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THE BEHAVIOR OF THE ν-INVARIANT OF A FIELD OF CHARACTERISTIC 2 UNDER FINITE EXTENSIONS

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ABSTRACT. Let F be a field of characteristic 2. We define $\nu(F)$ as the smallest integer n such that any n-fold quadratic Pfister form over F is isotropic. If L/F is any finite extension, we prove $\nu(F) \leq \nu(L) \leq \nu(F)+1$. The corresponding question for fields of characteristic $\neq 2$ is still open.

1. Introduction. The ν-invariant of a field F of characteristic $\neq 2$ was introduced in [**2**] as the number $\nu(F) = \operatorname{Min}\{n \mid I^n(F)$ is torsion free$\}$, where $I(F)$ denotes the maximal ideal of even dimensional quadratic forms over F in the Witt ring $W(F)$. If F is non real, then $\nu(F)$ is the smallest integer n such that any n-fold Pfister form over F is isotropic. Similarly, if F is a field of characteristic 2, let $W_q(F)$ be the Witt group of non singular quadratic forms over F and $W(F)$ the Witt ring of non singular symmetric bilinear forms over F. It is well known that $W_q(F)$ is a $W(F)$-module under the operation $b \cdot q(x \otimes y) = b(x,x)q(y)$ for any $x \in V =$ space of the bilinear form b, $y \in W =$ space of the quadratic form q. If $I(F) \subset W(F)$ is the maximal ideal of even-dimensional bilinear forms, then the chain of submodules $W_q(F) \supset IW_q(F) \supset I^2W_q(F) \supset \cdots$ plays an important role in the knowledge of the module structure of $W_q(F)$. If $a_1, \ldots, a_n \in F^*$, $b \in F$, then the quadratic n-fold Pfister form $\langle 1, a_1\rangle \ldots \langle 1, a_n\rangle[1,b]$ is a typical generator of $I^nW_q(F)$, where $\langle 1, a\rangle$ denotes the symmetric bilinear form $U^2 + aV^2$ and $[1,b]$ denotes the quadratic form $X^2 + XY + bY^2$. We shall usually write $\langle\langle a_1, \cdots, a_n, b]]$ instead of $\langle 1, a_1\rangle \cdots \langle 1, a_n\rangle[1,b]$. (We refer to [**1, 3**] for general facts on quadratic forms in characteristic 2). We define now, as in [**2**], the ν-invariant of a field F of characteristic 2 as

$$\nu(F) = \operatorname{Min}\{n \mid I^nW_q(F) = 0\}, \tag{1.1}$$

i.e., $\nu(F)$ is the smallest integer n such that any n-fold Pfister form over F is isotropic.

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In [**2**] it was conjectured that $\nu(L) \leq \nu(F) + 1$ for any non real field F of characteristic $\neq 2$ and any finite extension L/F. The authors proved $\nu(L) \leq \nu(F) + [L : F] - 1$, and recently Leep (unpublished) has shown the much better estimate $\nu(L) \leq \nu(F) + (\log_2([L : F]/3)) + 1$, which still depends on the degree $[L : F]$. In this paper we consider the same question for fields of characteristic 2 and prove that the above conjecture is true. Our main result is

THEOREM 1.2. *Let F be a field of characteristic* 2. *Then, for any finite extension L/F,*

$$\nu(F) \leq \nu(L) \leq \nu(F) + 1.$$

For the rest of this paper, F will denote a field of characteristic 2.

2. The separable case. Let L/F be a finite separable extension. In this section we will prove Theorem 1.2 under this assumption. For $a, b \in F$, let $[a, b]$ be the quadratic form $aX^2 + XY + bY^2$, so that if $a \neq 0$, we have $[a, b] \cong \langle a \rangle [1, ab]$. Obviously, for $a_1, a_2, b \in F$, we have in $W_q(F)$ the relation $[a_1 + a_2, b] = [a_1, b] + [a_2, b]$.

LEMMA 2.1. *For any $a, b, c \in F$ with $a, b, a+b \neq 0$, we have in $W_q(F)$*

$$\langle 1, a+b \rangle [1, c] = \langle 1, a \rangle \left[1, \frac{ac}{a+b}\right] + \langle 1, b \rangle \left[1, \frac{bc}{a+b}\right].$$

PROOF. We have $\langle 1, a+b \rangle [1, c] = [1, c] \perp \langle a+b \rangle [1, c] = [1, c] \perp \left[a+b, \frac{c}{a+b}\right]$. But in $W_q(F)$,

$$\begin{aligned}
&\left[a+b, \frac{c}{a+b}\right] \\
&= \left[a, \frac{c}{a+b}\right] + \left[b, \frac{c}{a+b}\right] \\
&= \langle a\rangle\left[1, \frac{ac}{a+b}\right] + \langle b\rangle\left[1, \frac{bc}{a+b}\right] \\
&= \left[1, \frac{ac}{a+b} + \frac{bc}{a+b}\right] + \langle 1, a\rangle\left[1, \frac{ac}{a+b}\right] + \langle 1, b\rangle\left[1, \frac{bc}{a+b}\right] \\
&= [1, c] + \langle 1, a\rangle\left[1, \frac{ac}{a+b}\right] + \langle 1, b\rangle\left[1, \frac{bc}{a+b}\right]
\end{aligned}$$

Inserting this in the first relation, the lemma follows. □

COROLLARY 2.2. *Let E/F be any extension, $\alpha \in E$ and $\beta = b_0 + b_1\alpha^2 + \cdots + b_m\alpha^{2m} \neq 0$ with $b_0, \cdots, b_m \in F$. Then, for any $\gamma \in E$, we have in $W_q(E)$*

$$\langle 1, \beta\rangle[1, \gamma] = \sum_{i=1}^{m}{}' \langle 1, b_i\rangle[1, \gamma_i]$$

with certain $\gamma_i \in E$ ($'$ means that the sum is taken over all i with $b_i \neq 0$).

Now for the finite separable extension L/F we have $L = F(\alpha^2)$ with some $\alpha \in L$, so that $1, \alpha^2, \cdots, \alpha^{2(n-1)}$ $(n = [L : F])$ is a basis of L over F. Thus any element $\beta \in L$ has the form $\beta = b_0 + b_1\alpha^2 + \cdots + b_{n-1}\alpha^{2(n-1)}$ with $b_0, \cdots, b_{n-1} \in F$. We conclude, from Corollary 2.2,

PROPOSITION 2.3. *Let L/F be a finite separable extension. For any $\beta \in L^*$, $\gamma \in L$ there exist $b_1, \cdots, b_m \in F^*$, $\gamma_1, \cdots, \gamma_m \in L$ $(m \geq 1)$ such that*

$$\langle 1, \beta\rangle[1, \gamma] = \sum_{i=1}^{m} \langle 1, b_i\rangle[1, \gamma_i]$$

in $W_q(L)$.

Iterating this result we obtain

COROLLARY 2.4. *Let* L/F *be a finite separable extension. Then, for any* $n \geq 0$, $I^n W_q(L)$ *is generated by the Pfister forms* $\langle\langle a_1, \cdots, a_n, \gamma]]$ *with* $a_1, \cdots, a_n \in F^*$, $\gamma \in L$.

Let $s : L \to F$ be any trace map, i.e., s is an F-linear map $\neq 0$. For any quadratic form q over L, let $s_*(q) = s \circ q$ be the transfer of q. s_* defines a homomorphism $s_* : W_q(L) \to W_q(F)$, which satisfies the usual Frobenius reciprocity law. We obtain, directly from Corollary 2.4,

COROLLARY 2.5. *Let* L/F *be a finite separable extension and* $s : L \to F$ *a trace map. Then, for any* $n \geq 0$,

$$s_*[I^n W_q(L)] \subseteq I^n W_q(F).$$

REMARK 2.6. If $\mathrm{ch}\,(F) \neq 2$, Corollary 2.5 is a well known result of Arason, but the proof in this case uses the Milnor-Scharlau exact sequence. Thus, for fields of characteristic 2, we have a completely elementary proof of this fact.

The above result can be improved. In fact we have

THEOREM 2.7. *Let* L/F *be a finite separable extension and* $s : L \to F$ *a trace map. Then, for any* $n \geq 0$, *we have*

$$s_*[I^n W_q(L)] = I^n W_q(F).$$

PROOF. From Corollaries 2.4, 2.5 and the Frobenius reciprocity law, it follows that we only need to consider the case $n = 0$, i.e., we must show that $s_* : W_q(L) \to W_q(F)$ is onto. Notice that this fact does not depend on the particular choice of the trace map s. We now consider several cases

(i) $[L : F] = 2$, i.e., $L = F(\alpha)$ with $\alpha^2 + \alpha = a \in F$. Using Frobenius reciprocity it suffices to show that $[1, b] \in \mathrm{Im}\,(s_*)$ for all $b \in F$. This follows from the direct computation $s_*([1,(1+\alpha)^2 b]) = [1,b]$, where $s = \mathrm{Tr}_{L/F}$.

(ii) $[L:F]$ odd. Let $L = F(\alpha)$ and define $s : L \to F$ by $s(1) = 1$, $s(\alpha) = \cdots = s(\alpha^{n-1}) = 0$, $n = [L:F]$. From Frobenius reciprocity law we get $s_*(q \otimes L) = s_*(\langle 1 \rangle) \cdot q$ for any $q \in W_q(F)$. But an easy computation shows that $s_*(\langle 1 \rangle) = \langle 1 \rangle$ in $W(F)$, so that $s_*(q \otimes L) = q$, i.e., s_* is onto.

(iii) L/F is Galois. According to (ii) we may assume that $[L:F]$ is even. Let $H < G = \mathrm{Gal}\,(L/F)$ be a 2-Sylow subgroup and denote by K the fixed field of H. Let $s_1 : L \to K$, $s_2 : K \to F$ be trace maps, so that $s = s_2 \circ s_1 \neq 0$, i.e., $s : L \to F$ is a trace map. Since $s_* = s_{2*} \circ s_{1*}$, and s_{2*} is onto by (ii), it suffices to show that s_{1*} is onto. But $H = \mathrm{Gal}\,(L/K)$ is a 2-group, so that we can find a chain of fields $K = K_0 \subset K_1 \subset \cdots \subset K_r = L$ with $[K_i : K_{i-1}] = 2$. We choose trace maps $t_i : K_i \to K_{i-1}$ such that $t = t_1 \circ \cdots \circ t_r \neq 0$, i.e., $t : L \to K$ is a trace map. Since $t_* = t_{1*} \circ \cdots \circ t_{r*}$ and any t_{i*} is onto by (i), we conclude that t_* is onto, and hence s_{1*}, too. This shows that s_* is onto.

(iv) Let L/F be any finite separable extension. Choose a finite extension N/L such that N/F is Galois, and trace maps $s_1 : N \to L$, $s : L \to F$ with $s \circ s_1 \neq 0$. By part (iii) $(s \circ s_1)_* = s_* \circ s_{1*}$ is onto, and therefore s_* is also onto. This concludes the proof of Theorem 2.7. □

COROLLARY 2.8. *Let L/F be a finite separable extension. Then $\nu(F) \leq \nu(L)$.*

PROOF OF THEOREM (1.2) FOR SEPARABLE EXTENSIONS. Let L/F be a finite separable extension. We will show $\nu(L) \leq \nu(F) + 1$. Let $L = F(\alpha^2)$, so that $1, \alpha^2, \cdots, \alpha^{2(n-1)}$ is a basis of L over F. For any $a \in F^*$, $\gamma \in L$ let us consider the quadratic form $\langle 1,a \rangle [1,\gamma] \neq 0$. We can write $\langle 1,a \rangle [1,\gamma] = \langle 1,a \rangle [1, (\gamma(\alpha^2 + a) \cdots (\alpha^2 + a^{2n-3}))/((\alpha^2 + a) \cdots (\alpha^2 + a^{2n-3}))]$, where we consider only factors of the form $\alpha^2 + a^{2i-1}$, $1 \leq i \leq n-1$. We may assume $a^{2i-1} \neq a^{2j-1}$ for all $i \neq j$, since otherwise we get $a^{2k-1} = 1$ for some integer k, because $\mathrm{Ch}F = 2$, and hence $\langle a \rangle = \langle 1 \rangle$, i.e., $\langle 1,a \rangle [1,\gamma] = 0$. Let $\gamma(\alpha^2 + a) \cdots (\alpha^2 + a^{2n-3}) = b_0 + b_1 \alpha^2 + \cdots + b_{n-1} \alpha^{2(n-1)}$ with

$b_0, b_1, \cdots, b_{n-1} \in F$. Because of the above assumption we have the following decomposition in partial fractions

$$\frac{b_0 + b_1\alpha^2 + \cdots + b_{n-1}\alpha^{2(n-1)}}{(\alpha^2+a)\cdots(\alpha^2+a^{2n-3})} = c_0 + \frac{c_1}{\alpha^2+a} + \cdots + \frac{c_{n-1}}{\alpha^2+a^{2n-3}}$$

with $c_0, c_1, \cdots, c_{n-1} \in F$. (We have $c_0 = b_{n-1}$ and the determinant of the linear system of equations defining $c_1, \cdots, c_{n-1}$ has the form $a^r(1+a)^s$ for some $r, s \geq 0$, which is $\neq 0$ since we assume $\langle a \rangle \neq \langle 1 \rangle$). Inserting the above expression in the form $\langle 1, a\rangle[1,\gamma]$ we obtain in $W_q(L)$

$$\langle 1,a\rangle[1,\gamma] = \langle 1,a\rangle[1,c_0] + \sum_{i=1}^{n-1}\langle 1,a\rangle\left[1,\frac{c_i}{\alpha^2+a^{2i-1}}\right].$$

But using lemma (2.1) we have

$$\begin{aligned}\langle 1,\alpha^2+a^{2i-1}\rangle\left[1,\frac{c_i}{a^{2i-1}}\right] &= \langle 1,a^{2i-1}\rangle\left[1,\frac{c_i a^{2i-1}}{a^{2i-1}(\alpha^2+a^{2i-1})}\right] \\ &\quad + \langle 1,\alpha^2\rangle\left[1,\frac{c_i\alpha^2}{a^{2i-1}(\alpha^2+a^{2i-1})}\right] \\ &= \langle 1,a\rangle\left[1,\frac{c_i}{\alpha^2+a^{2i-1}}\right]\end{aligned}$$

in $W_q(L)$, so it follows that

$$(2.9)\qquad \langle 1,a\rangle[1,\gamma] = \langle 1,a\rangle[1,c_0] + \sum_{i=1}^{n-1}\langle 1,\alpha^2+a^{2i-1}\rangle\left[1,\frac{c_i}{a^{2i-1}}\right].$$

Therefore, for any $m \geq 0$, $a_1, \cdots, a_{m+1} \in F^*$, $\gamma \in L$, we obtain in $W_q(L)$ applying (2.9) to $\langle 1, a_{m+1}\rangle[1,\gamma]$,

$$\begin{aligned}&\langle\langle a_1,\cdots,a_{m+1},\gamma]] \\ &= \langle\langle a_1,\cdots,a_{m+1},c_0]] + \sum_{i=1}^{n-1}\langle 1,\alpha^2+a_{m+1}^{2i-1}\rangle\left\langle\left\langle a_1,\cdots,a_m,\frac{c_i}{a_{m+1}^{2i-1}}\right]\right].\end{aligned}$$

The proof of Theorem 1.2 is now obvious. If $I^m W_q(F) = 0$, then, from the above formula and from the fact that $I^{m+1}W_q(L)$ is generated by the forms $\langle\langle a_1, \cdots, a_{m+1}, \gamma]]$ with $a_1, \cdots, a_{m+1} \in F^*$, $\gamma \in L$ (see Corollary 2.4), it follows that $I^{m+1}W_q(L) = 0$, i.e., $\nu(L) \leq \nu(F) + 1$. □

In fact we have shown the following general fact.

THEOREM 2.10. *Let L/F be a finite separable extension. Then*

$$I^{m+1}W_q(L) = I(L)i_*[I^m W_q(F)]$$

for all $m \geq 0$, where $i_*[I^m W_q(F)]$ is the image of $I^m W_q(F)$ under the natural homomorphism $i_* : W_q(F) \to W_q(L)$.

REMARK 2.11. It is easy to show that, for any quadratic separable extension L/F the equality $\nu(L) = \nu(F)$ holds. We just need to prove $\nu(L) \leq \nu(F)$. Assume $I^m W_q(F) = 0$. Let $L = F(\alpha)$, $\alpha^2 + \alpha = a \in F$ and define $s : L \to F$ the trace map given by $s(1) = 0$, $s(\alpha) = 1$, i.e., $s = \operatorname{Tr} L/F$. For any m-fold Pfister form over L, we have $s_*(q) \in I^m W_q(F) = 0$, i.e., $q \in \operatorname{Ker}(i_*)$. Hence $q \cong q_0 \otimes L$ with some form q_0 defined over F (see [**1**, V (4.10)]), and hence, using [**1**, (V, 4.14)], we conclude $q \cong q_1 \otimes L$ with an m-fold Pfister form defined over F, which by assumption is 0 in $W_q(F)$. This shows $I^m W_q(L) = 0$, i.e., $\nu(L) \leq \nu(F)$.

3. The purely inseparable case.

3. The purely inseparable case. The main result of this section is

THEOREM 3.1. *Let L/F be a finite purely inseparable extension. Then $\nu(F) = \nu(L)$.*

Since any finite purely inseparable extension L/F admits a chain of subfields $F = F_0 \subset F_1 \subset \cdots \subset F_m = L$ with $F_i = F_{i-1}(\sqrt{a_i})$, $a_i \in F^*_{i-1}$, to prove Theorem 3.1 it suffices to consider the case $L = F(\sqrt{l}), l \in F^*$. Let us write $L = F(\alpha), \alpha^2 = l$. Then we have

LEMMA 3.2. *Any n-fold Pfister form $q = \langle\langle \alpha_1, \cdots, \alpha_n, \beta]]$ over L is a linear combination in $W_q(L)$ of n-fold Pfister forms of the type*

(i) $\langle\langle a_1, \cdots, a_n, b]]$ *with* $a_1, \cdots, a_n b \in F^*$

(ii) $\langle\langle \alpha, a_1, \cdots, a_{n-1}, b]]$ *with* $a_1, \cdots, a_{n-1}, b \in F^*$.

PROOF. Let us consider first a 1-fold Pfister form $q = \langle 1, \beta\rangle[1, \gamma]$ over L. Since $\gamma \equiv \gamma^2(\operatorname{mod} \rho L)$, where $\rho L = \{x^2 + x \mid x \in L\}$ and $\gamma^2 \in F$ for all $\gamma \in L$, we may assume $\gamma = c \in F$. Let $\beta = a + b\alpha$, $a, b \in F$. From Lemma 2.1 we get $q = \langle 1, a + b\alpha\rangle[1, c] = \langle 1, a\rangle[1, ac/\beta] + \langle 1, b\alpha\rangle[1, cb/\beta] = \langle 1, a\rangle[1, c_1] + \langle 1, b\alpha\rangle[1, c_2]$ with some $c_1, c_2 \in F$. Hence $q = \langle 1, a\rangle[1, c_1] + \langle b\rangle\langle 1, b\rangle[1, c_2] + \langle b\rangle\langle 1, \alpha\rangle[1, c_2]$ in $W_q(L)$. Since $\langle 1, \alpha\rangle^2 = 0$ in $W(F)$, the lemma follows easily by induction. □

We proceed now to prove the theorem. As noticed above, we may assume $L = F(\alpha)$, $\alpha^2 = l \in F^*$. Suppose first $I^n W_q(F) = 0$. We will show $I^n W_q(L) = 0$. According to Lemma 3.2 we just need to consider forms of type (i), (ii), but since $I^n W_q(F) = 0$, then all forms of type (i) are 0.

Take a form $q = \langle\langle \alpha, a_1, \cdots, a_{n-1} b]]$ of type (ii). Since $q = \langle\langle a_1, \cdots, a_{n-1}, b]] \perp \langle \alpha\rangle\langle\langle a_1, \cdots, a_{n-1} b]]$, it suffices to show that any form $\langle\langle a_1, \cdots, a_{n-1}, b]]$ does represent α over L, because then q is isotropic and hence also hyperbolic over L. Notice that $I^n W_q(F) = 0$ implies that $p = \langle\langle a_1, \cdots, a_{n-1}, b]]$ represents any element of F^*. We set $\varphi = \langle\langle a_1, \cdots, a_{n-1}\rangle\rangle = \langle 1, a_1\rangle \cdots \langle 1, a_{n-1}\rangle$, and $[1, b] = Fe + Ff$ with $p(e) = 1$, $p(f) = b$, $b_p(e, f) = 1$, so that $p = \varphi \cdot [1, b] = \varphi \otimes e \oplus \varphi \otimes f$. Any vector of p has the form $z = x \otimes e + y \otimes f$ with $x, y \in \varphi$ and $p(z) = \varphi(x) + \varphi(x, y) + \varphi(y)b$. Over L, for $x, y \in \varphi \otimes L$ we write $x = x_0 + x_1\alpha$, $y = y_0 + y_1\alpha$ with $x_0, x_1, y_0, y_1 \in \varphi$ defined over F. Since φ is written in diagonal form we have $\varphi(x) = \varphi(x_0) + l\varphi(x_1)$, $\varphi(y) = \varphi(y_0) + l\varphi(y_1)$ so that, for $z = x \otimes e + y \otimes f \in p \otimes L$, we get

$$\begin{aligned} p(z) &= \varphi(x_0) + \varphi(x_0, y_0) + \varphi(y_0)b + l[\varphi(x_1) + \varphi(x_1, y_1) + \varphi(y_1)b] \\ &\quad + \alpha[\varphi(x_0, y_1) + \varphi(x_1, y_0)] \\ p(z) &= p(x_0 \otimes e + y_0 \otimes f) + p(x_1 \otimes e + y_1 \otimes f)l \\ &\quad + \alpha[\varphi(x_0, y_1) + \varphi(x_1, y_0)]. \end{aligned}$$

Choose $x_1 = (1, 0, \cdots, 0), y_1 = 0$, i.e., $p(x_1 \otimes e + y_1 \otimes f) = 1$. Since p represents all elements of F^*, we can find $x_0, y_0 \in \varphi$ such that $p(x_0 \otimes e + y_0 \otimes f) = l$. Setting $x = x_0 + x_1\alpha$, $y = y_0 + y_1\alpha = y_0$, we obtain, for $z = x \otimes e + y \otimes f \in p \otimes L$, $z \neq 0$,

$$p(z) = \alpha y_{0,1}, \tag{3.3}$$

where $y_{0,1}$ is the first coordinate of y_0. If $y_{0,1} = 0$, it follows that p is isotropic over L, and hence it represents α over L. If $y_{0,1} \neq 0$, then $y_{0,1}$ is represented by p over F, and since p is a Pfister form, it follows from (3.3), that p represents α over L. This proves $I^n W_q(L) = 0$. Thus we have $\nu(L) \leq \nu(F)$.

We now prove the converse, i.e., $\nu(F) \leq \nu(L)$. To this end we use

LEMMA 3.4. *Let $L = F(\alpha), \alpha^2 = l \in F^*$. Assume $I^n W_q(L) = 0$. Then*

(i) *Any n-fold Pfister form over F is of the type $\langle\langle l, a_1, \cdots, a_{n-1}, b]]$ with $a_1, \cdots, a_{n-1}, b \in F^*$*

(ii) *Any $(n-1)$-fold Pfister form over F is of the type $\langle\langle b_1, \cdots, b_{n-1}, lc^2]]$ with $b_1, \cdots, b_{n-1}, c \in F^*$.*

Let us proceed with the proof of $\nu(F) \leq \nu(L)$. Assume $I^n W_q(L) = 0$. Then any n-fold Pfister form over F has the form $q = \langle\langle l, a_1, \cdots, a_{n-1}, b]]$ (see Lemma 3.4(i)). Now using Lemma 3.4(ii) we can write $\langle\langle a_1, \cdots, a_{n-1}, b]] \cong \langle\langle b_1, \cdots, b_{n-1}, lc^2]]$ with some $c \in F^*$, i.e., $q = \langle\langle b_1, \cdots, b_{n-1}\rangle\rangle \cdot \langle 1, l\rangle[1, lc^2]$. But obviously $\langle 1, l\rangle[1, lc^2]$ is isotropic, so that $q = 0$ in $W_q(F)$. This proves $I^n W_q(F) = 0$, i.e., $\nu(F) \leq \nu(L)$, and Theorem 3.1 follows. □

For the proof of Lemma 3.4 we need the following general fact about Pfister forms over fields of characteristic 2.

PROPOSITION 3.5. *Let q be an n-fold Pfister form over F.*

(i) *If q contains a subform $[1, a], a \in F$, then $q \cong \langle\langle a_1, \cdots, a_n, a]]$ for some $a_1, \cdots, a_n \in F^*$.*

(ii) *Write $q = \varphi \otimes [1, b]$ with $\varphi = \langle\langle b_1, \cdots, b_n\rangle\rangle = \langle 1\rangle \perp \varphi'$,* i.e., *$q = [1, b] \perp \varphi' \cdot [1, b]$. If $l \in F^*$ is represented by $\varphi' \cdot [1, b]$, then $q \cong \langle\langle l, a_1, \cdots, a_{n-1}, c]]$ with some $a_1, \cdots, a_{n-1} \in F^*$.*

PROOF. Part (i) has been proved in [**1**, Chapter V] in a much more general setting, so that we omit the proof here. Let us prove (ii). Assume $n = 1$, $q = \langle 1, b_1\rangle[1, b] = [1, b] \perp \langle b_1\rangle[1, b]$. If l is represented by $\langle b_1\rangle[1, b], l = b_1(x^2 + xy + by^2)$, and since $\langle x^2 + xy + by^2\rangle[1, b] \cong [1, b]$, we get $q \cong [1,b] \perp \langle b_1(x^2+xy+by^2)\rangle[1,b] = [1,b] \perp \langle l\rangle[1,b] = \langle 1,l\rangle[1,b]$. Assume now $n > 1$. We use induction with respect to n. Write $\varphi = \langle 1,b_1\rangle \cdot \psi$, $\psi = \langle\langle b_2,\cdots,b_n\rangle\rangle$. Hence $\varphi' = \psi' \perp \langle b_1\rangle\psi, q = [1,b] \perp \psi' \cdot [1,b] \perp \langle b_1\rangle\psi[1,b]$, i.e., $\varphi'[1,b] = \psi'[1,b] \perp \langle b_1\rangle\psi[1,b]$. If $l \in F^*$ is represented by $\varphi'[1,b]$, we can write $l = c + b_1 d$ with c represented by $\psi'[1,b]$ and d represented by $\psi[1,b]$ (if $\neq 0$). (We may assume $c,d \neq 0$). By induction we have $\psi[1,b] \cong \langle 1,c\rangle\tau[1,b']$ with some $(n-2)$-fold bilinear Pfister form $\tau, b' \in F$. Moreover $\langle d\rangle\cdot\psi[1,b] \cong \psi[1,b]$. Therefore

$$q = \langle 1,b_1\rangle\psi[1,b] = \psi[1,b] \perp \langle b_1\rangle\psi[1,b],$$
$$q \cong \langle 1,c\rangle\tau[1,b'] \perp \langle b_1 d\rangle\langle 1,c\rangle\tau[1,b'],$$
$$q \cong \langle 1,c\rangle\langle 1,b_1 d\rangle\tau[1,b'],$$

But $\langle 1,c\rangle\langle 1,b_1 d\rangle = \langle 1,c,b_1 d,cb_1 d\rangle \cong \langle 1,c+b_1 d,x,x(c+b,d)\rangle\rangle \cong \langle 1,l\rangle\langle 1,x\rangle$, i.e., $q = \langle 1,l\rangle\langle 1,x\rangle\tau[1,b']$. This proves (ii). □

Now we prove Lemma 3.4. Let us assume $I^n W_q(L) = 0$. Let $q = \langle\langle a_1,\cdots,a_n,b]] = \varphi\cdot[1,b]$ be any n-fold Pfister form over F. Since $q\otimes L = 0$, we can find nonzero vectors $x = x_0 + x_1\alpha, y = y_0 + y_1\alpha \in \varphi\otimes L$ (see notation above) such that

$$q(x\otimes e + y\otimes f) = 0,$$

i.e.,

$$q(x_0\otimes e + y_0\otimes f) + lq(x_1\otimes e + y_1\otimes f) = 0,$$
$$b_q(x_0\otimes e + y_0\otimes f, x_1\otimes e + y_1\otimes f) = 0.$$

Let $u = x_0\otimes e + y_1\otimes f, v = x_1\otimes e + y_1\otimes f \in q$. Then $q(u) + lq(v) = 0$, $b_q(u,v) = 0$. Of course we may assume $q(u), q(v) \neq 0$, because otherwise q would be isotropic over F, and hence $q = 0$. Since $2 = 0$, we can find vectors $u_1, v_1 \in q$ with $b_q(u,u_1) = 1, b_q(v,v_1) = 1$, and $\langle u,u_1\rangle \perp \langle v,v_1\rangle$. Thus we have $\langle u,u_1\rangle \perp \langle v,v_1\rangle \subseteq q$. Let $a = q(v), a' = q(v_1), a'' = q(u_1)$. Then $[a,a'] \perp [al,a''] \subseteq q$, i.e., $\langle a\rangle[1,a_1] \perp \langle al\rangle[1,a_2] \subseteq q$ for some a_1, $a_2 \in F$. But $a = q(v)$ is represented by q, so that $\langle a\rangle q \cong q$, and therefore $[1,a_1] \perp \langle l\rangle[1,a_2] \subseteq q$. In particular $[1,a_1] \subset q$, so that, by Proposition 3.5(i), we have $q = \psi\cdot[1,a_1]$ with some n-fold bilinear Pfister form ψ. Since $q = [1,a_1] \perp \psi'[1,a_1]$, it follows by cancellation that $\langle l\rangle[1,a_2] \subseteq \psi'[1,a_1]$, and hence l is represented by $\psi'[1,a_1]$. Using Proposition 3.5(ii) we conclude $q \cong \langle\langle l, b_1, \cdots, b_{n-1}, b']]$ for some $b_1, \cdots, b_{n-1}, b' \in F^*$. This proves Lemma 3.4 (i).

Consider an $(n-1)$-fold Pfister form over $F, q = \langle\langle a_1, \cdots, a_{n-1}, b]]$ $= \varphi \cdot [1, b], \varphi = \langle\langle a_1, \cdots, a_{n-1}\rangle\rangle$. Since $I^n W_q(L) = 0$, it follows that $\langle 1, \alpha\rangle q = 0$ over L, i.e., q represents α over L. Therefore there exist $x = x_0 + x_1\alpha,\ y = y_0 + y_1\alpha \in \varphi \otimes L$ such that $q(x \otimes e + y \otimes f) = \alpha$.

This means $q(x_0 \otimes e + y_0 \otimes f) + lq(x_1 \otimes e + y_1 \otimes f) = 0, b_q(x_0 \otimes e + y_0 \otimes f, x_1 \otimes e + y_1 \otimes f) = 1$. Thus we have $u, v \in q$ with $q(u) + lq(v) = 0, b_q(u, v) = 1$. Hence $\langle u, v\rangle \subseteq q$ and $\langle u, v\rangle = [q(v), lq(v)] = \langle q(v)\rangle[1, lq(v)^2]$. But $\langle q(v)\rangle \cdot q \cong q$, so that $[1, lc^2] \subseteq q$, where $c = q(v)$. Now we apply Proposition 3.5(i) to conclude $q \cong \langle\langle b_1, \cdots, b_{n-1}, lc^2]]$, i.e., Lemma 3.4(ii). □

4. Proof of Theorem 1.2. Let L/F be a finite extension. Let F_s be the separable closure of F in L, $F \subset F_s \subset L$. Hence L/F_s is purely inseparable, and therefore (see Theorem 3.1) $\nu(L) = \nu(F_s)$. According to the results of §2 we have $\nu(F) \leq \nu(F_s) \leq \nu(F) + 1$, i.e., we have $\nu(F) \leq \nu(L) \leq \nu(F) + 1$.

5. An example. We will now construct a field F and a separable extension L/F with $[L : F] = 3$ and $\nu(L) = \nu(F) + 1$. In fact, for any n, it is possible to find a field F and a separable finite extension L/F with $\nu(F) = n, \nu(L) = n+1$, but we will just consider the simplest case $n = 0$. Let F be the quadratic separable closure of $\mathbf{F}_2(X)$. Obviously $\nu(F) = 0$. Since $W^3 + W + 1 \in F[W]$ is irreducible, let $L = F(\beta)$ with $\beta^3 = \beta + 1$. We want to show $\nu(L) = 1$, which is equivalent with $L \neq \rho L$. We assert $X\beta^2 \notin \rho L$. Otherwise there exist $y_0, y_1, y_2 \in F$ with $\rho(y_0 + y_1\beta + y_2\beta^2) = x\beta^2$, i.e., $y_0 + y_0^2 = 0,\ y_1 + y_2^2 = 0,\ y_1^2 + y_2 + y_2^2 = X$. Hence $Y^4 + Y^2 + Y = X$ has a solution in F. Let us show that this is impossible. Obviously there is no solution in $\mathbf{F}_2(X)$. Assume that $\mathbf{F}_2(X) \subset E \subset F$ is a subfield such that $Y^4 + Y^2 + Y = X$ has no solution in E. We will show that there is no solution in any quadratic separable extension $E(\alpha), \alpha^2 + \alpha = t \in E$ of E. Otherwise let $u + v\alpha (u, v \in E)$ be a solution. It follows that $u^4 + u^2 + u + v^4t^2 + v^4t + v^2t = X, v^4 + v^2 + v = 0$. But $v^3 + v + 1 = 0$ has no solution in F, and hence $v = 0$. Then $u^4 + u^2 + u = X$ in E, which is a contradiction. We conclude by induction, that there is no solution of $Y^4 + Y^2 + Y = X$ in F, and therefore we have $\nu(L) = 1$.

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