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EVEN POSITIVE DEFINITE UNIMODULAR QUADRATIC FORMS OVER REAL QUADRATIC FIELDS

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In spite of the numberous connections between even positive definite unimodular quadratic forms (henceforth referred to as even unimodular lattices) over $\mathbf{Q}$ with other subjects (e.g., finite group theory, geometry of numbers, combinatorial coding and design theories, automorphic functions, the explicit classification of these lattices has only been fully determined for a few cases, the most celebrated of them being undoubtedly the Leech-Niemeier-Witt [**4**] solution for the 24-dimensional $\mathbf{Z}$-lattices. We discuss here some results on the classification of even unimodular lattices over some real quadratic number fields.

Let $F = \mathbf{Q}(\sqrt{p}), p$ a prime, $R = \text{int}(F), d = d_F$ the field discriminant, and $e = e_F$ the fundamental unit. Let the rank of such an R-lattice be m. Then, m is even. If $p \equiv 3(\bmod 4)$, write $p = -1+4t, t > 0$. Clearly,

$$\begin{bmatrix} 2 & \sqrt{p} \\ \sqrt{p} & 2t \end{bmatrix}$$

defines a binary even unimodular lattice over F. On the other hand, if $p \equiv 1(\bmod 4)$ then it is not difficult to see from the local dyadic structures of the lattices that m must satisfy $m \equiv 0(\bmod 4)$. The same holds for $p = 2$.

I. Analytic mass formula. One way to get a crude estimate for the class number of such lattices is via Siegel's anlytic mass formula. Let $M_m(F)$ be the Minkowski-Siegel mass of the genus of an R-lattice over F of rank $m \equiv 0(\bmod 2)$ and determinant $+1$. One then has the following formula whose proof is analogous to that given in [**1**] for $\mathbf{Q}(\sqrt{5})$.

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LEMMA I.1.

$$M_m(F) = \frac{4^{1-m} L_F(m/2, \chi_m) \prod_{i=1}^{m/2-1} \zeta_F(2i)}{(\sqrt{d_F})^{-m(m-1)/2} \prod_{i=1}^{m} \pi^i \Gamma^{-2}(i/2)},$$

where

$$(1.1) \quad \chi_m(p) = \left(\frac{-1}{p}\right)^{m/2}, \qquad L_F(s, \chi_m) = \prod_p (1 - \chi_m(p) N p^{-s})^{-1}$$

and $\zeta_F(.)$ *is the Dedekind zeta function.*

Thus, for the fields $F = \mathbf{Q}(\sqrt{2}), \mathbf{Q}(\sqrt{5})$ we have a table:

(1.2) m	4	8	12	16
$M_m(\mathbf{Q}(\sqrt{2}))$	$4.3(10^{-4})$	$3.9(10^{-6})$	7.09	$> 10^{18}$
$M_m(\mathbf{Q}(\sqrt{5}))$	$6.9(10^{-5})$	$3.8(10^{-9})$	$1.15(10^{-6})$	$> 10^{6}$

Since the class number h_m satisfies the obvious inequality

$$h_m \geq 2M_m(F),$$

it is clear from (1.2) that explicit algebraic classification via enumeration for these two fields is "feasible" only for $m \leq 8, 12$ respectively for $F = \mathbf{Q}(\sqrt{2}), \mathbf{Q}(\sqrt{5})$.

II. Quadratic forms over $\mathbf{Q}(\sqrt{5})$. When $F = \mathbf{Q}(\sqrt{5})$, Maass [**2**] showed that $h_4 = 1$. The lattice, which we denote by F_4, may be identified with the unique (up to conjugacy class) maximal order in the quaternion algebra over F which is unramified everywhere at finite prime spots. Here the quadratic map is given by twice the reduced norm. Maass shows that $2F_4$ and E_8 are the only 8-dimensional lattices. For $m = 12$ we have the following classification – in the spirit of Leech-Niemeier-Witt:

THEOREM II.1. [**1**] (i) *There exist exactly* 15 *inequivalent classes of even unimodular lattices of dimension* 12 *over* $\mathbf{Q}(\sqrt{5})$ *which are* *distinguished by their root systems listed below (in order of increasing Coxeter numbers):*

$$\emptyset, 12A_1, 6A_2, 4A_3, 2A_4 + 2F_2, 6F_2, 3D_4, 2A_6, A_9 + F_3$$
$$D_6 + 2F_3, 4F_3, 2E_6, D_{12}, E_8 + F_4, \text{ and } 3F_4.$$

(ii) *For each dimension* $m \geq 12, m \equiv 0 (\bmod 4)$, *there is an even unimodular lattice over* $\mathbf{Q}(\sqrt{5})$ *which has an empty root system. In dimension* 12, *there is, up to isometry, a unique such lattice* $\mathcal{L}$ *whose automorphism group has order* $2^{10}3^4 5^3 7$. *This group is the central product of the double cover of the Hall-Janko simple group with the binary icosahedral group, i.e.,* $\mathrm{O}(\mathcal{L}) = \tilde{J}_2 \times_{\pm 1} \mathrm{SL}_2(\mathbf{F}_5)$.

REMARKS. (1) The root systems A_n $(n \geq 1)$, D_n $(n \geq 4), E_n$ $(n = 6, 7, 8)$ are the "old" classical root systems. The "new" root systems F_n $(n = 2, 3, 4)$ for $\mathbf{Q}(\sqrt{5})$ are given matricially by [**3**]:

$$F_2 = \begin{bmatrix} 2 & e \\ e & 2 \end{bmatrix}, F_3 = \begin{bmatrix} 2 & e & e \\ e & 2 & 1 \\ e & 1 & 2 \end{bmatrix}, F_4 = \begin{bmatrix} 2 & e & e & e \\ e & 2 & 1 & 1 \\ e & 1 & 2 & 1 \\ e & 1 & 1 & 2 \end{bmatrix}.$$

(2) The proof of Theorem II.1 is rather complicated. The full details may be found in [**1**]. It suffices to mention here only that our method is a combination of Venkov's method, Siegel's analytical theory, Hilbert modular forms, some coding theory, Kneser's neighborhood trick, and other algebraic arguments (e.g., the algebraic descent trick — see below — which bypasses some analytic difficulties, quaternion algebras, etc.).

III. Quadratic forms over $\mathbf{Q}(\sqrt{2})$. Aside from the "old" classical root systems of ADE-types, the "new" irreducible root systems over $\mathbf{Q}(\sqrt{2})$ are given by [**3**]:

$$\begin{aligned} \Delta_n \ (n \geq 2) &= \{z \in I_n \mid B(z, e_1 + \cdots + e_n) \equiv 0 (\bmod \sqrt{2})\} \\ &= \langle \sqrt{2}e_1, e_1 + e_2, \ldots, e_1 + e_n \rangle \end{aligned} \tag{3.1}$$

and

$$\begin{aligned} \Delta_4' &= \Delta_4 + \langle (e_1 + \cdots + e_4)/\sqrt{2} \rangle \\ &= \langle \sqrt{2}e_1, (e_1 + \cdots + e_4)/\sqrt{2}, e_1 + e_3, e_1 + e_4 \rangle \end{aligned} \tag{3.2}$$

where $\{e_i\}$ is an orthonormal basis and $I_n = \langle e_1, \ldots, e_n \rangle$. By direct calculations, one sees that

$$\det \Delta_n = 2, \quad \det \Delta_4' = 1, \tag{3.3}$$

Let L_Γ be the even unimodular lattice with root system Γ. [Sometimes we just denote it as Γ where there is no contextual confusion.] It is easy to see that

$$L_{\Delta_{4t}} = \Delta_{4t} + \langle (e_1 + \cdots + e_{4t})/\sqrt{2} \rangle, \qquad t \geq 2. \tag{3.4}$$

Algebraic descent. Given an R-lattice L of $\operatorname{rank}_R(L) = m$, the *algebraic descent* L_0 of L is the $\mathbf{Z}$-lattice $L_0 = L$ of $\operatorname{rank}_{\mathbf{Z}}(L_0) = 2m$ together with the quadratic form Q_0 defined by

$$Q_0(x) := \operatorname{Tr}_{F/\mathbf{Q}}(Q(x)/2e\sqrt{2}). \tag{3.5}$$

(Over $\mathbf{Q}(\sqrt{5})$ replace the toally positive generator $2e\sqrt{2}$ in (3.5) for the different by $e\sqrt{5}$, where e for $\mathbf{Q}(\sqrt{5})$ is $(1+\sqrt{5})/2$.) If (L, Q) is even R-unimodular, then (L_0, Q_0) is even $\mathbf{Z}$-unimodular.

We next examine the behavior of the root lattices under algebraic descent. Set $R = \mathbf{Z}[e]$. For $Q(x) = a + be, a, b \in \mathbf{Z}$ we have

$$Q_0(x) = \operatorname{Tr}_{F/\mathbf{Q}}((a+be)/2e\sqrt{2}) = a \tag{3.6}$$

so that

$$Q_0(x) = 2 \Leftrightarrow Q(x) \in \{2, 2+2e, 2(1+2e)\}. \tag{3.7}$$

For any two roots $u, v \in L$ we have $B(u, v) = 0, \pm 1, \pm\sqrt{2}$, which yield $B_0(u, v) = 0, \pm 1, \mp$ respectively. From this it follows that, when Γ is an "old" root system, then $\Gamma_0 = \Gamma + e\Gamma = 2\Gamma$. For "new" root systems, it is clear that if $\Gamma = \Delta_4'$, then $\Gamma_0 = E_8$ since E_8 is the unique 8-dimensional even unimodular $\mathbf{Z}$-lattice. Let $u_1 = \sqrt{2}e_1$, $u_2 = e_1 + e_2, \ldots, u_n = e_1 + e_n$ be the basis for Δ_n $(n \geq 2)$ as given in (3.1). Putting $u_i' = eu_i$ $(1 \leq i \leq n)$, then the inner product matrix of $((\Delta_n)_0, B_0)$ in the basis $\{-u_2 + u_1', u_1 + u_2, \ldots, u_1 + u_n; u_1', \ldots, u_n'\}$ is

$$\begin{bmatrix} 2 & 0 & & \\ 0 & 2 & & \\ & & \ddots & \\ & & & 2 \end{bmatrix} \tag{3.8}$$

where the blank spaces have 1 as entries. This matrix (3.8) is equivalent to D_{2n}. Note that $\det_{B_0}((\Delta_n)_0) = N_{F/\mathbf{Q}}(\det_B(\Delta_n)) = 4 = \det(D_{2n})$. This proves

PROPOSITION III.1. *Let Γ be an irreducible root system for $\mathbf{Q}(\sqrt{2})$. If Γ is "old" then the algebraic descent Γ_0 is 2 copies of Γ, whereas if $\Gamma = \Delta_n$ $(n \geq 2)$, Δ_4' then Γ_0 is D_{2n}, E_8 respectively.*

REMARKS. (i) Proposition III.1 shows that the Coxeter number of an irreducible "new" root system is not preserved under algebraic descent. Furthermore, Proposition III.3 below shows that, under algebraic descent, an empty root system lattice L_Φ can become one with the maximal 2-rank property (i.e., the roots span the entire space). These two properties, which fail for $\mathbf{Q}(\sqrt{2})$, do hold and are crucial in both the investigations [**7**] for $\mathbf{Q}$ and [**1**] for $\mathbf{Q}(\sqrt{5})$.

(ii) Explicitly, $\Delta_n = D_n \cup \{\pm\sqrt{2}e_i\}$ has $2n^2$ roots. D_n descends to $2D_n$, yielding $4n(n-1)$ roots. $\{\pm\sqrt{2}e_i\} = nA_1$ descends to $2nA_1$ giving $4n$ roots. Let $*$ denote an arbitrary sign. $\{*(ee_i * e_j) \mid j \neq i\}$ gives $4n(n-1)$ roots. Altogether we capture $4n(2n-1)$ roots which is the number of roots in D_{2n}. Similarly, Δ_4' has 32 roots from Δ_4 plus the 16 roots given by $\{(*e_1 * \cdots * e_4)/\sqrt{2}\}$. Upon descent, Δ_4 becomes D_8; $\{(*e_1 * \cdots * e_4)/\sqrt{2}\}$ descends to 32 roots; $\{(*e_i * i_j + e(*e_k * e_l))/\sqrt{2}\}$ provide another 96 roots, capturing all 240 roots in E_8.

We concentrate on 8-dimensional lattices over $\mathbf{Q}(\sqrt{2})$. There are certainly the lattices E_8, and $2\Delta_4'$. (3.4) gives also L_{Δ_8} (or just Δ_8). Takada [**6**] has also found a fourth lattice which we shall denote by either L_{2D_4} or just $2D_4$ (his notation is $N_{8,4}$). L_{2D_4} is obtained by the neighborhood method using the base lattice L_{Δ_8} and the vector $w := (ee_1 + \cdots + ee_4 + e_5 + \cdots + e_8)/2$. Note that $\{\pm\sqrt{2}e_i \mid 1 \leq i \leq 8\}$ and $\{*e_i * e_k \mid 1 \leq i \leq 4, 5 \leq k \leq 8\}$ are not in L_{2D_4} since their inner products with w are not integral. Thus, the root system is just $\Gamma \cup \Gamma'$, where

$$(3.9) \quad \Gamma := \{*e_i * e_j \mid 1 \leq i, j \leq 4\} \text{ and } \Gamma' := \{*e_k * e_l \mid 5 \leq k, l \leq 8\}.$$

Hence, we used the notation $2D_4$.

The algebraic descent of either E_8 or $2\Delta_4'$ is $2E_8$; L_{Δ_8} descends to D_{16}. What is $(L_{2D_4})_0$?

To answer this question we need to find the vectors of Q-length $2+2e$. Since $2D_4$ descends to $4D_4$ there should be $480-96=384$ such vectors, and they are given by

$$(\alpha): \quad (\alpha_1) := \{*ee_i * e_k \mid 1 \le i \le 4, 5 \le k \le 8\} \\ \cup \{*e_i * ee_k \mid 1 \le i \le 4, 5 \le k \le 8\} := (\alpha_2),$$

$$(\beta): \quad \Big\{1/2(e(a_1e_1 + \cdots + a_4e_4) + (a_5e_5 + \cdots + a_8e_8)) \mid a_i \in \{\pm 1\}, \prod_{i=1}^{8} a_i = 1\Big\},$$

$$(\gamma): \quad \Big\{1/2((a_1e_1 + \cdots + a_4e_4) + e(a_5e_5 + \cdots + a_8e_8)) \mid a_i \in \{\pm 1\}, \prod_{i=1}^{8} a_i = 1\Big\}.$$

Each category has 128 elements.

If ξ, ξ' are two vectors from category (β) then we can always find a third vector $\tilde{\xi}$ from (β) such that $B_0(\xi,\tilde{\xi}) \neq 0, B_0(\xi',\tilde{\xi}) \neq 0$. Therefore, all the vectors in (β) descend to roots that lie in the same indecomposable component. This component also contains vectors from (α_1) and the roots from $e\Gamma \cup \Gamma'$, yielding altogether the root system E_8. Similarly, $(\gamma) \cup (\alpha_2) \cup \Gamma \cup e\Gamma'$ constitute a second E_8 component orthogonal to the first one. This proves the following

PROPOSITION III.2. *The algebraic descent of L_{2D_4} is $2E_8$.*

Construction of L_Φ. The construction of this 8-dimensional empty root system lattice L_Φ is analogous to that of the 12-dimensional even unimodular lattice $\mathcal{L}$ over $\mathcal{Q}(\sqrt{5})$ described in [**1**]. Take the auxiliary lattice $K = \Delta_4'$, $\overline{K} = K/\pi K$ its reduction mod the prime $\pi = e\sqrt{2}$. Set

$\bar{K} = S \oplus S'$ its totally singular dual pair decomposition. Using 2 copies of K, let $T = \{(x, x) \mid x \in S\}, T' = T^{\perp} \cap S'^2$, and $\mathcal{C} = T + T' \subseteq \overline{K}^2$. Put

$$L_\Phi := \{v = (v_1, v_2) \in K^2 \mid \overline{v} \in \mathcal{C}\}$$

and set the quadratic form on L_Φ as

$$Q_\Phi := (1/\pi)Q^2,$$

where Q^2 is the quadratic map on K^2 induced from (K, Q). Then (L_Φ, Q_Φ) is an even 8-dimensional unimodular lattice.

We assert that L_Φ has no roots. Suppose $v = (v_1, v_2) \in L_\Phi$ has $Q_\Phi(v) = 2$. Then $Q^2(v) = 2\pi$. Set $v_i = x + y_i, x \in S, y_i \in S'$. If no v_i vanishes, then, by the inequality between arithmetic and geometric means, we have

$$N_{F/\mathbf{Q}}(\pi) = N\left(\frac{Q(v_1) + Q(v_2)}{2}\right) \geq \sqrt{N_{F/\mathbf{Q}}(Q(v_1)Q(v_2))} \geq 4.$$

But, $N_{F/\mathbf{Q}}(\pi) = 2$. This contradiction shows that some v_i, say v_1, must vanish. By construction, $\overline{v} = (0, y_2) \in \mathcal{C}$ so that $y_2 = 0$, implying that $v = \pi u$ for some $u \in K^2$. Now, $2\pi = Q^2(v) = \pi^2 Q^2(u)$ is absurd. Therefore, we have proved

PROPOSITION III.3. *There exists an* 8-*dimensional even unimodular lattice* L_Φ *over* $\mathbf{Q}(\sqrt{2})$ *which has an empty root system.*

Theta series. Given an even unimodular lattice L over $\mathbf{Q}(\sqrt{2})$, the *theta series* of L is given by

$$\begin{aligned}\Theta_L(z_1, z_2) : &= \sum_{x \in L} \exp\left(\pi i\left(\frac{Q(x)}{2e\sqrt{2}}\right) z_1 + \left(\frac{Q(x)}{2e\sqrt{2}}\right)' z_2\right) \\ &= \sum c_L(a + be)[a, b],\end{aligned}$$

where $c_L(a + be) = \text{Card}\,\{x \in L \mid Q(x) = 2(a + be)\}$,

$$[a, b] = \exp\left(2\pi i\left(\left(\frac{1 + be}{2e\sqrt{2}}\right) z_1 + \left(\frac{a + be}{2e\sqrt{2}}\right)' z_2\right)\right),$$

and $'$ denotes field conjugation. (Over $\mathbf{Q}(\sqrt{5})$, replace the totally positive generator above for the different by $e\sqrt{5}$, where $e_{\mathbf{Q}(\sqrt{5})} = (1+\sqrt{5})/2$.)

If $f(z_1, z_2) \in M_k(\mathrm{SL}_2(R))$, then the restriction of f along the diagonal yields an elliptic modular form $\tilde{f}(s) \in M_{2k}(\mathrm{SL}_2(\mathbf{Z}))$. The Fourier coefficients are related as follows. If

$$f(z_1, z_2) = f(z) = \sum_v c_f(v) e^{2\pi i \sigma(\frac{v}{2e\sqrt{2}} z)} = \sum c_f(a+be)[a,b]$$

and $\tilde{f}(s) = \sum_{n\geq 0} A(n) q^n$, then

$$A(n) = \sum_{n+me} c_f(n+me).$$

In particular, for $f(z) = \Theta_L(z)$, we have

$$A_{L_o} = 2c_L(1) + c_L(1+e).$$

It follows that the theta series of the five distinct lattices enumerated above are given by

$$\begin{aligned}
\Theta_{E_8}(z) &= 1 + 240([1,0]+[1,2]) + 0([1,1]) + \cdots \\
\Theta_{\Delta_8}(z) &= 1 + 128([1,0]+[1,2]) + 224([1,1]) + \cdots \\
\Theta_{2\Delta_4'}(z) &= 1 + 96([1,0]+[1,2]) + 288([1,1]) + \cdots \\
\Theta_{2D_4}(z) &= 1 + 48([1,0]+[1,2]) + 384([1,1]) + \cdots \\
\Theta_{L\phi}(z) &= 1 + 0([1,0]+[1,2]) + 480([1,1]) + \cdots .
\end{aligned}$$

While these theta series are distinct, they descend to the same classical Eisenstein series of weight 8.

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