ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 19, Number 2, Spring 1989

INVERSE LIMIT MEANS ARE NOT PRESERVED UNDER HOMEOMORPHISMS

B.E. WILDER

1. Introduction. A *mean* on a topological space, M, is a continuous mapping, μ, from $M \times M$ to M such that $\mu(x,y) = \mu(y,x)$ and $\mu(x,x) = x$ for all $x, y \in M$. If $M = \lim\{I_n, f_n\}$ is an inverse limit space, each coordinate space I_n admits a mean μ_n, and, for each n, the functional equation $f_n[\mu_{n+1}(x,y)] = \mu_n[f_n(x), f_n(y)]$ holds for all $x, y \in I_{n+1}$, then the sequence $\{\mu_n\}$ generates a mean, μ, on the space M, where $\mu(\{x_n\},\{y_n\}) = \{\mu_n(x_n, y_n)\}$. A mean generated in this manner is referred to as an *inverse limit mean* with respect to sequence $\{I_n, f_n\}$. Professor John Baker and the author [**1**] investigated this notion, and constructed an inverse limit continuum, M, in which each coordinate space, I_n, was the unit interval $[0,1]$ such that M did not admit an inverse limit mean. We were unable to show, however, that M did not admit a mean. This led us to formulate the following question: If $M = \lim\{I_n, f_n\}$, where each $I_n = [0,1]$ and M admits a mean, μ, is μ necessarily an inverse limit mean with respect to its sequence $\{I_n, f_n\}$? A related question would be if two such inverse limit spces are homeomorphic, and one admits an inverse limit mean with respect to its sequence, must the other space also admit on with respect to its sequence? The purpose of this paper is to answer both of these questions in the negative. For brevity, the definitions, terms and notations of [**1**] will be employed as in that paper and will not be repeated here. In what follows,g will denote the continuous function from $I = [0,1]$ onto I where

$$g(x) = \begin{cases} 2x & \text{if } 0 \le x \le 1/2, \\ 2 - 2x & \text{if } 1/2 < x \le 1. \end{cases}$$

2. A preliminary theorem. Theorem 4 of [**1**] established that there does not exist an *infinite* sequence $\{\mu_n\}$ of means on I such that,

1980 *Mathematics Subject Classification*: Primary 39B10, 39B30.
Received by the editors on January 29, 1986 and in revised form on February 27, 1986.

for all $n = 1, 2, 3, \ldots,$ $g(\mu_{n+1}(x, y)) = \mu_n(g(x), g(y))$ for all $x, y \in I$. We will now show that with appropriate modifications to the proof given in [**1**] for this theorem we can obtain a much stronger version.

THEOREM. *There exists a positive integer $N \geq 3$ such that there does not exist an N-term sequence $\mu_1, \mu_2, \ldots, \mu_N$ of means on I such that, for each $n \leq N - 1$, the functional equation*

$$g(\mu_{n+1}(x, y)) = \mu_n(g(x), g(y)) \quad \textit{for all } x, y \in I$$

holds.

PROOF. As the proof of the original theorem is quite lengthy, we will give here only the modifications to the proof in [**1**] necessary to yield the stronger result. As before, we will proceed by contradiction. The original supposition and equation (20) of that argument should be replaced by the following: Suppose that, for each positive integer $n \geq 2$, there exists an n-term sequence $\mu_{1,n}, \mu_{2,n}, \ldots, \mu_{n,n}$ of means on I such that, for each $j = 2, 3, \ldots, n$,

$$(20)' \qquad g(\mu_{j,n}(x, y)) = \mu_{j-1,n}(g(x)), g(y)) \text{ for all } x, y \in I.$$

Equation (21) now becomes

$$(21)' \qquad g^m \circ \mu_{n+m,q+m} = \mu_{n,q+m} \circ G^m \text{ for } q \geq n$$

Then, for $1/2 \leq x \leq 1$, $n \geq 1$ and $q > n$ we have

$$g(\mu_{n+1,q}(x - 1 - x)) = \mu_{n,q}(g(x), g(1 - x)) = \mu_{n,q}(g(x), g(x)) = g(x).$$

Hence $\mu_{n,q}(x, 1 - x) \in \{x, 1 - x\}$ whenever $1/2 \leq x \leq 1$ and $q > n \geq 2$.

Then, by replacing μ_n by $\mu_{n,q}$ and μ_k by $\mu_{k,q}$ everywhere they appear in the argument, we prove that

$$\mu_{k,q}(x, y) \to \max\{x, y\} \text{ as } k, q \to \infty, q > k,$$

and arrive at the same contradiction.

Thus some such integer N exists. To show that $N \geq 3$, we construct a two-term sequence which satisfies the functional equation. Let

$A = \{(x, y) \in I^2 : 0 \le x < 1/2,\ 0 \le y < 1/2\}$ and let $B = I^2 - \mathrm{cl}\,(A)$. Define means μ_1 and μ_2 on I as follows:

$$\mu_1(x, y) = \max\{x, y\}$$
$$\mu_2(x, y) = \begin{cases} (1/2)\mu_1(g(x), g(y)) & \text{if } (x, y) \in A \\ 1 - (1/2)\mu_1(g(x), g(y)) & \text{if } (x, y) \in B \\ 1/2 & \text{if } (x, y) \in \mathrm{cl}\,(A) \cap \mathrm{cl}\,(B) \end{cases}$$

Straightforward arguments show that μ_2 is a mean and that

$$g(\mu_2(x, y)) = \mu_1(g(x), g(y)) \quad \text{for all } x, y \in I.$$

□

3. The example. Let N be the positive integer given by the theorem, and let $M = \lim\{I_n, f_n\}$ where, for each n, $I_n = I$; for $n = 1, 2, \ldots, N$, $f_n = g$; and for $n > N, f_n$ is the identity map on I. Then M admits no inverse limit mean. But [**2**, p. 236], M is homeomorphic to the inverse limit space, M', obtained by using the subsequence $\{I_n, f_n\}$, $n = N + 1, N + 2, \ldots$. Letting each μ_n be the usual arithmetic mean, a straightforward argument will show that M' admits an inverse limit mean. Thus, inverse limit means are not preserved under homeomorphisms. However this homeomorphism induces a mean, μ, on M which cannot be an inverse limit mean, thus answering the original question in the negative also.

4. Remarks. This result now leaves open the possibility that the continuum, M, employed in [**1**] might still admit a mean. Also, to the author's knowledge, questions (1), (2), (4) and (5) listed at the end of that paper remain open.

REFERENCES

1. John A. Baker and B.E Wilder, *Constructing means on certain continua and functional equations,* Aequationes Math. **26** (1983), 89-103.

2. C.E. Capel, *Inverse limit spaces,* Duke Math. J. **21** (1954), 233-245.

DEPARTMENT OF MATHEMATICAL SCIENCES, BERRY COLLEGE, MOUNT BERRY, GA 30149