensional representations, $G_{0,2n}$ has only one irreducible unitary representation T, of dimension 2^n. The group $G_{0,2n}$ and its irreducible representation T are also discussed in detail in [**50**; p.252, p.276] and [**29**; p.42, pp. 94-96].

Groups of the type $G_{s,0}$ have also occurred in the physics literature. They were studied, for instance, in §15 of the paper of Jordan, von Neumann and Wigner [**23**] in connection with the mathematical foundations of quantum mechanics. Eddington's work [**11**, **12**] on "E-numbers" and sets of pairwise anticommuting matrices was likewise motivated by the quantum mechanical formalism. Eddington was particularly interested in "pentads" of 4×4 matrices which give rise to complex representations of $G_{5,0}$, and noted that these representations of $G_{5,0}$ are closely related to the 4-dimensional real representations of $G_{2,3}$ (cf. [**11**], [**13**, p.270]). Eddington's work was subsequently generalized by Newman [**33**]. In 1934, Littlewood [**28**] explicitly defined the family of finite groups $G_{s,t}$, and, using the general theory of group characters, gave a much more conceptual treatment of the results of Eddington and Newman on sets of anticommuting matrices.

Another important source of the groups $G_{s,t}$ is provided by the Hurwitz Problem on the composition of quadratic forms (see [**20**, **21**] and [**43**]). The paper of Radon [**37**] and the posthumous paper of Hurwitz [**21**] were both devoted to solving the so-called Hurwitz equations:

$$\begin{cases} A_j^t A_j = I_n, \\ A_j + A_j^t = 0, \\ A_j A_k + A_k A_j = 0, \ j \neq k, \end{cases} \tag{1.6}$$

where $A_1, \ldots, A_s$ are (real or complex) $n \times n$ matrices. Using purely matrix theoretic tools, Hurwitz and Radon determined the exact relationship between s and n for the system of equations (1.6) to be solvable. This relationship is expressed by the "Radon function" which has since proved to be important in many branches of mathematics. In 1942, apparently unaware of Littlewood's paper, Eckmann defined in [**10**] the groups $G_{s,0}$, and observed that a set of solutions to the Hurwitz equations (1.6) over a field F amounts to an n-dimensional *orthogonal* representation T of $G_{s,0}$ over F with the property that $T(\epsilon) = -I_n$. Eckmann determined the irreducible orthogonal representations of $G_{s,0}$ over $F = \mathbf{R}$, and, using this, he arrived at a purely group-theoretic proof of the theorem of Hurwitz-Radon on the composition of sums of squares. (By working carefully enough, it is not difficult to generalize Eckmann's work to any field F of characteristic not 2.) Because of its clarity and elegance, Eckmann's paper [**10**] has been a popular reference for authors since 1942.

In view of the above historical survey, we propose to call $G_{s,t}$ the *Clifford-Littlewood-Eckmann groups*, abbreviated as CLE-groups in this paper.

In the topology literature, it is well-known that the Clifford algebras $C(\varphi_{s,t})$ obey a certain periodicity formula modulo 8, namely

$$(1.7) \qquad C(\varphi_{s+8,t}) \cong C(\varphi_{s,t+8}) \cong \mathbf{M}_{16}(C(\varphi_{s,t})).$$

This reduces the computation of $C(\varphi_{s,t})$ to low-dimensional cases, and leads to the explicit tables of Clifford algebras in [**1**] in the case $F = \mathbf{R}$. (See [**27**, pp. 128-129] for the case of an arbitrary ground field F of characteristic not 2.) Since the CLE-group $G_{s,t}$ spans the algebra $C(\varphi_{s,t})$, it seems natural to ask if there is also a periodicity mod 8 phenomenon for the groups $G_{s,t}$, and, if so, whether the groups $G_{s,t}$ can be determined explicitly in terms of just a few basic groups.

As it turned out, this is indeed possible, although the details for such a theory have apparently never appeared before. In §2, we shall fill this gap by describing a complete decomposition theory for the groups $G_{s,t}$ in terms of four basic groups, namely, the two abelian groups of order 4 and the two non-abelian groups of order 8 (Theorem 2.10). This analysis entails, in particular, a periodicity law mod 8 for the groups $G_{s,t}$, thus enabling us to account for the periodicity phenomenon at the more primitive level of groups (rather than algebras). In §5, we shall show that the periodicity of the groups $\{G_{s,t}\}$ *implies* that of the graded Clifford algebras $\{C(\varphi_{s,t})\}$. One distinct advantage of this new derivation of Clifford algebra periodicity is that, in contrast to the usual derivation (as in, e.g., [**27**, pp.126-128]), it does not depend on using the notion of graded tensor products of Clifford algebras.

The decomposition theory mentioned in the paragraph above has many consequences. Indeed, almost all facts about the groups $G_{s,t}$, their representations, and their abelian subgroup structures can be deduced simply and systematically from this theory. In this approach, the determination of the Frobenius-Schur classification of the irreducible representations of $G_{s,t}$ becomes especially transparent. This extends Eckmann's results from the $G_{s,0}$'s to the $G_{s,t}$'s, and, in the meantime, provides a purely group-theoretic alternative to the combinatorial treatment in [**10**]. Some applications of this nature will be given in §3.

From the viewpoint of the classification theory of finite p-groups, the CLE-groups form a class of 2-groups which is slightly larger than the class of extra-special 2-groups. Recall that a finite p-group G is called *extra-special* ([**39**, p.140], [**9**, p.179]) if its center $Z(G)$ and its commutator subgroup G' both have order p (and therefore $Z(G) = G'$). The relation between extra-special 2-groups and the CLE-groups is given by

PROPOSITION 1.8. (1) *Any extra-special* 2*-group* G *is isomorphic to some* CLE*-group* $G_{s,t}$ *where* $s+t$ *is even.* (2) *If* $s+t$ *is even,* $G_{s,t}$ *is an extra-special* 2*-group.* (3) *If* $s+t$ *is odd, then* $G_{s,t} \cong G_{s-1,t} \dot{\times} Z(G_{s,t})$ *whenever* $s \geq 1$, *and* $G_{s,t} \cong G_{s,t-1} \dot{\times} Z(G_{s,t})$ *whenever* $t \geq 1$. (*See the beginning of* §2 *for the definition of* "$\dot{\times}$".) *Here,* $G_{s-1,t}$ *and* $G_{s,t-1}$ *are extra-special by* (2), *and* $Z(G_{s,t})$ *has order* 4.

Part (2) and Part (3) of this Proposition will be clear from (2.3), (2.4) and the definition of "$\dot{\times}$". Part (1) is an easy exercise in linear algebra which we shall leave to the reader. (If G is an extra-special 2-group, its commutator quotient group G/G' may be viewed as an $\mathbf{F}_2$-vector space. Let $Z(G) = \{1, \epsilon\}$ and try to find an $\mathbf{F}_2$-basis $\{x_1 G', \ldots, x_n G'\}$ of G/G' such that $x_j x_k = \epsilon x_k x_j$ $(\forall j \neq k)$.)

In group theory, it is well-known that any extra-special p-group is a "central product" of copies of the two non-abelian p-groups of order p^3 [**39**, p.141]. In view of Proposition 1.8, this already implies that any CLE-group $G_{s,t}$ can be decomposed into a "product" of a number of very small 2-groups. However, the *explicit* decomposition of the $G_{s,t}$'s in terms of the parameters s and t does not seem to have been given before, and their periodicity was never noted. What we shall do in §2 may therefore be viewed as a *concrete realization* of the central product decomposition of the $G_{s,t}$'s. To make our treatment self-contained, this decomposition will be achieved entirely within the framework of the CLE-groups, without reference to the class of extra-special p-groups. In particular, to understand the proof of our Decomposition Theorem

2.10 requires only a bare minimum of group theory.

As pointed out to us by Bruno Kahn, the decomposition of the groups $G_{s,t}$ can be interpreted in terms of the quadratic form theory over the field $\mathbf{F}_2$. The two germane references here are the papers of Wall [**49**] and Quillen [**35**]. From the viewpoint of quadratic form theory, the classification of the groups $G_{s,t}$ amounts essentially to the determination of the rank, the defect, and the Arf invariant (if defined) of the quadratic form

$$(1.9) \qquad q_{s,t} := x_1^2 + \cdots + x_s^2 + \sum_{1\le i<j\le r} x_i x_j, \qquad r := s+t.$$

over $\mathbf{F}_2$. These quadratic form invariants were computed in [**49**] and [**35**] in the special case $t = 0$: Wall did this by constucting an explicit "symplectic basis" for the commutator quotient group of $G_{s,0}$, while Quillen did it by using the structure of the Clifford algebra $C(\varphi_{s,0})$ as determined in the paper of Atiyah, Bott and Shapiro [**1**]. For us, the quadratic form invariants of $q_{s,t}$ (for any (s,t)) are easy to write down, since we have completely determined the groups $G_{s,t}$ in §2. The details of this beautiful connection between group theory and quadratic forms will be explained in §4 of the paper.

We shall now conclude this introductory section with a remark. Concerning the groups $G_{0,2n}$ which arise from the consideration of the commutation relations (1.4) and (1.5), we have been informed by Professor Harry Morrison that it is also of interest to physicists to consider countably infinite systems of creation and annihilation operators $\{x_j, y_j : j = 1,2,3,\dots\}$ satisfying (1.4). The mathematical problem of analyzing the representations of these commutation relations by operators on Hilbert spaces has been dealt with in two papers of Gårding and Wightman [**17**, **18**]. As these authors pointed out, earlier work of Friedrichs and von Neumann already showed that the Jordan-Wigner result (about the uniqueness of an irreducible unitary representation) no longer holds in the case of infinitely many operators. We thank Professor Morrison for pointing out to us the references [**17**] and [**18**], but we shall not use or discuss the results of these papers here since the consideration of infinite groups and infinite dimensional representations is beyond the scope of our paper.

2. Classification and periodicity of the groups $\mathbf{G_{s,t}}$. Besides $G_{0,0} \cong \mathbf{Z}_2$ (the cyclic group of two elements), the four most basic groups among the $G_{s,t}$'s are:

$$(2.1)\qquad \begin{cases} C := G_{1,0} \text{ (the cyclic group of order 4)}, \\ K := G_{0,1} \text{ (the Klein 4-group)}, \\ Q := G_{2,0} \text{ (the quaternion group of order 8)}, \\ D := G_{0,2} \text{ (the dihedral group of order 8)}. \end{cases}$$

It is also easy to see that $D \cong G_{1,1}$. Let us now show how to use the four groups above to compute all of the CLE-groups. The method we use is completely elementary. First we note the following (easy) computation for the commutator subgroup and the center of $G_{s,t}$:

$$(2.2)\qquad G'_{s,t} = [G_{s,t}, G_{st}] = \langle \epsilon \rangle \quad \text{if } (s,t) \neq (0,0), (1,0), (0,1);$$

$$(2.3)\qquad Z(G_{s,t}) = \begin{cases} \langle \epsilon \rangle & \text{if } r = s+t \text{ is even}, \\ \langle \epsilon,\ a_1 \cdots a_s b_1 \cdots b_t \rangle & \text{if } r = s+t \text{ is odd}. \end{cases}$$

Another easy computation yields the following squaring formulas in $G_{s,t}$:

$$(2.4)\qquad \left(a_{i_1} \cdots a_{i_p} b_{j_1} \cdots b_{j_q}\right)^2 = \begin{cases} \epsilon^p & \text{if } p+q \equiv 0, 1 (\operatorname{mod} 4), \\ \epsilon^{p+1} & \text{if } p+q \equiv 2, 3 (\operatorname{mod} 4), \end{cases}$$

where $i_1, \ldots, i_p$ are distinct and $j_1, \ldots, j_q$ are distinct.

Now consider the category $\mathcal{C}$ whose objects are groups with a distinguished central involution ϵ, and whose morphisms are ϵ-preserving group homomorphisms. For two groups (G, ϵ) and (H, ϵ) in this category $\mathcal{C}$, we define $G \dot{\times} H$ to be the quotient group of $G \times H$ obtained by identifying $(\epsilon, 1)$ with $(1, \epsilon)$. Then $G \dot{\times} H$ is an object in $\mathcal{C}$ with distinguished central involution $\epsilon = (\epsilon, 1) = (1, \epsilon)$. This group has the following universal property: if $\alpha : G \to K$, $\beta : H \to K$ are morphisms in $\mathcal{C}$ such that $\alpha(G)$ commutes elementwise with $\beta(H)$, then there is a unique morphism $\gamma : G \dot{\times} H \to K$ which makes the following diagram commutative:

$$\begin{tikzcd}
 & G \arrow[dl, "p"'] \arrow[dr, "\alpha"] & \\
G\dot{\times}H \arrow[rr, "\gamma"] & & K \\
 & H \arrow[ul, "q"] \arrow[ur, "\beta"'] &
\end{tikzcd}$$

where p, q are the obvious inclusion maps. This definition of $G\dot{\times}H$ is a modified version of the central product construction in group theory (cf. [**39**: p.141], [**9**: p.179]). If G and H are finite groups in $\mathcal{C}$, we clearly have $|G\dot{\times}H| = |G|\cdot|H|/2$. The group $\mathbf{Z}_2$ (with the non-identity element as ϵ) serves as the trivial object in $\mathcal{C}$, with $\mathbf{Z}_2\dot{\times}H \cong H$ for any H in $\mathcal{C}$. The classification of the CLE-groups is made possible by the following isomorphism theorem for $G_{s,t}\dot{\times}G_{m,n}$.

THEOREM 2.5. *Let $G = G_{s,t}\dot{\times}G_{m,n}$, and $r = s+t$. Then G is isomorphic to the groups in the right-hand column of the chart below depending on the parity of s and on the congruence class of r modulo 4:*

	s	$r\,(\mathrm{mod}\,4)$	G
(1)	odd	0	
			$G_{s+n,t+m}$
(2)	even	2	
(3)	odd	2	
			$G_{s+m,t+n}$
(4)	even	0	
(5)	odd	1	
		$G_{s,t}\dot{\times}G_{n,m}$	
(6)	even	3	

PROOF. As usual, we think of $G_{s,t} = \langle\epsilon,\ a_1,\dots,a_s,\ b_1,\dots,b_t\rangle$

and $G_{m,n} = \langle \epsilon, a'_1, \ldots, a'_m,\ b'_1, \ldots, b'_n \rangle$ as subgroups of G. Let $z = a_1 \ldots a_s\ b_1 \ldots b_t$, and consider the group

$$K := \langle \epsilon,\ za'_1, \ldots, az'_m,\ zb'_1, \ldots, zb'_n \rangle \subseteq G.$$

Note that $G_{s,t}$ and K generate G. Moreover, $(za'_i)^2 = z^2\epsilon$, $(zb'_j)^2 = z^2$ and any two distinct elements from $\{za'_1, \ldots, za'_m,\ zb'_1, \ldots, zb'_n\}$ "anticommute". In cases (5) and (6), r is odd, so z is central in $G_{s,t}$ (by (2.3)) and K commutes elementwise with $G_{s,t}$. Furthermore, by (2.4), $z^2 = \epsilon$ in these two cases, so $(za'_i)^2 = 1$ and $(zb'_j)^2 = \epsilon$. This gives $K \cong G_{n,m}$ and we see easily that $G \cong G_{s,t} \dot{\times} K \cong G_{s,t} \dot{\times} G_{n,m}$. In the cases (1), (2), (3) and (4), z is not central in $G_{s,t}$ and it anticommutes with each a_i and b_j. Hence, any two distinct elements from

$$\{a_1, \ldots, a_s,\ b_1, \ldots, b_t,\ za'_1, \ldots, za'_1, \ldots, za'_m,\ zb'_1, \ldots, zb'_n\}$$

anticommute. In the cases (1) and (2), we have (by (2.4)) $(zb'_j)^2 = z^2 = \epsilon$ and $(za'_i)^2 = z^2\epsilon = 1$, so $G \cong G_{s+n,t+m}$. In the cases (3) and (4), we have $(za'_i)^2 = z^2\epsilon = \epsilon$ and $(zb'_j)^2 = z^2 = 1$, so $G \cong G_{s+m,t+n}$. □

Note that, in the two cases (7) $s =$ odd, $r \equiv 3 (\mathrm{mod}\, 4)$, and (8) $s =$ even, $r \equiv 1 (\mathrm{mod}\, 4)$, the argument above does not yield any new information on the group $G_{s,t} \dot{\times} G_{m,n}$. This is why we did not include these two cases in the formulation of Theorem 2.5.

Using the notation in (2.1), we have the following special cases of the theorem:

COROLLARY 2.6.

(1) $C \dot{\times} G_{m,n} \cong C \dot{\times} G_{n,m}$ $\quad (s = 1,\ t = 0:$ Case (5)).

(2) $Q \dot{\times} G_{m,n} \cong G_{n+2,m}$ $\quad (s = 2,\ t = 0:$ Case (2)).

(3) $D \dot{\times} G_{m,n} \cong G_{n,m+2}$ $\quad (s = 0,\ t = 2:$ Case (2)).

(4) $D \dot{\times} G_{m,n} \cong G_{m+1,n+1}$ $\quad (s = 1,\ t = 1:$ Case (3)).

We shall refer to (4) above as the "Diagonal Law". By (2) and (3) here, we have

$$G_{m+4,n} \cong Q \dot{\times} G_{n,m+2} \cong Q \dot{\times} (D \dot{\times} G_{m,n}),$$

$$G_{m,n+4} \cong D\dot{\times}G_{n+2,m} \cong D\dot{\times}(Q\dot{\times}G_{m,n}).$$

Since the product "$\dot{\times}$" is commutative as well as associative (up to natural isomorphisms), we have proved

COROLLARY 2.7. *$G_{m+4,n} \cong G_{m,n+4} \cong D\dot{\times}Q\dot{\times}G_{m,n}$. In particular, for $m = n = 0$, we get $G_{4,0} \cong G_{0,4} \cong D\dot{\times}Q$.*

The isomorphism $G_{4,0} \cong G_{0,4}$ was observed by Coxeter in [**6**, §6], although Coxeter did not point out that the (Dirac) group $G_{0,4}$ is in fact isomorphic to $D\dot{\times}Q$. Letting $(m,n) = (2.0)$ in (2.6) (4), we see that the Dirac group is also isomorphic to $G_{3,1}$. This explains why physicists often represent a set of generators of the Dirac group by four pairwise anticommuting matrices, one with square I and the three others with square $-I$. (This is clearly more convenient for space-time considerations in relativity.) There are at least three such representations in use in the physics literature: the Dirac representation, the chiral representation, and the Majorana representation (cf. [**5**, pp. 688-689]).

COROLLARY 2.8.

(1) $Q\dot{\times}Q \cong D\dot{\times}D \cong G_{2,2}$.

(2) $Q\dot{\times}C \cong D\dot{\times}C \cong G_{2,1} \cong G_{0,3}$.

(3) $Q\dot{\times}K \cong G_{3,0}$.

(4) $C\dot{\times}C \cong K\dot{\times}C$.

(These isomorphisms, incidentally, show the lack of a cancellation law for the modified product "$\dot{\times}$".)

PROOF. By (2.6)(2) with $(m,n) = (2,0)$, we have $Q\dot{\times}Q \cong G_{2,2}$ and by (2.6)(3) with $(m,n) = (0,2)$, we have $D\dot{\times}D \cong G_{2,2}$. This proves (1). Similarly, we have $Q\dot{\times}C \cong Q\dot{\times}G_{1,0} \cong G_{2,1}$ and $D\dot{\times}C \cong D\dot{\times}G_{1,0} \cong G_{0,3}$, and (2.6) (1) shows that $D\dot{\times}C \cong Q\dot{\times}C$. This proves (2). By (2.6) (2), $Q\dot{\times}K \cong Q\dot{\times}G_{0,1} \cong G_{3,0}$, so we have (3). Finally, $C\dot{\times}C$ and $K\dot{\times}C$ are both isomorphic to the ordinary direct product of C and $\mathbf{Z}_2$, so we have (4). □

From now on, to simplify the notation, we shall write GH for $G\dot{\times}H$. Applying (2.7) twice and (2.8) (1), we get

COROLLARY 2.9. (PERIODICITY mod 8) $G_{m+8,n} \cong D^4 G_{m,n} \cong G_{m,n+8}$.

(Here, and in the following, D^k means $DD\cdots D$ with k factors.)

We can now prove the

DECOMPOSITION THEOREM 2.10. *Let $G = G_{s,t}$ and $r = s+t$. Then G is isomorphic to exactly one of the following products:*

$$D^i,\ D^{i-1}Q,\ D^iK,\ D^{i-1}QK \quad \text{or} \quad D^iC.$$

We shall call these the canonical forms. The first two canonical forms occur when r is even; the last three canonical forms occur when r is odd.

PROOF. We first determine $G_{s,0}$ for $0 \le s \le 7$. Beyond $G_{0,0} \cong \mathbf{Z}_2$, $G_{1,0} \cong C$ and $G_{2,0} \cong Q$, we have $G_{3,0} \cong QK$ by Corollary 2.8 (3) and $G_{4,0} \cong DQ$ by Corollary 2.7. Moreover,

$$\begin{aligned}
G_{5,0} &\cong DQG_{1,0} \cong DQC \cong D^2C && \text{by (2.7) and (2.8)(2),}\\
G_{6,0} &\cong DQG_{2,0} \cong DQQ \cong D^3 && \text{by (2.7) and (2.8)(1),}\\
G_{7,0} &\cong DQG_{3,0} \cong DQQK \cong D^3K && \text{by (2.7) and (2.8)(1).}
\end{aligned}$$

Similarly, we can derive canonical forms for $G_{0,t}$ for $0 \le t \le 7$. The remaining groups $G_{s,t}$ can then be determined by the Periodicity Law (2.9) and the Diagonal Law (2.6) (4). We display the canonical forms for the groups $G_{s,t}$ with $s,\ t \le 7$ in the following chart:

(2.11)

D^3C	D^3Q	D^3QK	D^4Q	D^5C	D^6	D^6K	D^7
D^2Q	D^2QK	D^3Q	D^4C	D^5	D^5K	D^6	D^6C
DQK	D^2Q	D^3C	D^4	D^4K	D^5	D^5C	D^5Q
DQ	D^2C	D^3	D^3K	D^4	D^4C	D^4Q	D^4QK
DC	D^2	D^2K	D^3	D^3C	D^3Q	D^3QK	D^4Q
D	DK	D^2	D^2C	D^2Q	D^2QK	D^3Q	D^4C
K	D	DC	DQ	DQK	D^2Q	D^3C	D^4
$\mathbf{Z}_2$	C	Q	QK	DQ	D^2C	D^3	D^3K

Noting that $|D^i| = |D^{i-1}Q| = 2^{2i+1}$ and $|D^iK| = |D^{i-1}QK| = |D^iC| = 2^{2i+2}$, we see that, for $r = 2i$, $G_{s,t}$ is isomorphic to one of D^i, $D^{i-1}Q$, while, for $r = 2i+1$, $G_{s,t}$ is isomorphic to one of D^iK, $D^{i-1}QK$ or D^iC. (We note, incidentally, that along the line $s+t = 2i$, the two groups $A = D^i$, $B = D^{i-1}Q$ occur with the "$AABB$" pattern, while, along the line $s+t = 2i+1$, the three groups $X = D^iK$, $Y = D^{i-1}QK$ and $Z = D^iC$ occur with the "$XZYZ$" pattern.)

Finally, to prove the *uniqueness* of the canonical form, we must show that the five groups A, B, X, Y and Z are mutually non-isomorphic. This can be done as follows. For any finite group H, write $I(H)$ for the number of elements $h \in H$ such that $h^2 = 1$. For instance, $I(D) = 6$, $I(K) = 4$, and $I(C) = I(Q) = 2$. Using these, and an induction on i, we can easily compute $I(H)$ for the five groups under consideration:

H	D^i	$D^{i-1}Q$	D^iK	$D^{i-1}QK$	D^iC
$I(H)$	$2^i(2^i+1)$	$2^i(2^i-1)$	$2^{i+1}(2^i+1)$	$2^{i+1}(2^i-1)$	2^{2i+1}

Since the five numbers listed are different, no two of the five groups can be isomorphic. □

REMARK 2.13. It is also possible to give a purely representation-theoretic proof of the last conclusion above. In fact, in the next section, we shall show that X, Y, Z have "types" I, II, III respectively (in the sense of representation theory), so no two of them are isomorphic. Similarly, it will be clear that A has "type" I and B has "type" II, so A, B are also not isomorphic.

It is worth noting that the canonical forms of the groups $G_{s,0}$ and $G_{0,t}$ $(0 \leq s, t < 8)$ are formally related by the formula

$$G_{0,t} \cong D^{t-4}G_{s,0} \tag{2.14}$$

whenever $s+t = 8$. Here, in case $t < 4$, the negative powers of D are to be "cancelled" by the D-powers in the canonical form for $G_{s,0}$. In fact, for any $s, t \geq 0$ such that $s+t = 8$, we have, by Corollaries 2.9 and 2.6 (4),

$$D^4G_{m,n+t} \cong G_{m+8,n+t} \cong D^tG_{m+s,n},$$

so formally

$$G_{m,n+t} \cong D^{t-4} G_{m+s,n}. \tag{2.15}$$

For $m = n = 0$, we get the special case (2.14).

For purposes of studying the spin groups Spin (n) and their representations, it is also useful to look at the subgroup $G^+_{s,t}$ of $G_{s,t}$ consisting of all elements $\epsilon^k a_{i_1} \cdots a_{i_p} b_{j_1} \cdots b_{j_q}$ where $p + q$ is *even*. However, it can be easily seen that

$$G^+_{s,t} \cong \begin{cases} G_{s-1,t} & \text{if } s \geq 1, \\ G_{t-1,s} & \text{if } t \geq 1. \end{cases} \tag{2.16}$$

Therefore, our results also yield the complete determination of the groups $G^+_{s,t}$. (The symmetry property $G_{s-1,t} \cong G_{t-1,s}$ implied by (2.16) is already clear from (3) and (4) of Corollary 2.6.)

As a final remark, we should point out that, in the definition of the groups $G_{s,t}$, one can also try to replace some or all of the anticommuting relations between $\{a_1, \ldots, a_s,\ b_1, \ldots, b_t\}$ by *commuting* relations (and keep all the other relations). For any group G obtained in this way, one can show (with a little bit of work) that there is an isomorphism $G \cong G_{m,n} \dot{\times} Z(G)$ for some m, n such that $m+n$ is even, and, moreover, $Z(G)$ is isomorphic to either $K \cdots\cdot K$ or $K \cdots\cdot K \cdot C$. Thus, such groups G are also completely determined by the results of this paper. The "canonical" forms of these groups will be similar to those of the groups $G_{m,n}$, except that we may now have more than one factor of K. The reader can check easily that passing from the $G_{m,n}$'s to the new groups corresponds exactly to forming the "closure" of the class of groups $\{G_{m,n}\}$ with respect to the product "$\dot{\times}$". For instance, $G = G_{1,2} \dot{\times} G_{3,0}$ is a group of the new type. This product corresponds to one of the two cases not treated in Theorem 2.5; nevertheless, the group G has the canonical form $DQKK$. It is of interest to note that the six matrices furnished by Veblen on p.509 of **[47]** satisfy exactly the commuting and anticommuting relations (as well as the squaring relations) between the generators of $G = G_{1,2} \dot{\times} G_{3,0}$, so Veblen's matrices (without the scalar factors) correspond precisely to a 4-dimensional irreducible complex representation of G.

3. Applications. The first application of the decomposition theory in §2 is the immediate determination of the irreducible real as well as complex representaions of the CLE-group $G_{s,t}$. Let $\mathcal{C}$ continue to denote the category defined at the beginning of §2. A representation T of a group G in $\mathcal{C}$ will be called an *ϵ-representation* if $T(\epsilon) = -I$. First let us determine the irreducible ϵ-representations of $G_{s,t}$ over $\mathbf{C}$. These are just the irreducible representations of $G_{s,t}$ which do not "come from" representations of $G/\langle\epsilon\rangle$. Except when $G_{s,t} \cong \mathbf{Z}_2$, C or K, these are just the irreducible representations of $\mathbf{C}$-dimension > 1.

It is well-known that each of D, Q has a unique irreducible 2-dimensional ϵ-representation over $\mathbf{C}$, and that each of C, K has two 1-dimensional ϵ-representations over $\mathbf{C}$. By taking tensor products (note that the tensor product of an ϵ-representation for G and an ϵ-representation for H gives, in a natural way, an ϵ-representation for $G \dot{\times} H$), we can therefore construct an irreducible 2^i-dimensional ϵ-representation module V for $G_{s,t}$ in case $G_{s,t}$ has the canonical form D^i or $D^{i-1}Q$ (i.e., when $r = s + t = 2i$), and two different 2^i-dimensional ϵ-representation modules V, V' for $G_{s,t}$ in case $G_{s,t}$ has the canonical forms D^iK, $D^{i-1}QK$ or D^iC (i.e., when $r = s+t = 2i+1$). By counting the number of linear characters, it follows easily that the irreducible representations of $G_{s,t}$ over $\mathbf{C}$ are either 1-dimensional, or isomorphic to V or V'. In particular, we can deduce

COROLLARY 3.1. *The matrices for the irreducible representations of $G_{s,t}$ over $\mathbf{C}$ can be written down with entries from $\{0, \pm 1, \pm i\}$.*

PROOF. This fact is well-known for D, Q, K and C, so it also follows for $G_{s,t}$ since matrices with entries from $\{0, \pm 1, \pm i\}$ are closed under tensor products. □

Recall that a complex irreducible representation V is said to have type I if V is equivalent to a real representation, type II if V is not equivalent to a real representation but the character χ_V is real-valued, and type III if χ_V is not real-valued. Letting $s(V) = |G|^{-1} \sum_{g \in G} \chi_V(g^2)$ be the Frobenius-Schur index of V, it is well-known that $s(V) = 1$ if V has type I, $s(V) = -1$ if V has type II, and $s(V) = 0$ if V has type III (see [**40**: §13.2]).

It will be clear in a moment that, for the group $G_{s,t}$, the ϵ-representations V and V' constructed above have the same type, which we may then call the type of $G_{s,t}$ for short. For instance, D and K have type I, Q has type II, and C has type III. Since the Frobenius-Schur index is multiplicative over tensor products of irreducible representations, we see immediately that

(3.2) $\qquad D^i$ *has type* I, $D^{i-1}Q$ *has type* II.

(3.3) D^iK *has type* I, $D^{i-1}QK$ *has type* II, *and* D^iC *has type* III.

In view of the Diagonal Law (2.6) (4), the type of $G_{s,t}$ depends only on $t-s$, since additional factors of D in the canonical form of $G_{s,t}$ do not change the type. (Note that $t-s$ is the *signature* of the quadratic form $s\langle -1\rangle \perp t\langle 1\rangle$.) By inspection of the canonical forms of $G_{s,0}$ and $G_{0,t}$ in (2.11), it follows that:

THEOREM 3.4.

$$G_{s,t} \text{ has } \begin{cases} \text{type I} & \text{if } t-s \equiv 0,1,2 \pmod 8, \\ \text{type II} & \text{if } t-s \equiv 4,5,6 \pmod 8, \\ \text{type III} & \text{if } t-s \equiv 3,7 \pmod 8. \end{cases}$$

In the special case when $t = 0$, this result was first proved by Eckmann [**10**, p.364] who used a trick on binomial coefficients to compute explicitly the Frobenius-Schur index of V.

Let us now derive an equation which relates the type of $G = G_{s,t}$ to the number $I(G)$ of elements of order ≤ 2 in G. Since V affords an ϵ-representation, the summation $\sum_{g\in G}\chi_V(g^2)$ has $I(G)$ terms equal to $\dim V$ and $|G| - I(G)$ terms equal to $-\dim V$. Therefore,

$$\begin{aligned} |G|\cdot s(V) &= I(G)\cdot \dim V - (|G| - I(G))\dim V \\ &= (2I(G) - |G|)\dim V, \end{aligned}$$

and so

$$I(G) = \frac{|G|}{2}\left(1 + \frac{s(V)}{\dim V}\right).$$

In particular, $I(G)$ *is larger than, less than, or equal to* $|G|/2$ *according as* G *has type* I, II *or* III (cf. [**49**]). Note that this new formula for $I(G)$ also provides another derivation of the earlier table (2.12).

Next, we turn our attention to the real representations of $G_{s,t}$. Using the same method as in the complex case, we can also construct a real orthogonal ϵ-representation of the smallest dimension for $G_{s,t}$. For the four basic groups K, C, D and Q, this smallest dimension is 1,2,2 and 4 respectively. By taking tensor products again, we can construct explicitly a real orthogonal ϵ-representation U for $G_{s,t}$ such that

(3.5) *If* $G_{s,t}$ *has type* I, *then* $\dim_{\mathbf{R}} U = \dim_{\mathbf{C}} V$;

(3.6) *If* $G_{s,t}$ *has type* II *or type* III, *then* $\dim_{\mathbf{R}} U = 2 \dim_{\mathbf{C}} V$.

(Here, it is important to notice that, in the canonical form of $G_{s,t}$, there is *at most one* factor not of type I.) By extending scalars from $\mathbf{R}$ to $\mathbf{C}$, it follows easily that U has already the least dimension among all real (orthogonal) ϵ-representations of $G_{s,t}$; in particular, U is irreducible. Using the knowledge of the irreducible complex representations of $G_{s,t}$, it follows further that $G_{s,t}$ has at most one other irreducible real (orthogonal) ϵ-representation, U', and that if U' exists, we have $\dim_{\mathbf{R}} U = \dim_{\mathbf{R}} U'$. (*In fact,* U' *exists if and only if* $G_{s,t}$ *has the canonical form* D^iK *or* $D^{i-1}QK$. The proof of this fact will be left to the reader.)

Note that the real orthogonal ϵ-representations of dimension 1,2,2 and 4 for K, C, D and Q can all be written down using matrices with entries from $\{0, \pm 1\}$. For $K = \langle \epsilon, b_1 \rangle$, we have two choices, sending b_1 to 1 or -1. (This explains, incidentally, why we have U and U' in case the canonical form of G involves a factor of K.) For $C = \langle \epsilon, a_1 \rangle$, we represent a_1 by a 90° plane rotation. For $D = \langle \epsilon, b_1, b_2 \rangle$, we represent b_1, b_2 by the two flips $(x, y) \mapsto (-x, y)$ and $(x, y) \mapsto (x, -y)$. For $Q = \langle \epsilon, a_1, a_2 \rangle$, we represent a_1, a_2 by left multiplications of i and j on the real quaternions. Since U is constructed by taking tensor products of these representations, it follows (as in (3.1)) that the ϵ-representation U (and U') for $G_{s,t}$ can be written down using matrices with entries from $\{0, \pm 1\}$.

The next result was first obtained by Kawada and Iwahori [**25**] in the context of Clifford modules. We offer here a group-theoretic rendition of it, generalizing the work of Eckmann [**10**].

THEOREM 3.7. *Let $n = 2^m n_0$ be a given positive integer, where n_0 is odd. Then the necessary and sufficient condition for $G_{s,t}$ to have a real (orthogonal) ϵ-representation of dimension n is as follows (where $r = s + t$):*

$t - s \pmod 8$	0	1	2	3
NASC	$m \geq \frac{r}{2}$	$m \geq \frac{r-1}{2}$	$m \geq \frac{r}{2}$	$m \geq \frac{r+1}{2}$

$t - s \pmod 8$	4	5	6	7
NASC	$m \geq \frac{r+2}{2}$	$m \geq \frac{r+1}{2}$	$m \geq \frac{r+2}{2}$	$m \geq \frac{r+1}{2}$

PROOF. The chart follows by combining the following information:

(1) The real (orthogonal) ϵ-representations of $G_{s,t}$ are exactly those obtained by taking direct sums of U and U'.

(2) The type of $G_{s,t}$ is determined by the signature $t - s \pmod 8$ as in (3.4).

(3) The dimensions of U and U' are determined by the type of $G_{s,t}$ as in (3.5) and (3.6).

(4) $\dim_{\mathbf{C}} V$ is $2^{r/2}$ if r is even, and is $2^{(r-1)/2}$ if r is odd.

For instance, when $t - s \equiv 6 \pmod 8$, $G_{s,t}$ has type II; since $r = s + t$ is even in this case, $\dim_{\mathbf{R}} U = 2\dim_{\mathbf{C}} V = 2 \cdot 2^{r/2} = 2^{(r+2)/2}$. Thus, $G_{s,t}$ has a real (orthogonal) ϵ-representation of dimension $n = 2^m n_0$ if and only if $m \geq (r+2)/2$. The other cases are similar. □

Putting together the columns with the same bounds on m, and replacing r by $s+t$, the necessary and sufficient conditions in the above theorem can be reformulated as follows:

(3.8)

$t - s \pmod 8$	4, 6	3, 5, 7	0, 2	1
NASC	$s \leq 2m - 2 - t$	$s \leq 2m - 1 - t$	$s \leq 2m - t$	$s \leq 2m + 1 - t$

where the upper bounds for s in the four cases are arranged as four consecutive integers.

Finally, for purposes of applications, it is also desirable to have a "single" expression for the lower bounds of m in (3.7) in terms of the parameters t and $\sigma := t - s$. (The integer σ may be called the "signature" of the group $G_{s,t}$ see (4.11).) Let us define an arithmetic function $\varphi : \mathbf{Z} \to \mathbf{Z}$ as follows: For $k = 8q + e$ where $q \in \mathbf{Z}$ and $0 \le e \le 7$, define

$$\varphi(k) = \begin{cases} 4q & \text{if } e = 0 \\ 4q+1 & \text{if } e = 1 \\ 4q+2 & \text{if } e = 2,3 \\ 4q+3 & \text{if } e = 4,5,6,7. \end{cases} \tag{3.9}$$

Then, a straightforward calculation with the lower bounds on m in (3.7) shows

THEOREM 3.7$'$. *The smallest possible dimension for a real (orthogonal) ϵ-representation of the group $G_{s,t}$ is $2^{t+\varphi(-\sigma)}$, where $\sigma := t - s$, and $\varphi : \mathbf{Z} \to \mathbf{Z}$ is as defined above.*

(For instance, when $-\sigma = 8q + 5$, the critical value of m in (3.7) is $(r+1)/2 = (s+t+1)/2 = (2t-\sigma+1)/2 = (2t+8q+6)/2 = t+4q+3$.)

The function φ in (3.9) (not to be confused with Euler's totient function) is well-known to topologists in the case when the argument k is ≥ 0. In the topology literature, φ is usually defined by

$$(*)\quad \varphi(k) = \operatorname{Card}\{i \in \mathbf{Z} :\ 1 \le i \le k,\ i \equiv 1, 2, 4, 8 \pmod 8\} \quad \text{for } k \ge 0.$$

One of the most significant facts about φ in topology is that the reduced KO-group (or the J-group) of the real projective space $\mathbf{P}^k$ is a cyclic group of order $2^{\varphi(k)}$ (with generator given by the class of the Hopf line bundle on $\mathbf{P}^k$). By (3.7$'$), this order is precisely the smallest possible dimension of a real (orthogonal) ϵ-representation of the group $G_{k,0}$. But now our work above has shown how to extend the domain of definition of the function φ from non-negative integers to *all* integers (cf. (3.9)). Of course, it is also possible to describe φ for negative arguments in the style of (*): a straightforward check from (3.9) shows that, for $k \le 0$,

$$(**)\qquad \varphi(k) = -\operatorname{Card}\{i \in \mathbf{Z} :\ k \le i \le -1,\ i \equiv 0, 1, 3, 7 \pmod 8\}.$$

Here, the "markers" 0,1,3,7 are one less than 1,2,4,8, the first four powers of 2 used in (*). We do not know of any topological or K-theoretic interpretation for $\varphi(k)$ when $k \le 0$, but is may be of interest to find one. (Can one perhaps eventually make some sense out of the statement that the reduced KO-group of a projective space of negative dimension is a cyclic group of fractional order ?)

Next, we shall try to tackle the following problem which generalizes the classical problem on the Hurwitz equations (1.4): *Let $F = \mathbf{C}$ or $\mathbf{R}$ and let $n \geq 1$ and $t \geq 0$ be given. What is the largest integer s such that there exist skew symmetric matrices $\mathbf{A}_1, \ldots, \mathbf{A}_s$* and *symmetric matrices $\mathbf{B}_1, \ldots, \mathbf{B}_t$ in the orthogonal group $O_n(F)$ such that $\{\mathbf{A}_1, \ldots, \mathbf{A}_s,\ \mathbf{B}_1, \ldots, \mathbf{B}_t\}$ pairwise anticommute?* This amounts to finding the largest s (if it exists) such that $G_{s,t}$ has an orthogonal ϵ-represention over F of dimension n. In the representation theory of finite groups, it is well-known that any complex orthogonal representation (i.e., a representation by matrices in $O_n(\mathbf{C}) = \{M \in \mathbf{M}_n(\mathbf{C}) : M^t M = I_n\}$) is always equivalent to a real represention. Thus, our problem has the same solution over the real field and the complex field. For this reason, we shall assume $F = \mathbf{R}$ in the following.

In order to solve our problem, we fix the two integers n, t and write $n = 2^m n_0$ where n_0 is odd, and $m - t = 4a + b$ where a is an integer (possibly negative!) and $b \in \{0, 1, 2, 3\}$. Then, in the necessary and sufficient condition of (3.8), we have

$$2m - t = 2(t + 4a + b) - t = t + 8a + 2b.$$

We can now easily determine from (3.8) whether s exists and if so, what is its largest possible value. The outcome is summarized in the following result.

THEOREM 3.10. *For n and t as above, the largest s for which there exist skew symmetric matrices $\mathbf{A}_1, \ldots, \mathbf{A}_s$ and symmetric matrices $\mathbf{B}_1, \ldots, \mathbf{B}_t$ in $O_n(\mathbf{R})$ (or $O_n(\mathbf{C})$) such that $\mathbf{A}_1, \ldots, \mathbf{A}_s,\ \mathbf{B}_1, \ldots, \mathbf{B}_t$ pairwise anticommute is as follows:*

(3.11)

b	0	1	2	3
s	$t + 8a$ $= 2m - t$	$t + 8a + 1$ $= 2m - 1 - t$	$t + 8a + 3$ $= 2m - 1 - t$	$t + 8a + 7$ $= 2m + 1 - t$

Here, we must use the following interpretation: If the value of s given in the chart is negative, then s does not exist (i.e., the matrices

$\{\mathbf{A}_1, \ldots, \mathbf{A}_s,\ \mathbf{B}_1, \ldots, \mathbf{B}_t\}$ *do not exist). For the maximal* s *(in case* s *exists), a system of such matrices* $\{\mathbf{A}_1, \ldots, \mathbf{A}_s,\ \mathbf{B}_1, \ldots, \mathbf{B}_t\}$ *can be found with entries from* $\{0, \pm 1\}$; *in particular, they are defined over any field.*

In this theorem, we stress again that a may be a negative integer. For instance, if $n = 2^7$ and $t = 8$, then $m = 7 = t + 4a + b$ with $a = -1$ and $b = 3$. The largest possible value of s is $s = t + 8a + 7 = 7$ so that we can find a system $\{\mathbf{A}_1, \ldots, \mathbf{A}_7,\ \mathbf{B}_1, \ldots, \mathbf{B}_8\}$ in $O_{2^7}(\mathbf{R})$ (with entries from $\{0, \pm 1\}$).

In a completely analogous manner, we can prove the following parallel result.

THEOREM 3.12. *Let* s *and* $n = 2^m n_0$ *be given, where* n_0 *is odd. Let* $m - s = 4a' + b'$ *where* $a' \in \mathbf{Z}$ *and* $b' \in \{0, 1, 2, 3\}$. *Then the largest* t *for which there exist skew symmetric matrices* $\mathbf{A}_1, \ldots, \mathbf{A}_s$ *and symmetric matrices* $\mathbf{B}_1, \ldots, \mathbf{B}_t$ *in* $O_n(\mathbf{R})$ *(or* $O_n(\mathbf{C})$*) such that* $\{\mathbf{A}_1, \ldots, \mathbf{A}_s,\ \mathbf{B}_1, \ldots, \mathbf{B}_t\}$ *pairwise anticommute is as follows*:

(3.13)

b'	0	1	2	3
	$s + 8a' + 1$ $= 2m + 1 - s$	$s + 8a' + 2$ $= 2m - s$	$s + 8a' + 3$ $= 2m - 1 - s$	$s + 8a' + 5$ $= 2m - 1 - s$

Here, again, a negative value of t *means that no* $\{\mathbf{A}_1, \ldots, \mathbf{A}_s,\ \mathbf{B}_1, \ldots, \mathbf{B}_t\}$ *can exist. For the maximal* t *(in case* t *exists), a system of such matrices* $\{\mathbf{A}_1, \ldots, \mathbf{A}_s,\ \mathbf{B}_1, \ldots, \mathbf{B}_t\}$ *can be found with entries from* $\{0, \pm 1\}$.

The two theorems above have been reported before in Wolfe's work on amicable orthogonal designs [**51**], [**16**, pp. 220-227], and have been further generalized in the work on (s,t)-families by D. Shapiro [**41**, **42**]. Note that if we add 1 to the values of s in (3.11) and to the values of t in (3.13), we'll get, respectively, the $\rho_t(n)$ and $\sigma_s(n)$ in the cited references. (If a value of s in (3.11) is negative, we set $\rho_t(n) = 0$, and similarly for $\sigma_s(n)$. This convention was, unfortunately, not explicitly mentioned in [**51**] and [**16**].) Here, ρ_t and σ_s are the *generalized Radon functions*, the usual Radon function being ρ_0 with

$\rho_0(2^{4a+b}.\text{odd}) = 8a + 2^b$ where $b \in \{0, 1, 2, 3\}$. The functions ρ_t, σ_s are related to the function φ in (3.7′) through the following triangle, where n is any positive integer of the form $2^m \cdot (\text{odd})$, and $s, t \geq 0$:

$$\begin{tikzcd}[column sep=small]
 & t + \varphi(s-t) \leq m \arrow[dl, Leftrightarrow] \arrow[dr, Leftrightarrow] & \\
s \leq \rho_t(n) - 1 \arrow[rr, Leftrightarrow] & & t \leq \sigma_s(n) - 1
\end{tikzcd}$$

This generalizes the classical relation $\varphi(s) \leq m \Leftrightarrow s \leq \rho_0(n) - 1$ which is well-known to topologists.

Let us now make some comments about our approach to Theorems (3.10) and (3.12). Our formulation of these results distinguishing four cases depending on the classes of $m - t (\text{mod}\, 4)$ and $m - s (\text{mod}\, 4)$ as in [**41**, **42**] is simpler than that in [**51**, **16**] (which distinguishes 16 cases) and is easier to remember since it conforms completely with the classical formulation. The following properties of ρ_t and σ_s noted by Wolfe:

$$(3.14) \qquad \begin{cases} \rho_t(2n) = \rho_{t-1}(n) + 1, & \rho_t(n) = \rho_{t+8}(16n), \\ \sigma_s(2n) = \sigma_{s-1}(n) + 1, & \sigma_s(n) = \sigma_{s+8}(16n), \end{cases}$$

are now immediate from *the form* of the maximal values of s and t as computed in (3.11) and (3.13), and *do not* require tedious case-by-case verifications as in [**51**, Proposition 2.3]. Last but not least, our complete decomposition of the groups $G_{s,t}$ in terms of K, C, D and Q led to a quicker and more intrinsic proof of the fact that the systems of matrices $\{\mathbf{A}_1, \ldots, \mathbf{A}_s, \mathbf{B}_1, \ldots, \mathbf{B}_t\}$ can be found with entries from $\{0, \pm 1\}$. This fascinating combinatorial fact has been observed by Wolfe [**51**] in the general (s,t)-setting and by K.Y. Lam [**26**], Gabel [**14**], and Geramita-Pullman [**15**] in the classical Hurwitz-Radon setting with $t = 0$.

From (3.8), we see that, for a fixed $n = 2^m n_0$ (and varying s, t), the largest possible $r = s + t$ for which $G_{s,t}$ has a real (orthogonal) ϵ-representation on $\mathbf{R}^n$ is $2m + 1$. The following consequence of (3.10) shows that this maximal value of r is *always* realizable, and also gives all pairs (s, t) which realize this value.

COROLLARY 3.15. (Cf. [**42**, (2.6)]) *Let $n = 2^m n_0$ be given, where n_0 is odd. Let s, t be non-negative integers with $s+t = 2m+1$. Then $G_{s,t}$ has a* *real (orthogonal) ϵ-representation on $\mathbf{R}^n$ if and only if $t \equiv m+1 \pmod 4$, if and only if the "signature" $t - s$ is $\equiv 1 \pmod 8$.*

PROOF. The two congruence conditions are clearly equivalent since

$$(t-s) - 1 = t - (2m + 1 - t) - 1 = 2(t - m - 1).$$

Suppose the said ϵ-representation exists. Then, by (3.8), we must be in the case $t - s \equiv 1 \pmod 8$ because this is the *only* case which does not contradict $s = 2m + 1 - t$. Conversely, if $t - s \equiv 1 \pmod 8$, the last column of (3.8) implies that $G_{s,t}$ has the desired real ϵ-representation on $\mathbf{R}^n$. □

To conclude this section, we shall now apply the decomposition theory in §2 to compute the sizes of the maximal abelian and maximal elementary abelian subgroups of $G_{s,t}$. These results are probably not new; on the other hand, they do not seem to be easily available in the literature. In the case of extra-special p-groups G for p an *odd* prime, Pham Anh Minh [**30**] has recently shown that the maximal elementary abelian subgroups of G are also maximal abelian (and they all have size p^{n+1} when $|G| = p^{2n+1}$). For 2-groups, however, the situation turned out to be different, as the example of the quaternion group Q shows. In the following, we shall prove that, for $G = G_{s,t}$, a maximal elementary abelian subgroup E of G is maximal abelian if G has type I, and E lies as a subgroup of index 2 in a maximal abelian subgroup A of G if and only if G has type II or type III. The indices $[G : E]$ and $[G : A]$ turn out to be exactly the dimensions of the irreducible ϵ-representations of $G_{s,t}$, respectively over $\mathbf{R}$ and over $\mathbf{C}$.

We begin with an elementary lemma.

LEMMA 3.16. *Let H be any subgroup of $G = G_{s,t}$ containing the commutator subgroup G'. If $|H| = 2^{m+1}$, then $[G : C_G(H)] \le 2^m$.*

PROOF. Let $h_1, \dots, h_m \in H$ be such that their images in G/G' form a $\mathbf{Z}_2$-basis of H/G'. Since $|G'| \le 2$, each h_i has at most two conjugates in G. Therefore, $[G : C_G(h_i)] \le 2$ for each i. Taking the intersection of the $C_G(h_i)$'s, we get $[G : C_G(H)] \le 2^m$. □

THEOREM 3.17. *Let A be a maximal abelian subgroup of $G = G_{s,t}$ and write $r = s + t$ as $2i$ or $2i + 1$. Then $[G : A] = 2^i$; in other words, $[G : A]$ is exactly the dimension of an irreducible ϵ-representation of G over $\mathbf{C}$. The isomorphism type of A is given by the following chart:*

G	D^i	$D^{i-1}Q$	D^iK	$D^{i-1}QK$	D^iC
A	K^i or $K^{i-1}C$	$K^{i-1}C$	K^{i+1} or K^iC	K^iC	K^iC

PROOF. Let $[G : A] = 2^k$. Clearly, A must contain $Z(G) \subseteq G'$, so A is normal in G. By Itô's Theorem in group representation theory [**7**, p. 365], we have $2^i \mid [G : A]$, so $i \leq k$. On the other hand, since $|A| = 2^{r-k+1}$, the Lemma above yields $[G : C_G(A)] \leq 2^{r-k}$. Since $C_G(A) = A$, this amounts to $k \leq r - k$. Therefore, $2k \leq r$ and so $k \leq i$. This completes the proof that $[G : A] = 2^i$. To determine the isomorphism type of A, note that A has exponent ≤ 4 and $A^2 \subseteq \langle \epsilon \rangle$ has order ≤ 2, so A is isomorphic to either $K \oplus \cdots \oplus K$ or $K \oplus \cdots \oplus K \oplus C$. If G has type I (viz., $G \cong D^i$ or D^iK), clearly both of these possiblities can occur, for K^i and $K^{i-1}C$ are both abelian (and hence maximal abelian) subgroups of D^i, and K^{i+1} and K^iC are both abelian (and hence maximal abelian) subgroups of D^iK. Now assume that G has type II or type III. If A were of exponent 2, there would exist a one-dimensional real representation $T : A \to \{\pm 1\} \subset \mathbf{R}^*$ with $T(\epsilon) = -1$. Inducing T up to G, we would get a real ϵ-representation of dimension $[G : A] = 2^i$. This would imply that G has type I, a contradiction. Thus, A must have exponent 4 and hence $A \cong K \oplus \cdots \oplus K \oplus C$. □

THEOREM 3.18. *Let E be a maximal elementary abelian subgroup of G and write $r = s + t$ as $2i$ or $2i + 1$. Then $[G : E]$ is given by the following chart:*

G	D^i	$D^{i-1}Q$	D^iK	$D^{i-1}QK$	D^iC
$[G : E]$	2^i	2^{i+1}	2^i	2^{i+1}	2^{i+1}

In other words, $[G : E]$ is exactly the dimension of an irreducible real ϵ-represention of G. (In the notation of (3.7'), $[G : E] = 2^{t+\varphi(s-t)}$.)

PROOF. Let A be any maximal abelian subgroup of G containing E. First let us assume that G has type II or type III. By the preceding theorem, A has the isomorphism type $K \oplus \cdots \oplus K \oplus C$. Since E is also a maximal elementary abelian subgroup of A, we clearly have $|E| = |A|/2$ and so $[G : E] = 2[G : A] = 2^{i+1}$. Next, we assume that G has type I. In this case we must prove that $E = A$, i.e., E is already maximal abelian. To do this, it is enough to treat the case $G = D^i$. For, if $G = D^i K$, the group E must contain the factor K and so $E = E_0 K$, where E_0 is a maximal elementary abelian subgroup of D^i. If we know that $[D^i : E_0] = 2^i$, then $[D^i K : E_0 K] = 2^i$ as well.

Consider now the case $G = D^i$, and assume that $E \subsetneq A$. Then $|E| = |A|/2 = 1/2 \cdot 2^{i+1} = 2^i$. By Lemma 3.16, we have $[G : C_G(E)] \leq 2^{i-1}$ and, by Theorem 3.17, $[G : A] = 2^i$; hence $C_G(E) \supsetneq A$. In particular, $C_G(E)$ cannot be abelian, so there exist two non-commuting elements $x, y \in C_G(E)$. The group H generated by x and y is of order 8, so we have either $H \cong D$ or $H \cong Q$. Consider the group $L := C_G(H)$. Clearly, $L \supseteq E$, $L \cap H = Z(H) = \langle \epsilon \rangle$, and by Lemma 3.16 again, $[G : L] \leq 4$. From this, we see easily that $G = L \dot{\times} H$ and hence $Z(L) = Z(G) = \langle \epsilon \rangle$. Furthermore, $L' \subseteq G' = \langle \epsilon \rangle$, so L is *extra-special*. By Proposition 1.8 (1) and Theorem 2.10 (for r even), it follows that either $L \cong D^{i-1}$ or $L \cong D^{i-2}Q$. If $L \cong D^{i-2}Q$, then, by our results in the type II case, the size of a maximal elementary abelian subgroup of L is 2^{i-1}. We have now a contradiction since $E \subseteq L$ and $|E| = 2^i$. Now assume $L \cong D^{i-1}$. In this case we must have $H \cong D$; for if $H \cong Q$, the group $G = L \dot{\times} H \cong D^{i-1}Q$ would have type II instead. Fix a non-central element x of order 2 in the dihedral group H. Then $x \notin E$ since $E \cap H \subseteq L \cap H = \langle \epsilon \rangle$, and x commutes elementwise with $L \supseteq E$. Thus, $\langle E, x \rangle$ is a *bigger* elementary abelian subgroup than E. This is again a contradiction, so we must have $E = A$. □

REMARK 3.19. From Theorem 3.17, it follows that an irreducible ϵ-representation of $G = G_{s,t}$ over $\mathbf{C}$ can be obtained by inducing to G any 1-dimensional ϵ-representation $A \to \{\pm 1, \pm i\} \subseteq \mathbf{C}^*$, where A is any maximal abelian subgroup of G. Similarly, from Theorem 3.18, it follows that an irreducible ϵ-representation of G over $\mathbf{R}$ can be obtained by inducing to G any 1-dimensional ϵ-representation $E \to \{\pm 1\} \subseteq \mathbf{R}^*$, where E is any maximal elementary abelian subgroup of G.

REMARK 3.20. In the case when $G = G_{s,0}$, Yuzvinsky [**53**] has constructed certain explicit examples of maximal elementary abelian subgroups $E \subseteq G$ in terms of the generators a_i's. These subgroups E were used in [**53**] to construct new monomial pairings satisfying the norm identities of Hurwitz [**20, 21**]. However, the proofs of Theorems 1 and 2 in [**53**] both require corrections, while the truth of Theorem 3 there remains in doubt (cf. [**45**]).

4. Relations to quadratic forms over $\mathbf{F}_2$. The decomposition theory of the groups $G = G_{s,t}$ developed in §2 turns out to have a very natural interpretation in terms of the quadratic form theory over the field $\mathbf{F}_s = \{0, 1\}$. In this (mainly expository) section, we shall try to make this connection explicit. Basic references for the material in this section are the papers of Quillen [**35**] and Wall [**49**].

For any two elements $x,\ y \in G = G_{s,t}$, define $B(x, y) \in \mathbf{F}_2 = \{0, 1\}$ by the equation $[x, y] = x^{-1}y^{-1}xy = \epsilon^{B(x,y)}$. It is easy to see that $B(x, y)$ depends only on the images of x and y in the commutator quotient group $W := G/G'$. Thus, B gives a pairing $B_G : W \times W \to \mathbf{F}_2$. Here and in the following, we shall view W as an $\mathbf{F}_2$-vector space (of dimension $r = s + t$). Using the fact that $G' \subseteq Z(G)$, a straightforward calculation shows that B_G is an *alternating* $\mathbf{F}_2$-bilinear form on W.

For $x \in G_{s,t}$, define $q(x) \in \mathbf{F}_2 = \{0, 1\}$ by the equation $x^2 = \epsilon^{q(x)}$. Again, $q(x)$ depends only on the image of x in W, so q defines a function $q_G : W \to \mathbf{F}_2$. Since

$$(xy)^2 = x \cdot yx \cdot y = x \cdot xy\epsilon^{B(x,y)} \cdot y = \epsilon^{q(x)}\epsilon^{q(y)}\epsilon^{B(x,y)},$$

it follows that q_G is an $\mathbf{F}_2$-quadratic form on W with B_G as its associated symmetric bilinear form. Using the images of $a_1, \ldots, a_s,\ b_1, \ldots, b_t$ as a basis of W, we can express q_G by the upper triangular matrix with ones above the diagonal, and with s ones and t zeros on the diagonal. In other words, q_G can be coordinatized in the form

$$q_{s,t} = q_{s,t}(x_1, \ldots, x_r) = x_1^2 + \cdots + x_s^2 + \sum_{1 \le i < j \le r} x_i x_j. \tag{4.1}$$

In particular, the four basic groups K, C, D and Q have the associated quadratic forms $0 \cdot x^2,\ x^2,\ xy$ and $x^2 + xy + y^2$ respectively.

The following proposition shows that the isometry class of the quadratic form q_G determines the group G. This is a special case of a result of C.T.C. Wall [**49**, Proposition 10]. Since the proof is fairly short, we include it here for the sake of completeness. (See also the beginning remarks in §4 of [**35**].)

PROPOSITION 4.2. *$G_{s,t}$ is isormorphic to $G_{s',t'}$ if and only if $q_{s,t}$ is isometric to $q_{s',t'}$.*

PROOF. The "only if" part is clear. For the "if" part, let φ be an isometry from $G_{s,t}/G'_{s,t}$ to $G_{s',t'}/G'_{s',t'}$. Choose $c_1, \ldots, c_s$ and $d_1, \ldots, d_t \in G_{s',t'}$ such that $\varphi(a_i G'_{s,t}) = c_i G'_{s',t'}$ and $\varphi(b_j G'_{s,t}) = d_j G'_{s',t'}$ for all i, j. The fact that φ is an isometry implies that $\{c_i, d_j, \epsilon\}$ satisfy the defining relations (1.1) between $\{a_i, b_j, \epsilon\}$. Therefore, $\Phi(a_i) = c_i$, $\Phi(b_j) = d_j$ and $\Phi(\epsilon) = \epsilon$ gives a well-defined homomorphism $\Phi : G_{s,t} \to G_{s',t'}$ lifting φ. If $\ker\Phi \neq \{1\}$, it would contain a nonidentity *central* element of $G_{s,t}$. However, $\Phi(\epsilon) = \epsilon \neq 1$ and $\Phi(\epsilon^k a_1 \cdots a_s\, b_1 \cdots b_t) = \epsilon^k c_1 \cdots c_s\, d_1 \cdots d_t \neq 1$ (unless $k = s = t = 0$). Thus, $\ker\Phi$ must be $\{1\}$, and since $s + t = s' + t'$, $\Phi : G_{s,t} \to G_{s',t'}$ is an isomorphism. □

Let us now recall some basic notions associated with quadratic forms over a field F of characteristic 2. Let (W, q) be a quadratic space over F with the associated alternating bilinear form B. We can associate with (W, q, B) the following two radicals:

$$\text{(4.3)} \qquad \mathrm{rad}_B\, W = \{w \in W :\ B(w, W) = 0\},$$

$$\text{(4.4)} \qquad \mathrm{rad}_q\, W = \{w \in W :\ B(w, W) = q(w) = 0\}.$$

The quotient space $W/\mathrm{rad}_B W$ inherits from B a non-degenerate alternating form, so it has a symplectic basis and is even-dimensional. Following Chevalley [**3**, p.12], we call $\dim_F W/\mathrm{rad}_B W$ the *rank* of q, and $\dim_F \mathrm{rad}_B\, W/\mathrm{rad}_q\, W$ the *defect* of q. In the case when q has defect 0, the *Arf invariant* of q, denoted by $\Delta(q)$, is defined as follows. Since $q\,|\,\mathrm{rad}_B W \equiv 0$, q induces a quadratic form $\overline{q}$ on $W/\mathrm{rad}_B W$. Let $\{\overline{e}_1, \overline{f}_1, \ldots, \overline{e}_n, \overline{f}_n\}$ be a symplectic basis for $W/\mathrm{rad}_B\, W$. Then

$$\text{(4.5)} \qquad \Delta(q) \in F/\{x^2 + x :\ x \in F\}$$

is defined to be $\sum \overline{q}(\overline{e}_j)\overline{q}(\overline{f}_j)$. It can be shown that the class of this value in $F/\{x^2 + x : x \in F\}$ is independent of the choice of the symplectic basis (see [**24**, pp.31-32], [**44**, p.340]), and that if q' is another quadratic form of defect 0, then $\Delta(q \perp q') = \Delta(q) + \Delta(q')$. *If q has positive defect, the Arf invariant of q is undefined.*

We shall only be interested here in the case when $F = \mathbf{F}_2$. In this special case, q is an $\mathbf{F}_2$-linear functional on $\mathrm{rad}_B W$ (since $x^2 = x$ for any $x \in \mathbf{F}_2$), so the defect of q is either 1 or 0. And, in case q has defect 0, the value of $\Delta(q)$ is either 1 or 0 since $\mathbf{F}_2/\{x^2 + x : x \in \mathbf{F}_2\}$ is just $\mathbf{F}_2$. The classification of quadratic forms over $\mathbf{F}_2$ (with possibly non-zero B-radicals) is fairly well-known and is, for instance, nicely explained in [**49**, Theorem 11]: If q has defect 1, then the isometry type of q is determined by its dimension and its rank; if q has defect 0, then the isometry type of q is determined by its dimension, rank and the Arf invariant $\Delta(q) \in \mathbf{F}_2$.

Now, returning to the CLE-groups, our classification of the $G_{s,t}$'s in §2 leads quickly to the determination of all the isometry invariants of the associated forms $q_{s,t}$, as follows.

THEOREM 4.6. *Let $G = G_{s,t}$, and $r = s + t \in \{2i, 2i + 1\}$. Then the associated quadratic form q_G has rank $2i$, and its defect and Arf invariant are determined as in the following chart:*

	$t - s \pmod 8$	Defect	Arf Invariant
(1) (Type I)	0, 1, 2	0	0
(2) (Type II)	4, 5, 6	0	1
(3) (Type III)	3, 7	1	undefined

PROOF. For $W = G/G'$, the two radicals for the quadratic space (W, q, B) ($q = q_G$, $B = B_G$) are clearly

$$\mathrm{rad}_B W = Z(G)/G', \tag{4.7}$$

$$\mathrm{rad}_q W = \{x \in Z(G) : x^2 = 1\}/G'. \tag{4.8}$$

From (2.3), it follows that $\mathrm{rad}_B W$ has dimension 0 or 1 according as $r = 2i$ or $2i + 1$. In either case, the rank of q is $2i$. In Cases (1) and (2), the canonical form of G is D^i, $D^{i-1}K$, D^iQ or $D^{i-1}QK$, and therefore $Z(G)$ is either $\langle\epsilon\rangle$ or $\cong K$. From (4.8), it follows that $\mathrm{rad}_B\, W = \mathrm{rad}_q\, W$, so q has defect 0. In Case (3), $Z(G) \cong C$, so $\dim \mathrm{rad}_B\, W = 1$, $\dim \mathrm{rad}_q\, W = 0$ and therefore q has defect 1.

Finally, to compute the Arf invariant of q, note that if we have an isomorphism $G_1 \dot{\times} G_2 \cong G_3$ where G_i's are CLE-groups, then q_{G_3} is isometric to the orthogonal sum $q_{G_1} \perp q_{G_2}$, and therefore $\Delta(q_{G_3}) = \Delta(q_{G_1}) + \Delta(q_{G_2})$ if q_{G_1} and q_{G_2} both have defect 0. This leads immediately to the calculation of $\Delta(q_{s,t})$ in Case (1) and Case (2) since $q_D \cong xy$ has Arf invariant 0 and $q_Q \cong x^2 + xy + y^2$ has Arf invariant 1. □

REMARK 4.9. Let $\mathbf{H}$ denote the "hyperbolic plane" xy. Then the five canonical forms D^i, $D^{i-1}Q$, D^iK, $D^{i-1}QK$ and D^iC have, respectively, the associated quadratic forms

$$i \cdot \mathbf{H}, \qquad (i-1)\cdot \mathbf{H} \perp (x^2+xy+y^2), \qquad i\cdot\mathbf{H} \perp 0 \cdot z^2,$$

$$(i-1)\cdot\mathbf{H} \perp (x^2+xy+y^2) \perp 0\cdot z^2 \quad \text{and} \quad i\cdot\mathbf{H}\perp z^2.$$

These represent precisely all the possible isometry classes of forms over $\mathbf{F}_2$ of rank $2i$ and dimension $2i$ or $2i+1$. The special group isomorphisms in Corollary 2.8 used for the derivation of the five canonical forms all have natural quadratic form analogues over $\mathbf{F}_2$; we shall leave to the reader the pleasant task of writing down the various form isometries predicted by Corollary 2.8 and finding direct proofs for these isometries.

It is worth noting that the Arf invariants $\Delta(q_{s,t})$ can also be expressed by the following rather symmetrical tables:

(4.10)

$r \backslash s$	0	1	2	3
0	0	1	1	0
2	0	0	1	1
4	1	0	0	1
6	1	1	0	0

$(r = s + t = \text{even})$

$r \backslash s$	0	1	2	3
1	0	*	1	*
3	*	0	*	1
5	1	*	0	*
7	*	1	*	0

$(r = s + t = \text{odd})$

where the values r are tabulated mod 8 and the values of s are tabulated mod 4. ("$*$" means "undefined".)

In the special case of the groups $G_{s,0}$, Wall has given a different calculation of the Arf invariants $\Delta(q_{s,0})$ in [**49**]. In this Wall paper, an explicit symplectic basis is constructed for $G_{s,0}/Z(G_{s,0})$, from which $\Delta(q_{s,0})$ can be computed without difficulty. A moment's reflection shows that Wall's construction of the symplectic basis works also in the (s,t)-setting. In fact, if we relabel $a_1, \ldots, a_s,\ b_1, \ldots, b_t \in G_{s,t}$ as $x_1, \ldots, x_r$, where $r = s + t \in \{2i, 2i+1\}$, then

$$\begin{cases} e_1 = x_1, & e_2 = x_1x_2x_3, \ \ldots\ldots\ldots,\ e_i = x_1 \cdots x_{2i-2}x_{2i-1}, \\ f_1 = x_2, & f_2 = x_1x_2x_4, \ \ldots\ldots\ldots,\ f_i = x_1 \cdots x_{2i-2}x_{2i}, \end{cases}$$

will give a symplectic basis in $G_{s,t}/Z(G_{s,t})$. The calculation of $\sum q(e_j)q(f_j)$ (for $q = q_{s,t}$) is a little messy because of the "transition" from the a_i's to the b_j's, but after the necessary case considerations are made, one gets the following formulas for $\Delta(q_{s,t})$ in the defect 0 cases:
(4.11)

$$\Delta(q_{s,t}) = \begin{cases} [(\sigma-1)^2 - 1]/8 & \textit{if } \sigma := t - s = \textit{even}, \\ (\sigma^2 - 1)/8 & \textit{if } \sigma := t - s = \textit{odd} \not\equiv 3, 7 (\operatorname{mod} 8). \end{cases}$$

It is easy to check that this is indeed consistent with the values of $\Delta(q_{s,t})$ given in (4.6). Also, since the form $q_{s,t}$ is defined over $\mathbf{Z}$ (cf. (4.1)), the computation of $\Delta(q_{s,t})$ can presumably be done also by Wadsworth's formulas in [**48**], at least in the case $r =$ even (when the B-radical is zero).

In closing, we observe that the results in Theorems 3.17 and 3.18 concerning the sizes of the maximal abelian subgroups and the maximal elementary abelian subgroups in $G = G_{s,t}$ are *also* capable of quadratic form theoretic interpretations. In fact, as observed by Quillen [**35**, p.204], the maximal abelian subgroups $A \subseteq G$ correspond to maximal totally B_G-isotropic subspaces $A/G' \subseteq G/G'$, while the maximal elementary abelian subgroups $E \subseteq G$ correspond to maximal totally q_G-isotropic subspaces $E/G' \subseteq G/G'$. Thus, Theorems 3.17 and 3.18 may be interpreted as giving the dimensions of the maximal totally B_G-isotropic and the maximal totally q_G-isotropic subspaces of the quadratic space $(G/G', q_G)$. In the case when this space has a zero B_G-radical, it is also known generally that the maximal totally q_G-isotropic subspaces are conjugate under the action of the orthogonal group $O(q_G)$

(see [**3**, p.17]). For a different computation of the dimension of a maximal totally q_G-isotropic subspace of $(G/G', q_G)$, see [**35**, Corollary 2.7]. Note that this dimension has a cohomological meaning, since Quillen has shown in general that, for any finite p-group G, the maximal rank of an elementary abelian p-subgroup of G is exactly the Krull dimension of the commutative mod p cohomology ring of G (see [**36**]). In the case when G is an extra-special p-group, this cohomology ring was studied in [**35**] for $p = 2$, and in [**46**] and [**30**] for p odd. The Schur multiplicator group of an extra-special p-group was computed by Blackburn and Evens [**2**].

5. Clifford algebra periodicity. In this section, we shall show how the results in §2 can be used to calculate quickly the graded Clifford algebra $C(\varphi_{m,n})$ of the form $\varphi_{m,n} = m\langle -1\rangle \perp n\langle 1\rangle$ over a field F of characteristic not 2. In the following, we shall abbreviate $C(\varphi_{m,n})$ by $C^{m,n}$. Our calculations of these will be in terms of the four basic Clifford algebras $X := C^{1,0} \cong F\langle\sqrt{-1}\rangle$, $Y := C^{2,0} \cong \left\langle\frac{-1,-1}{F}\right\rangle$, $Z := C^{0,1} \cong F\langle\sqrt{1}\rangle$ and $W := C^{0,2} \cong \left\langle\frac{1,1}{F}\right\rangle$, with notations as in [**27**, p.127].

All graded algebras in this section are understood to be $\mathbf{Z}_2$-graded algebras. For two such algebras A and B over F, we can form the usual tensor product $A \otimes_F B$ and the graded tensor product $A \widehat{\otimes}_F B$ (see [**27**, p.77]). Both are graded algebras; however, in general, they are not isomorphic even as ungraded algebras. For two quadratic forms q and q' over F, one has a graded algebra isomorphism

$$C(q \perp q') \cong C(q)\widehat{\otimes}_F C(q') \quad [\mathbf{27}, \text{p.105}];$$

in particular, $C^{m+p,n+q} \cong C^{m,n}\widehat{\otimes}_F C^{p,q}$. The latter isomorphism is the key to the computation of $C^{m,n}$ in [**27**, pp.126-129]. It is, therefore, somewhat suprising that the alternative computation of $C^{m,n}$ below does not depend on this isomorphism, and in fact, *does not depend on using the graded tensor product* $\widehat{\otimes}$ *at all.* To emphasize this point, we shall now drop the notation $\widehat{\otimes}$ altogether: all the $\otimes$'s below shall denote the ordinary tensor product of graded algebras (over F).

Let $G = G_{m,n}$, with its usual generators ϵ, $a_1, \ldots, a_m$ and $b_1, \ldots, b_n$. We write FG for the group algebra of G over F, and let $\overline{FG}$ denote $FG/(\epsilon + 1)$. We shall view this as a *graded* algebra by assigning degree 0 to ϵ and elements of F, and degree 1 to the elements

$a_1, \ldots, a_m,\ b_1, \ldots, b_n$. It is easy to see that $\overline{FG}$ is isomorphic to the graded Clifford algebra $C^{m,n}$.

Note that the definition of $\overline{FG}$ as a graded algebra depends on the specific choice of the generators of G. For instance, although $G_{0,2} \cong G_{1,1}$ as groups (both being isomorphic to the dihedral group D), $\overline{FG_{0,2}}$ and $\overline{FG_{1,1}}$ may not be isomorphic as *graded* algebras. In fact, the even part of $\overline{FG_{0,2}} \cong C^{0,2} \cong \left\langle \frac{(1,1)}{F} \right\rangle$ is $F\langle\sqrt{-1}\rangle$ (as an ungraded algebra), but, according to the following lemma, the even part of $\overline{FG_{1,1}} \cong C^{1,1} \cong \left\langle \frac{(-1,1)}{F} \right\rangle$ is $F \times F$. If $-1 \notin F^2$, these are not isomorphic.

LEMMA 5.1. *We have a graded algebra isomorphism* $\left\langle \frac{(-1,1)}{F} \right\rangle \cong \widehat{\mathbf{M}}_2(F)$. (*For any graded algebra* $A, \widehat{\mathbf{M}}_n(A)$ *denotes the matrix algebra* $\mathbf{M}_n(A)$ *equipped with the check-board grading, see* [**27**, p.81].)

PROOF. By definition, $\left\langle \frac{(-1,1)}{F} \right\rangle \cong (F \cdot 1 \oplus F \cdot k) \oplus (F \cdot i \oplus F \cdot j)$ with $i^2 = -1$, $j^2 = 1$ and $k = ij$. Clearly, $i \mapsto \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, $j \mapsto \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ gives the desired graded algebra isomorphism. □

Now consider the group $G_{s,t} \dot{\times} G_{m,n}$. We shall regard this group as generated by the given generators of $G_{s,t}$ and those of $G_{m,n}$. With this choice of generators, $\overline{F(G_{s,t} \dot{\times} G_{m,n})}$ acquires the structure of a graded algebra. Since the two subalgebras $\overline{FG_{s,t}}$ and $\overline{FG_{m,n}}$ commute elementwise, we see that, as graded algebras,

$$\overline{F(G_{s,t} \dot{\times} G_{m,n})} \cong \overline{FG_{s,t}} \otimes \overline{FG_{m,n}} \cong C^{s,t} \otimes C^{m,n}.$$

Using the computation of $G_{s,t} \dot{\times} G_{m,n}$ in Theorem 2.5, we shall now deduce

THEOREM 5.2. $C^{s,t} \otimes C^{m,n}$ *is isomorphic to the algebras listed in the last column in the six cases below* (*where* $r := s + t$):

	s	$r(\bmod 4)$	$C^{s,t}\otimes C^{m,n}$
(1)	odd	0	$C^{s+n,t+m}$
(2)	even	2	
(3)	odd	2	$C^{s+m,t+n}$
(4)	even	0	
(5)	odd	1	$C^{s,t}\otimes C^{n,m}$
(6)	even	3	

In Cases (1),(2),(3),(4), *we have graded algebra isomorphisms. In Cases* (5),(6), *we have only ungraded algebra isomorphisms.*

PROOF. Since r is even in the first four cases, the group isomorphisms constructed in the proof of Theorem 2.5 preserve the degrees of the generators of $G_{s,t}\dot{\times}G_{m,n}$ and those of $G_{s+n,t+m}$ (respectively $G_{s+m,t+n}$). Thus, the induced algebra isomorphisms are graded algebra isomorphisms. In the cases (5) and (6), however, r is odd, so we only get ungraded algebra isomorphisms. □

COROLLARY 5.3. (1) (DIAGONAL LAW) $C^{m+1,n+1}\cong\widehat{\mathbf{M}}_2(C^{m,n})$.
(2) $C^{n,n}\cong\widehat{\mathbf{M}}_{2^n}(F)$.

PROOF. By Case (3) of Theorem 5.2, $C^{m+1,n+1}\cong C^{1,1}\otimes C^{m,n}$ as graded algebras. By Lemma 5.1, we therefore have $C^{m+1,n+1}\cong\widehat{\mathbf{M}}_2(F)\otimes C^{m,n}$. This gives (1) since $\widehat{\mathbf{M}}_p(F)\otimes A\cong\widehat{\mathbf{M}}_p(A)$ for any graded algebra A. (2) follows from (1) since $\widehat{\mathbf{M}}_p(\widehat{\mathbf{M}}_q(A))\cong\widehat{\mathbf{M}}_{pq}(A)$. □

(Here, and in the following, all isomorphisms are graded algebra isomorphisms.)

THEOREM 5.4. (1) $C^{4,0}\cong C^{0,4}$.
(2) (*Periodicity* mod 8) $C^{m+8,n}\cong\widehat{\mathbf{M}}_{16}(C^{m,n})\cong C^{m,n+8}$.

PROOF. (1) Let $G_{4,0} = \langle \epsilon, a_1, \ldots, a_4 \rangle$ and $G_{0,4} = \langle \epsilon, b_1, \ldots, b_4 \rangle$. We have an explicit group isomorphism $\varphi : G_{0,4} \to G_{4,0}$ by $\varphi(\epsilon) = \epsilon$, $\varphi(b_i) = za_i$, where $z := a_1a_2a_3a_4$. Since φ preserves degrees, we get a graded algebra isomorphism $\overline{FG_{0,4}} \cong \overline{FG_{4,0}}$, and so $C^{0,4} \cong C^{4,0}$. For (2), first note that

$$
\begin{aligned}
C^{8,0} &\cong C^{4,0} \otimes C^{4,0} && \text{(by Theorem 5.2 (4))}\\
&\cong C^{4,0} \otimes C^{0,4} && \text{(by (1) above)}\\
&\cong C^{4,4} && \text{(again by Theorem 5.2 (4))}\\
&\cong \widehat{\mathbf{M}}_{16}(F) && \text{(by Corollary 5.3 (2)).}
\end{aligned}
$$

Therefore, by Theorem 5.2 (4) once more, $C^{m+8,n} \cong C^{8,0} \otimes C^{m,n} \cong \widehat{\mathbf{M}}_{16}(F) \otimes C^{m,n} \cong \widehat{\mathbf{M}}_{16}(C^{m,n})$, and similarly for $C^{m,n+8}$. □

In view of the Diagonal Law 5.3 (1) and the Periodicity Law 5.4 (2), the computation of $C^{m,n}$ is reduced to the cases $C^{n,0}$ and $C^{0,n}$ for $n \leq 7$. We shall now compute these in terms of X, Y, Z and W (defined at the beginning of this section), as follows (cf. [**27**, p.128]):

n	0	1	2	3
$C^{n,0}$	F	X	Y	$Y \otimes Z$
$C^{0,n}$	F	Z	W	$X \otimes W$

n	4	5	6	7
$C^{n,0}$	$Y \otimes W$	$\widehat{\mathbf{M}}_2(X \otimes W)$	$\widehat{\mathbf{M}}_4(W)$	$\widehat{\mathbf{M}}_8(Z)$
$C^{0,n}$	$Y \otimes W$	$\widehat{\mathbf{M}}_2(Y \otimes Z)$	$\widehat{\mathbf{M}}_4(Y)$	$\widehat{\mathbf{M}}_8(X)$

In fact, by Theorem 5.2 (4), $C^{p+2,0} \cong C^{2,0} \otimes C^{0,p} \cong Y \otimes C^{0,p}$. This computes $C^{3,0}$ and $C^{4,0}$. Similarly, $C^{0,p+2} \cong C^{0,2} \otimes C^{p,0} \cong W \otimes C^{p,0}$; this computes $C^{0,3}$ and $C^{0,4}$. Finally, for $p \leq 4$, we have

$$
\begin{aligned}
C^{p+4,0} &\cong C^{4,0} \otimes C^{p,0} && \text{(by Theorem 5.2 (4))}\\
&\cong C^{0,4} \otimes C^{p,0} && \text{(by Theorem 5.4 (1))}\\
&\cong C^{p,4} && \text{(again by Theorem 5.2 (4))}\\
&\cong \widehat{\mathbf{M}}_{2^p}(C^{0,4-p}) && \text{(by Corollary 5.3 (1)).}
\end{aligned}
$$

For $p = 1,2,3$, this computes $C^{5,0}, C^{6,0}, C^{7,0}$, and the graded algebras $C^{0,5}, C^{0,6}, C^{0,7}$ are computed similarly, by using $C^{0,p+4} \cong \widehat{\mathbf{M}}_{2^p}(C^{4-p,0})$. □

The computations of the $C^{m,n}$'s as ungraded algebras now follow from the above as in [**27**, pp.128-129].

To conclude this final section, we shall now say a few words about the possible generalizations of the groups $G_{s,t}$ and the Clifford algebras $C^{s,t}$. As early as 1934, Littlewood [**28**] had already observed that one can extend the definition of the $G_{s,t}$'s to get an analogous class of finite p-groups for any prime p. This class is slightly larger than the class of extra-special p-groups, and is susceptible to the treatment given in the present paper. A further generalization can be made by replacing the role of the prime p by that of an arbitrary integer $n \geq 2$. The groups G which arise out of this generalization will have generators c_i's $(1 \leq i \leq r)$ with the relations $c_i^n = \omega^{k(i)}$ and $c_i c_j = \omega c_j c_i$ whenever $i < j$, where ω is a (fixed) central element such that $\omega^n = 1$. If F is any field which contains a primitive n^{th} root of unity ω_0, we can then define as before a finite dimensional F-algebra $\overline{FG} := FG/(\omega - \omega_0)$. To see the connection between these algebras and the $C^{s,t}$'s, let us look at the case when n is *even*, say $n = 2m$, and assume that the first s of the $k(i)$'s are equal to m, and the reamining t of the $k(i)$'s are zero, where $s + t = r$. Renaming $c_1, \ldots, c_r$ as $a_1, \ldots, a_s, b_1, \ldots, b_t$, the algebra $\overline{FG}$ then has generators $\overline{a}_1, \ldots, \overline{a}_s$ and $\overline{b}_1, \ldots, \overline{b}_t$ with relations $\overline{a}_i^n = \omega_0^m = -1$, $\overline{b}_j^n = 1$ and $\overline{a}_i \overline{a}_j = \omega_0 \overline{a}_j \overline{a}_i$ (for any $i < j$), $\ldots$, etc. This is precisely the "generalized Clifford algebra" studied by Yamazaki [**52**], Morris [**31**, **32**] and Popovici-Ghéorghe [**34**]. (For $n = 2$ and with $\omega_0 = -1 \neq 1$, we get back the classical Clifford algebras $C^{s,t}$.) The generalized groups G and their representations can be investigated by using techniques similar to those used in this paper, so that one can prove again the existence of a periodicity law modulo 8 and ascertain the explicit structure of the groups G as well as that of the generalized Clifford algebras $\overline{FG}$, in particular recapturing and clarifying the results of Morris in [**31**, **32**]. The details of this will be reported in [**45**]. Finally, we might add that commutation relations of the kind $\overline{a}_i \overline{a}_j = \omega_0 \overline{a}_j \overline{a}_i$ (where ω_0 is a primitive n^{th} root of unity) are also of significance to physicists. Indeed, a considerable number of papers written on generalized Clifford algebras seem to have physical motivations. Since a collection of some of these papers has appeared in the book by Ramakrishnan [**38**], we shall refer the reader to this work for a list of the relevant papers, and also for an exposition of how ω_0-commutation relations and generalized Clifford algebras are used in physics.

Note added June, 1987. After the completion of this paper, we were informed by Professors A. Hahn and A.O. Morris that a closely related work by H.W. Braden ("N-dimensional spinors: Their properties in terms of finite groups") has appeared in J. Math. Physics **26** (1985), 613-620. In this work, Braden also described completely the decomposition theory of the groups $G_{s,t}$ and discussed their real and complex representations. Braden obtained his explicit decompositions of $G = G_{s,t}$ by working with the quadratic form $q_{s,t}$ on G/G', so his methods were close in spirit to those of Wall [**49**]. We feel, however, that our methods are more elementary, and that they led to a quicker proof and a better view of the periodicity of the groups $G_{s,t}$. Since, also, various other results and viewpoints in our paper are not covered by those of Braden, we believe it is still best to publish this work in its original form.

REFERENCES

1. M. Atiyah, R. Bott and A. Shapiro, *Clifford modules*, Topology **3** (1964), 3-38.

2. N. Blackburn and L. Evens, *Schur multipliers of p-groups*, J. reine angew. Math. **309** (1979), 100-113.

3. C. Chevalley, *The Algebraic Theory of Spinors*, Columbia Univ. Press, 1954.

4. W.K. Clifford, *Mathematical Papers*, Macmillan, London, 1882.

5. J.F. Cornwell, *Group Theory in Physics*, Vol. 2, Academic Press, New York, 1984.

6. H.S.M. Coxeter, *The Dirac matrix group and other generalizations of the quaternion group*, Communications on Pure and Applied Mathematics **26** (1973), 693-698.

7. C.W. Curtis and I. Reiner, *Representation Theory of Finite Groups and Associative Algebras*, Interscience Publishers, J. Wiley & Sons, New York, 1962.

8. P.A.M. Dirac, *The quantum theory of the electron*, Proc. Royal Soc. (London) **117** (1928), 610-624.

9. L. Dornhoff, *Group Representation Theory*, Part A, Marcel Dekker, Inc., New York, 1971.

10. B. Eckmann, *Gruppentheoretischer Beweis des Satzes von Hurwitz-Radon über die Komposition der quadratischen Formen*, Comment. Math. Helv. **15** (1942/43), 358-366.

11. A.S. Eddington, *On sets of anticommuting matrices*, J. London Math. Soc. **7** (1932), 58-68.

12. ———, *On sets of anticommuting matrices*, Part II: *The Foundations of E-numbers*, J. London Math. Soc. **8** (1933), 142-152.

13. ———, *New Pathways in Science*, Cambridge Univ. Press, 1935.

14. M.R. Gabel, *Generic orthogonal stably free projectives*, J. Alg. **29** (1974), 477-488.

15. A.V. Geramita and N.J. Pullman, *A theorem of Hurwitz and Radon and orthogonal projective modules*, Proc. Amer. Math. Soc. **42** (1974), 51-56.

16. ——— and J. Seberry, *Orthogonal Designs*, Lecture Notes in Pure and Applied Math., Vol. **45**, Marcel Dekker, Inc., N.Y., 1979.

17. L. Gårding and A. Wightman, *Representations of the anticommutation relations*, Proc. Natl. Acad. Sci. U.S.A. **40** (1954), 617-621.

18. ——— and ———, *Representations of the commutation relations*, Proc. Natl. Acad. Sci. U.S.A. **40** (1954), 622-626.

19. G. Haenzel, *Die Diracsche Wellengleichung und das Ikosaeder*, J. reine angew. Math. **183** (1941), 232-242.

20. A. Hurwitz, Über die Komposition der quadratischen Formen von beliebig vielen Variabeln, Nach. Ges. Wiss. Göttingen, Math.-Phys. Kl. (1898), 309-316, reprinted in Math. Werke II, 565-571.

21. ———, *Über die Komposition der quadratischen Formen*, Math. Ann. **88** (1923), 1-25, reprinted in Math. Werke II, 641-666.

22. P. Jordan and E. Wigner, *Über das Paulische Äquivalenzverbot*, Z. für Physik **47** (1928), 631-651.

23. ———, J. von Neumann and E. Wigner, *On an algebraic generalization of the quantum mechanical formalism*, Ann. Math. **35** (1934), 29-64.

24. I. Kaplansky, *Linear Algebra and Geometry*, Allyn and Bacon, Boston, 1969.

25. Y. Kawada and N. Iwahori, *On the structure and representations of Clifford algebras*, J. Math. Soc. Japan **2** (1950), 34-43.

26. K.Y. Lam, *Non-singular bilinear forms and vector bundles over P^n*, Doctoral Dissertation, Princeton Univ., 1966.

27. T.Y. Lam, *The Algebraic Theory of Quadratic Forms*, Benjamin, Addison-Wesley, 1973 (Revised Printing: 1980).

28. D.E. Littlewood, *Note on the anticommuting matrices of Eddington*, J. London Math. Soc. **9** (1934), 41-50.

29. J.S. Lomont, *Applications of Finite Groups*, Academic Press, N.Y., 1959.

30. P.A. Minh, *Modular invariant theory and cohomology algebras of extra-special p-groups*, Pacific J. Math. **124** (1986), 345-363.

31. A.O. Morris, *On a generalized Clifford algebra*, Quart. J. Math. Oxford **18** (1967), 7-12.

32. ———, *On a generalized Clifford algebra*, II, Quart. J. Math. Oxford **19** (1968), 289-299.

33. M.H.A. Newman, *Note on an algebraic theorem of Eddington*, J. London Math. Soc. **7** (1932), 93-99. [Corrigenda *ibid.* **7** (1932), 272.]

34. I. Popovici and C. Ghéorghe, *Algèbres de Clifford généralisées*, C. R. Acad. Sci. Paris, Sér. A-B **262** (1966), 682-685.

35. D. Quillen, *The* mod 2 *cohomology ring of extra-special* 2*-groups and the spinor groups*, Math. Ann. **194** (1971), 197-212.

36. ———, *The spectrum of an equivariant cohomology ring*, I, II, Ann. Math. **94** (1971), 549-572 and 573-602.

37. J. Radon, *Lineare Scharen orthogonaler Matrizen*, Abh. Math. Sem. Univ. Hamburg **1** (1922), 1-14.

38. A. Ramakrishnan, *L-matrix Theory or the Grammar of Dirac Matrices*, Tata McGraw-Hill Publishing Co., Bombay-New Delhi, 1972.

39. D. Robinson, *A Course in the Theory of Groups*, Graduate Texts in Math., Vol. **80**, Springer-Verlag, Berlin-Heidelberg-New York, 1982.

40. J.-P. Serre, *Linear Representations of Finite Groups*, Graduate Texts in Math., Vol. **42**, Springer-Verlag, Berlin-Heidelberg-New York, 1977.

41. D. Shapiro, *Spaces of similarities*, I: *the Hurwitz Problem*, J. Alg. **46** (1977), 148-170.

42. ———, *Spaces of similarities*, IV: (s,t)-families, Pacific. J. Math. **69** (1977), 223-244.

43. ———, *Products of sums of squares*, Expo. Math. **2** (1984), 235-261.

44. W. Scharlau, *Quadratic and Hermitian Forms*, Springer-Verlag, Berlin-Heidelberg-New York, 1985.

45. T. Smith, *Some* 2*-groups arising in quadratic form theory and their generalizations*, Doctoral dissertation, Univ. of Calif., Berkeley, 1988.

46. M. Tezuka and N. Yagita, *The varieties of the* mod *p cohomology rings of extra-special p-groups for an odd prime p*, Proc. Camb. Phil. Soc. **94** (1983), 449-459.

47. O. Veblen, *Geometry of four-component spinors*, Proc. Natl. Acad. Sci. U.S.A. **19** (1933), 503-517.

48. A. Wadsworth, *Discriminants in characteristic two*, Lin. and Multilin. Alg. **17** (1985), 235-263.

49. C.T.C. Wall, *On groups consisting mostly of involutions*, Proc. Camb. Phil. Soc. **67** (1970), 251-262.

50. H. Weyl, *The Theory of Groups and Quantum Mechanics*, Dover Publications, Inc., New York. (Translated by H.P. Robertson from "Gruppentheorie und Quantenmechanik", 1931.)

51. W. Wolfe, *Amicable orthogonal designs–Existence*, Canad. J. Math. **28** (1976), 1006-1020.

52. K. Yamazaki, *On projective representations and ring extensions of finite groups*, J. Fac. Sci. Univ. Tokyo, Sect. I, **10** (1964), 147-195.

53. S. Yuzvinsky, *A series of monomial pairings*, Lin. and Multilin. Alg. **15** (1984), 109-119.

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF CALIFORNIA, BERKELEY, CA 94720