ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 19, Number 3, Summer 1989

ORTHOGONAL DECOMPOSITIONS OF INDEFINITE QUADRATIC FORMS

DONALD G. JAMES

Introduction. A well known theorem of Milnor (see [**8**] or [**9**]) classifies the unimodular indefinite quadratic forms over **Z**. Either the form represents both even and odd numbers, in which case the form diagonalizes as $\langle \pm 1, \dots, \pm 1\rangle$; or the form only represents even numbers, in which case it decomposes into an orthogonal sum of hyperbolic planes and 8-dimensional unimodular definite forms. We give here some generalizations of this theorem for indefinite forms with square free discriminant of rank at least three.

Let L be a **Z**-lattice on an indefinite regular quadratic **Q**-space V of finite dimension $n \geq 3$ with associated symmetric bilinear form $f : V \times V \to \mathbf{Q}$. Assume, for convenience, that $f(L, L) = \mathbf{Z}$ and that the signature $s = s(L)$ of the form is non-negative. Let $x_1, \dots, x_n$ be a **Z**-basis for L and put $d = dL = \det f(x_i, x_j)$, the discriminant of the lattice L. We assume that d is square free. Let $\langle a_1, \dots, a_n\rangle$ denote the **Z**-lattice $\mathbf{Z}x_1 \perp \cdots \perp \mathbf{Z}x_n$ with an orthogonal basis where $f(x_i) = f(x_i, x_i) = a_i$, $1 \leq i \leq n$. Most of our notation follows O'Meara [**7**]. Thus L_p denotes the localization of L at the prime p.

The lattice L is called *even* if $f(x) \in 2\mathbf{Z}$ for all $x \in L$; otherwise the lattice is *odd.* The condition that L is an odd lattice is equivalent to the local condition that L_2 diagonalizes over the 2-adic integers (since d is not divisible by 4).

Odd lattices. While not all odd indefinite lattices have an orthogonal basis, we can get very close to this.

THEOREM 1. *Let L be an odd indefinite* **Z***-lattice of rank $n \geq 3$ with square free discriminant d. Then*

$$L = \langle \pm 1, \dots, \pm 1\rangle \perp B,$$

This research was partially supported by the National Science Foundation.

where B is a binary lattice. Moreover, if d is even, then B can be chosen to be definite or indefinite.

PROOF. It suffices, by induction, to prove that L represents both 1 and -1, for then we can split off a one-dimensional orthogonal component, although we must be careful that the other component remains odd. Also, it is enough to consider $n = 3$. Since d is square free, by Kneser [**6**], the genus and the class of L coincide. Thus it remains to show that the localization L_p represents both 1 and -1 for each prime p. This is clear for the odd primes, since the Jordan form of L_p has a unimodular component of rank at least two. It remains to consider L_2.

Assume first that d is even. Then

$$L_2 = \mathbf{Z}_2 x_1 \perp \mathbf{Z}_2 x_2 \perp \mathbf{Z}_2 x_3 = \langle \epsilon_1, \epsilon_2, 2\epsilon_3 \rangle$$

with ϵ_i all 2-adic units. Thus $f(x_1) = \epsilon_1 \equiv 1 \bmod 2$. We can choose a_3 equal to 0 or 1 such that $f(x_1 + a_3 x_3) \equiv \pm 1 \bmod 4$ (both choices of sign are possible). Finally, choose a_2 equal to 0 or 1 so that $f(x_1 + 2a_2 x_2 + a_3 x_3) \equiv \pm 1 \bmod 8$. By Hensel's Lemma, L_2 now represents both 1 and -1. Note that the orthogonal complement of $x_1 + 2a_2 x_2 + a_3 x_3$ in L_2 is not an even lattice and hence there is no problem in proceeding by induction. The choice of sign in ± 1 is made to ensure the complement remains indefinite (except at the last step).

Now assume that d is odd. Then either $L_2 = \langle \epsilon \rangle \perp H$ with H a hyperbolic plane, or

$$L_2 = \mathbf{Z}_2 x_1 \perp (\mathbf{Z}_2 x_2 + \mathbf{Z}_2 x_3) = \langle \epsilon \rangle \perp \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$$

(see O'Meara [**7**; §93]). In the first case L_2 clearly represents both 1 and -1. For the second case, the unit ϵ can be changed by a square and we may assume $\epsilon \in \{\pm 1, \pm 3\}$. If $\epsilon = 3$, then $f(x_1 + x_2 + x_3) \equiv 1 \bmod 8$ and L_2 represents 1. If $\epsilon = -3$, then $f(x_1 + x_2) = -1$ and L_2 represents -1. Thus L_2 always represents either 1 or -1, and if $\epsilon = \pm 1$ then L_2 represents both 1 and -1. This would complete the proof except that in the induction step from $n = 4$ to $n = 3$ it is necessary for L_2 to represent both 1 and -1. However, for $n = 4$, since $\langle 3, 3 \rangle \cong \langle -1, -1 \rangle$, the lattice $\langle 3, 3 \rangle \perp \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$ represents both 1 and -1. □

REMARK. The problem of determining for which values of the discriminant $d = dL$ the odd lattice L always has an orthogonal basis is studied in [**5**]. The answer is complicated and depends on the factorization of d and on the Legendre symbols of some of the prime factors. For example, if $|d|$ is a prime $q \equiv 3 \bmod 4$, then L can always be diagonalized, while if $|d|$ is a prime $p \equiv 1 \bmod 4$, then L cannot always be diagonalized.

Even lattices. We now study even lattices with square free discriminants and try to construct them as far as possible from hyperbolic planes H and the eight dimensional even definite unimodular form E_8. Recall we are assuming that the signature s is non-negative.

THEOREM 2. *Let L be an even indefinite $\mathbf{Z}$-lattice of rank $n \geq 3$ with square free discriminant d. Then*

$$L = H \perp \cdots \perp H \perp E_8 \perp \cdots \perp E_8 \perp M,$$

where $\operatorname{rank} M \leq 7$. *Moreover, if d is odd, then* $\operatorname{rank} M \leq 6$.

PROOF. The idea for this proof was suggested by John Hsia. We may assume $n \geq 5$. Then L is isotropic and is split by a hyperbolic plane H (see [**4**, p. 18]). Hence $L = H \perp \cdots \perp H \perp L'$, where either $\operatorname{rank} L' \leq 4$ and we are finished, or L' is positive definite. We need only consider $\operatorname{rank} L' \geq 8$. It now suffices to show that E_8 is an orthogonal summand of $H \perp L'$. Moreover, since d is square free, the class and genus coincide and it suffices to establish this locally. For the prime 2 the localization of E_8 is the sum of four hyperbolic planes, while L'_2 must contain at least three hyperbolic planes. Hence $(H \perp L')_2$ is split by $(E_8)_2$. For odd primes the localization of E_8 is $\langle 1, 1, \ldots, 1 \rangle$ which clearly splits the localization $(H \perp L')_p = \langle 1, 1, \ldots, 1, \epsilon, \epsilon d \rangle$ (ϵ some local unit). This completes the proof, for if d is odd then L_2 must have even rank. □

REMARK. Theorems 1 and 2 are sharper results for forms with square free discriminants of the general results of Watson [**10**] and Gerstein [**3**] on the orthogonal decomposition of indefinite forms. Our bounds on rank M are, in general, best possible. For odd d, $L = H \perp E_6$ is an example with $d = -3$. For even d take $L = \begin{pmatrix} 10 & 1 \\ 1 & -2 \end{pmatrix} \perp E_7$ so that $s = 7$ and $d = -42$; that $L \neq E_8 \perp \langle -42 \rangle$ follows from computing Hasse symbols at 3. See also the remarks following Theorem 3.

For special classes of d the component M in Theorem 2 can be more specifically described. For $|d| = 2q$ with $q \equiv 3 \bmod 4$ a prime, this was done in [**4**]. We consider now the case where $|d|$ is a prime.

THEOREM 3. *Let L be an even indefinite $\mathbf{Z}$-lattice with rank $n \geq 3$, signature $s \geq 0$ and discriminant $\pm q$, where $q \equiv 3 \bmod 4$ is a prime. Then n is even and $s \equiv 2 \bmod 4$. Moreover, for each compatible choice of n and s, there exists a unique such $\mathbf{Z}$-lattice L with an orthogonal decomposition as in Theorem 2 with $M = \begin{pmatrix} \frac{1}{2}(q+1) & q \\ q & 2q \end{pmatrix}$ when $s \equiv 2 \bmod 8$, and $M = E_6^q$ a definite even lattice of rank 6 and discriminant q when $s \equiv 6 \bmod 8$.*

PROOF. Let L be an even indefinite $\mathbf{Z}$-lattice with discriminant $\pm q$. Then $N = L \perp \mathbf{Z}x$ where $f(x) = -1$ is an odd lattice and, by [**5**], must diagonalize as $\mathbf{Z}x_1 \perp \cdots \perp \mathbf{Z}x_{n+1} = \langle \pm 1, \dots, \pm 1, \pm q \rangle$. Let $x = \sum a_i x_i$. Then all the coefficients a_i must be odd integers since the orthogonal complement of x in N is the even lattice L. Hence

$$-1 = f(x) = \sum a_i^2 f(x_i) \equiv \sum f(x_i) \equiv \pm 1 \pm \cdots \pm 1 \pm q \bmod 8.$$

Since the signature $s(N) = s(L) - 1 = s - 1$, it follows that $s \equiv 2 \bmod 4$. This fact also follows from the general results of Chang [**2**].

We now construct even indefinite lattices with discriminant $\pm q$ and even rank for each possible signature $s \equiv 2 \bmod 4$. If $s = 2 + 8r$ take

$$L = H \perp \cdots \perp H \perp E_8 \perp \cdots \perp E_8 \perp \begin{pmatrix} \frac{1}{2}(q+1) & q \\ q & 2q \end{pmatrix}$$

with r copies of E_8 and the number of hyperbolic planes chosen to match the required even rank.

Next consider $s \equiv 6 \bmod 8$ and $q \equiv 3 \bmod 8$. Let

$$N = \perp \mathbf{Z}x_i = \langle 1, 1, 1, 1, 1, -1, q \rangle$$

and put $z = a_1x_1 + a_2x_2 + a_3x_3 + x_4 + x_5 + qx_6 + x_7$ so that $f(z) = a_1^2 + a_2^2 + a_3^2 - q^2 + q + 2$. By Gauss' Three Square Theorem, there exist odd integers a_i such that $f(z) = -1$. Let E_6^q be the orthogonal complement of z in N. Then E_6^q is an even definite lattice with rank 6 and discriminant q (since any vector orthogonal to z must have an even number of its coefficients odd, and hence have even length). By adjoining copies of H and E_8 to E_6^q we obtain lattices of all possible ranks and signatures in this case. For $s \equiv 6 \bmod 8$ and $q \equiv 7 \bmod 8$ a similar argument can be used to obtain E_6^q by starting with $N = \langle 1,1,1,1,1,1,-q\rangle$.

Only the uniqueness remains to be shown. Now let N be an even indefinite $\mathbf{Z}$-lattice with the same rank n, signature s and discriminant as L. Then $dN = dL = (-1)^{(n-s)/2}q$. Locally at the prime 2 we have either $N_2 = H \perp \cdots \perp H$ or $N_2 = H \perp \cdots \perp H \perp \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$. In the first case $dN_2 \equiv (-1)^{n/2} \equiv \pm q \bmod 8$ and hence $q \equiv 7 \bmod 8$, while in the second case $dN_2 \equiv -3(-1)^{n/2} \equiv \pm q \bmod 8$ and $q \equiv 3 \bmod 8$. Thus, locally, $N_2 \cong L_2$, and hence the Hasse symbols S_2N and S_2L are equal. For all odd primes $p \neq q$ we have $S_pN = S_pL = 1$. At the infinite prime we already know $S_\infty N = S_\infty L$ since the signatures match. By Hilbert Reciprocity it follows that $S_qN = S_qL$. Thus N and L can be viewed as lying on the same quadratic space over $\mathbf{Q}$. Finally, N and L are locally isometric at all primes and hence globally isometric. □

REMARKS. (i) Theorem 3 can be strengthened if we assume the lattice L has Witt index $i(L) \geq 2$. Then, from uniqueness,

$$H \perp H \perp E_6^q \cong E_8 \perp \begin{pmatrix} -\frac{1}{2}(q+1) & q \\ q & -2q \end{pmatrix}$$

since these two even lattices have the same rank, signature and discriminant. Then, in Theorem 3, we have

$$L = H \perp \cdots \perp H \perp E_8 \perp \cdots \perp E_8 \perp \begin{pmatrix} \pm\frac{1}{2}(q+1) & q \\ q & \pm 2q \end{pmatrix}.$$

(ii) In general, for Theorem 2, if $\operatorname{rank} M \geq 5$ and M is indefinite, we can split another hyperbolic plane from M and reduce the rank of M by two. If $\operatorname{rank} M \geq 6$, M is definite and $i(L) \geq 2$, then $H \perp H \perp M$ is split by E_8. Thus, for $i(L) \geq 2$ in Theorem 2, L has a splitting of the type

$$L = H \perp \cdots \perp H \perp E_8 \perp \cdots \perp E_8 \perp M$$

with rank $M \leq 5$ if d is even, and rank $M \leq 4$ if d is odd. If d is even and $i(L) \geq 3$, we can further strengthen this to rank $M \leq 3$.

(iii) For $|d| \equiv 1 \bmod 4$, Satz 2 in Chang [**2**] implies that $s \equiv 0 \bmod 4$. Hence, for this case, we again have rank $M \leq 4$ in Theorem 2. If, moreover, $|d| = p$ is prime, then Theorem 3 has an analogue with $M = \begin{bmatrix} \frac{1}{2}(p-1) & p \\ p & 2p \end{bmatrix}$ if $s \equiv 0 \bmod 8$, and M an even definite lattice of rank 4 and discriminant p when $s \equiv 4 \bmod 8$.

(iv) For $|d| = 2$ we have, from [**2**], that n is odd and $s \equiv \pm 1 \bmod 8$. In Theorem 2 we can now take $M = \langle \pm 2 \rangle$.

REFERENCES

1. J.W.S. Cassels, *Rational Quadratic Forms*, Academic Press, London, 1978.

2. K.S. Chang, *Diskriminanten und Signaturen gerader quadratischer Formen*, Arch. Math. **21** (1970), 59-65.

3. L.J. Gerstein, *Orthogonal splitting and class numbers of quadratic forms*, J. Number Theory **5** (1973), 332-338.

4. D.G. James, *Even indefinite quadratic forms of determinant* $\pm 2p$, A Spectrum of Mathematics, Oxford University Press, 1971, 17-21.

5. ———, *Diagonalizable indefinite integral quadratic forms*, Acta Arith. **50** (1988), 309-314.

6. M. Kneser, *Klassenzahlen indefiniter quadratisher Formen in drei oder mehr Veranderlichen*, Arch. Math. **7** (1956), 323-332.

7. O.T. O'Meara, *Introduction to Quadratic Forms*, Springer-Verlag, New York, 1963.

8. J.P. Serre, *A Course in Arithmetic*, Springer-Verlag, New York, 1973.

9. C.T.C. Wall, *On the orthogonal groups of unimodular quadratic forms*, Math. Ann. **147** (1962), 328-338.

10. G.L. Watson, *Integral Quadratic Forms*, Cambridge University Press, 1960.

DEPARTMENT OF MATHEMATICS, THE PENNSYLVANIA STATE UNIVERSITY, UNIVERSITY PARK, PA 16802