ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 19, Number 1, Winter 1989

T-INVARIANT ALGEBRAS ON RIEMANN SURFACES II

ANDRÉ BOIVIN

1. Introductions. In [**7**], T.W. Gamelin has introduced a subclass of planar uniform algebras which are by definition invariant under the so-called Vitushkin localization operators T_φ. In [**7, 8**] (see also [**5**]) he has developed the theory and, in particular, he has proved that a planar T-invariant algebra always has the Banach approximation property.

DEFINITION. A Banach space B has the *Banach approximation property* (BAP) if there exists a sequence $\{P_n\}_{n=1}^{\infty}$ of finite dimensional linear operators on B such that $P_n f$ converges to f for all $f \in B$.

More recently, J.A. Cima and R.M. Timoney [**4**] have shown that all planar T-invariant algebras also have the Dunford-Pettis property.

DEFINITION. A Banach space B has the *Dunford-Pettis property* (DPP) if, whenever $\{f_n\}_{n=1}^{\infty}$ is a sequence in B^* tending weakly to 0, then

$$\lim_{n\to\infty} F_n(f_n) = 0.$$

We have, in [**2**], suggested a generalization of the Vitushkin operators to arbitrary non-compact Riemann surfaces and we have then proceeded to outline the development of a theory of T-invariant algebras in this context. We now continue our study by establishing the BAP and the DPP for T-invariant algebras on Riemann surfaces.

REMARK. In 1972, A. Sakai had already used the Behnke-Stein kernel (see [**10**]) in order to define a Cauchy transform and study some of the properties of the T_φ-operators on non-compact Riemann surfaces. We would like to thank the referee who has brought this to our attention.

Research supported in part by an NSERC of Canada postdoctoral scholarship.
Received by the editors on October 10, 1986.

Acknowledgement. It will be evident that I owe a great debt to the work of T.W. Gamelin. I wish however to emphasize his kind permission to publish for the first time some of his results in this article. I thank A.G. O'Farrell for many valuable conversations and references. My appreciation also goes to J.A. Cima and R.M. Timoney who made available their manuscript prior to publication. Finally, I wish to express my gratitude to J.M. Anderson and the Department of Mathematics at University College London for their hospitality while this work was carried out.

Notation. Following [**2**], R will denote an arbitrary non-compact Riemann surface, and K a compact subset of R. $C(K)$ will be the set of all continuous functions on K with the topology of uniform convergence; $\mathrm{Hol}(K)$ will denote the set of all holomorphic functions on K, that is, holomorphic in a neighborhood of K, and $M(K)$ will be the algebra of functions in $C(K)$ which can be uniformly approximated on K by meromorphic functions on R with poles off K. If K is a compact subset of the complex plane $\mathbf{C}, M(K)$ is just the closure in $C(K)$ of the rational functions with poles off K, usually denoted by $R(K)$.

For complex-valued functions f and φ on R, f bounded and φ smooth with compact support, define

$$(T_\varphi f)(q) = \varphi(q)f(q) + \frac{1}{2\pi i}\int\int_R f(p)F(p,q)\ \bar{\partial}\varphi(p) \wedge \partial\rho(p),\ \ q \in R,$$

where ρ is a globally holomorphic but locally univalent function on R and $F(p,q)$ is a Cauchy kernel on R, that is, F is a meromorphic function on $\mathbf{R} \times \mathbf{R}$ such that $F(p,q) = -F(q,p)$ and the only singularities of $F(\cdot, q)$ are simple poles on the diagonal with residues $+1$. For the existence of ρ and F, we refer the reader to [**2**].

DEFINITION. We say that a closed subalgebra A of $C(K)$ is T-invariant if it satisfies the following properties:

(1) $M(K) \subseteq A$

(2) $T_\varphi f \in A$, for all $f \in A$ and all φ smooth with compact support. Here T_φ is regarded as an operator on $C(K)$. In other words, $f \in C(K)$

is extended to be zero off K, the operator T_φ is applied to the extension and the result is restricted to K.

The expression "planar T-invariant algebra" will refer to the case $R = \mathbf{C}$, $\rho = z$, with the Cauchy kernel $1/(z-w)$. Remark, however, that the definition of T-invariance is independent of a particular choice of ρ or F. It will nevertheless be convenient to let ρ be fixed and assume that all references to coordinate systems are made with respect to ρ.

2. On the Banach approximation property. In this section we will prove

THEOREM I. *A T-invariant subalgebra of $C(K)$ has the Banach approximation property* (BAP).

We will establish first a few simple lemmas, and proofs will be given for the sake of completeness.

LEMMA 1. Let A be a T-invariant subalgebra of $C(K)$ and let D be a closed parametric disk on R. Let A_1 be the uniform closure on $K \cap D$ of the restriction of A to $K \cap D$ (denoted by $A|_{K\cap D}$). Then A_1 is a planar T-invariant algebra. (Of course, we are identifying $p \in K \cap D$ and $z = \rho(p) \in \mathbf{C}$.)

PROOF. First of all, we have to show that $R(K \cap D) \subset A_1$. By Runge's theorem [**1**], $R(K \cap D) = M(K \cap D)$, but, by [**11**], since no components of $R \backslash (K \cap D)$ are contained in K, every function in $M(K \cap D)$ can be approximated uniformly on $K \cap D$ by meromorphic functions with poles off K, i.e., by functions in $M(K)|_{K\cap D}$. Thus, by property (1), $R(K \cap D) \subset A_1$. □

Let us now introduce the following notation. T_φ will denote, as above, the Vitushkin localization operator on $C(K)$. T_φ^1 will denote the same operator but now acting on $C(K \cap D)$; in other words, $f \in C(K \cap D)$ is extended to be zero off $K \cap D$ before the operator T_φ is applied.

Finally, by $T_\varphi^{\mathbf{C}}$, we will denote the operator defined by

$$(T_\varphi^{\mathbf{C}} f)(w) = \varphi(w)f(w) + \frac{1}{2\pi i}\iint \frac{f(z)}{z-w}\,\frac{\partial\varphi}{\partial\bar z}\,d\bar z\wedge dz, \qquad w \in \mathbf{C}.$$

On $K \cap D$, $T_\varphi f$ and $T_\varphi^1 f$ differ by

$$(*) \qquad \frac{1}{2\pi i}\iint_{K\setminus D} f(p)F(p,q)(\bar\partial\varphi\wedge\partial\rho)(p).$$

We remark that $(*)$ is the Cauchy transform of a measure of the form $h\omega$, where ω is the area measure on R and $h \in L_c^\infty(\omega)$. In particular, the support of h is contained in $K\setminus D$ and thus, by [**2**, Theorem 3.3], $(*) \in M(K\cap D)$. This implies, with property (**2**), that $T_\varphi^1 f \in A_1$, for all $f \in A$ and, by density, for all $f \in A_1$.

Now we can assume that, on $D\cap K$, F is of the form $1/(z-w)+h(z,w)$ where $p, q \in R$ and $z = \rho(p)$, $w = \rho(q) \in \mathbf{C}$ are identified and where h is holomorphic. Thus T_φ^1 and $T_\varphi^{\mathbf{C}} f$ differ only by

$$\frac{1}{2\pi i}\iint f(z)h(z,w)\frac{\partial\varphi}{\partial\bar z}(z)\,d\bar z\wedge dz$$

which is holomorphic on $K\cap D$ and so $T_\varphi^{\mathbf{C}} f \in A_1$ for all $f \in A_1$. □

LEMMA 2. *Let A be a T-invariant subalgebra of $C(K)$ and let D be a closed parametric disk on R. Let A_1 be the uniform closure on $K\cap D$ of the restriction of A to $K\cap D$. Suppose $f \in A_1$ and φ is a smooth function with support contained in D. Then $T_\varphi f \in A$.*

PROOF. We first extend f to be continuous on D and note that there exists a sequence $\{f_n\}$ of functions in A and continuous in D that converges uniformly on D to f. Then, by [**2**, Lemma 4.4], $(T_\varphi f_n)|_k \in A$. But

$$\begin{aligned}
\|T_\varphi f - T_\varphi f_n\|_K &= \|T_\varphi(f-f_n)\|_K \\
&= \|T_\varphi((f-f_n)|_D)\|_K \\
&\le \|f-f_n\|_D \\
&\quad + \|f-f_n\|_D\Big(\frac{1}{2\pi}\iint |F(p,q)|\,|\bar\partial\varphi\wedge\partial\rho|(p)\Big),
\end{aligned}$$

where $||\cdot||_K$ and $||\cdot||_D$ denote the sup norm on K and D respectively. Thus $(T_\varphi f_n)|_K$ converges uniformly on K to $(T_\varphi f)|_K$. Since A is closed, this proves the Lemma. □

LEMMA 3. *Let A be a T-invariant subalgebra of $C(K)$, let $f \in A$ and denote again, by f, a continuous extension of f with support in a pre-compact neighborhood V of K. Let $\{\varphi_i, D_i\}_{i=1}^N$ be a finite partition of unity of V where each D_i is a parametric disk on R, each function φ_i is smooth with support contained in D_i, $0 \le \varphi_i \le 1$ and $\sum_{i=1}^N \varphi_i = 1$ on V. The the equality*

$$f(q) = \sum_{i=1}^{N} (T_{\varphi_i} f)(q)$$

holds for all $q \in R$.

PROOF. By [**2**, Theorem 4.2], $f - \Sigma T_{\varphi_i} f$ is holomorphic on R. But, if $q \notin \cup_{i=1}^N D_i$,

$$\begin{aligned}(f - \Sigma T_{\varphi_i} f)(q) = f(q) - \sum_{i=1}^{N} (\varphi_i(q) f(q) \\ + \frac{1}{2\pi i} \iint f(p) F(p,q)(\overline{\partial}(\Sigma \varphi_i) \wedge \partial \rho)(p)) \\ = 0\end{aligned}$$

since supp$f \in \cup_{i=1}^N D_i$ and $\overline{\partial}(\Sigma\varphi_i) = 0$ on supp$f$. □

We need one more result which can be found in [**8**].

THEOREM A. (GAMELIN). *Let Q be a compact subset of the complex plane $\mathbf{C}$ and let A be a planar T-invariant subalgebra of $C(Q)$. Then A has the Banach approximation property.*

Before sketching the proof of Theorem A, let us introduce some notation. Let E be any subset of the complex plane. The partition of unity $\{\Delta_j, g_j\}$ is called a Vitushkin δ-cover of E if each Δ_j is a disk of radius δ centered at a point $z_j \in E$, each g_j is supported on $\Delta_j, \Sigma g_j = 1$ in a neighborhood of $\overline{E}, |\partial g_j/\partial\overline{z}| \leq c_1/\delta$ and no point z belongs to more than c_2 of the disks, where c_1 and c_2 are some universal constants independent of E and δ.

If f is analytic at infinity, define

$$\alpha(f) = \lim_{z\to\infty} z(f(z) - f(\infty)).$$

If moreover, f satisfies $f(\infty) = \alpha(f) = 0$, then define

$$\beta(f) = \lim_{z\to\infty} z^2 f(z).$$

Let $\mathbf{C}^* = \mathbf{C} \cup \{\infty\}$ denote the extended complex plane. If E is a bounded subset of $\mathbf{C}$, then define

$$\begin{aligned}\alpha_A(E) = \sup\{|\alpha(f)| : f \in C(\mathbf{C}^*), f \text{ analytic off a compact subset of } \\ E, f|_Q \in A,\ |f| \leq 1 \text{ and } f(\infty) = 0\},\\ \beta_A(E) = \sup\{|\beta(f)| : f \in C(\mathbf{C}^*), f \text{ analytic off a compact subset of } \\ E, f|_Q \in A,\ |f| \leq 1 \text{ and } f(\infty) = \alpha(f) = 0\}.\end{aligned}$$

We are now ready for:

SKETCH OF PROOF. (of THEOREM A). Let $f \in A$. We can extend f in such a way that the extension is continuous on $\mathbf{C}$, has compact support and depends linearly on f. Fix $\delta > 0$ and let $\{\Delta_j, g_j\}$ be a Vitushkin δ-cover of supp$f$. Choose functions $F_j, G_j \in C(\mathbf{C}^*)$ such that $|F_j| \leq 1$, $|G_j| \leq 1$, $F_j|_Q \in A$, $G_j|_Q \in A$, F_j and G_j are analytic off a compact subset of Δ_j, $F_j(\infty) = G_j(\infty) = \alpha(G_j) = 0$ and

$$\begin{aligned}\alpha(F_j) &\geq \alpha_A(\Delta_j)/2\\ \beta(G_j) &\geq \beta_A(\Delta_j)/2.\end{aligned}$$

Set $f_j = T_{g_j}^{\mathbf{C}} f$. Note that $f_j(\infty) = 0$. Finally put

$$\begin{aligned}\theta_j(f) &= \alpha(f_j)/\alpha(F_j)\\ \psi_j(f) &= \beta(f_j - \theta_j(f)F_j)/\beta(G_j).\end{aligned}$$

Note that θ_j and ψ_j are continuous linear functionals on A and that $f_j - \theta_j F_j - \psi_j G_j$ has a triple zero at infinity. Set

$$P_\delta f = \Sigma(\theta_j(f)F_j + \psi_j(f)G_j).$$

Then $P_\delta f$ is a continuous linear operator on A with finite dimensional range. The usual estimates (see for example [**6**, Chapter VIII]) show that

$$||f - P_\delta f|| \leq c\omega_f(2\delta),$$

where c is a constant and $\omega_f(b) = \sup\{|f(z) - f(\zeta)| : z, \zeta in \mathbf{C}, |z - \zeta| \leq b\}$ is the modulus of continuity of f. Consequently the P_δ^i converge strongly to the identity operator as δ tends to zero. □

PROOF OF THEOREM I. Let $f \in A$. Denote again by f an extension of f which is continuous and has compact support. Let $\{D_i, \varphi_i\}_{i=1}^N$ be a finite partition of unity of the support of f such that each D_i is a closed parametric disk. Let A_i be the closure of $A|_{K\cap D_I}$ on $K \cap D_i$. By Lemma 1, each A_i can be regarded as a planar T-invariant subalgebra of $C(K \cap D_i)$. By Theorem A, there exists for each i, a sequence P_δ^i of finite rank continuous linear operators which converges strongly to the identity operator on $C(K \cap D_i)$. Each P_δ^i is of the form

$$P_\delta^i f = \sum_{k=1}^{M_i} \lambda_k^i(f) H_k^i,$$

with each $H_k^i \in A_i$. Set

$$P_\delta = \sum_{i=1}^{N} T_{\varphi_i} P_\delta^i = \sum_{i=1}^{N} \sum_{k=1}^{M_i} \lambda_k^i(f) T_{\varphi_i} H_k^i.$$

By Lemma 2, $T_{\varphi_i} H_k^i \in A$. Thus P_δ is a continuous linear operator on A with finite dimensional range and

$$\begin{aligned} ||f - P_\delta f||_K &= ||\Sigma T_{\varphi_i} f - \Sigma T_{\varphi_i} P_\delta^i f||_K \\ &= ||\Sigma T_{\varphi_i}(f - P_\delta^i f)||_K \\ &\leq \Sigma ||f - P_\delta^i f||_{K\cap D_i} ||T_{\varphi_i} 1||_{K\cap D_i}. \end{aligned}$$

This proves Theorem I. □

3. On the Dunford-Pettis property. In this section, as an easy consequence of [**3**], [**4**] and [**9**], we will establish

THEOREM II. *Let A be a T-invariant subalgebra of $C(K)$. Then both A and A^* have the Dunford-Pettis property* (DPP).

Let Q be a compact subset of the complex plane and let A be a planar T-invariant subalgebra of $C(Q)$. In [**4**], J.A. Cima and R.M. Timoney have used a recent result of J. Bourgain [**3**] to show that both A and A^* have the Dunford-Pettis Property. In [**9**], T.W. Gamelin has extended this result to all compactly tight algebras.

DEFINITION. (see [**5**]). We say that a uniform algebra A on a compact Hausdorff space X is *tight* on X if the operators $S_g : A \to C(X)/A$ defined by $S_g(f) = gf + A$ are weakly compact for all $g \in C(X)$. We say that A is *compactly tight* if each of the operators S_g is compact.

Let Y be a subspace of $C(X)$ and let the operators S_g be defined with respect to Y, i.e., for each $g \in C(X)$, the operator $S_g : Y \to C(X)/Y$ is defined by $S_g(f) = gf + Y$. The double dual of S_g is the operator $S_g^{**} : Y^{**} \to C(X)^{**}/Y^{**}$ given by $S_g^{**}(F) = gF + Y^{**}$. Following [**4**], define the Bourgain algebras with respect to Y to be

$$Y_B = \{g \in C(X) : f_n \in Y, f_n \to 0 \text{ weakly implies } S_g f_n \to 0 \text{ in norm}\}.$$

$$Y_b = \{g \in C(X) : F_n \in Y^{**}, F_n \to 0 \text{ weakly implies } S_g^{**} F_n \to 0 \text{ in norm}\}.$$

THEOREM B. (BOURGAIN [**3**], [**4**, Theorem 3]). *Let X be a compact Hausdorff space and Y a closed subspace of $C(X)$.*

(i) *If $Y_B = C(X)$, then Y and Y^* both have the Dunford-Pettis property.*

item(ii) *If $Y_b = C(X)$, then Y has the Dunford-Pettis property.*

As an immediate consequence, we have

COROLLARY 1. (GAMELIN [**9**]. *If A is a compactly tight algebra, then both A and A^* have the* DPP.

PROOF. Since A is compactly tight, the operators S_g are compact operators and so, map weakly null sequences onto null sequences. Thus A_B (and A_b) coincides with $C(X)$. □

To complete the proof of Theorem II, it thus suffices to establish the following.

PROPOSITION. (See [**5**, Theorm 17.7]). *Any T-invariant subalgebra A of $C(K)$ is compactly tight on K.*

PROOF. Let g be a smooth function supported on a compact subset of R and let $f \in A$. Define

$$(R_q f)(q) = \frac{1}{2\pi i} \iint f(p)F(p,q)(\bar{\partial} g \wedge \partial \rho)(p), \; q \in R.$$

The same proof as in [**5**, Lemma 6.1] shows that the operator R_g is compact. Note that $fg = T_g f - R_g f$ and that, by definition, $T_g f \in A$. Hence S_g is compact. Approximating an arbitrary $g \in C(K)$ uniformly by smooth functions, we see that each operator S_g is compact. □

REFERENCES

1. H. Behnke and K. Stein, *Entwicklung analytischer Funktionen auf Riemannschen Flächen*, Math. Ann. **12** (1949), 430-461.

2. A. Boivin, *T-invariant algebras on Riemann surfaces* I, Mathematika **34** (1987), 160-171.

3. J. Bourgain, *The Dunford-Pettis property for the ball-algebras, polydisc-algebras and the Sobolev spaces*, Studia Math. **77** (1984), 245-253.

4. J.A. Cima and R.M. Timoney, *The Dunford-Pettis property for some planar uniform algebras*, Michigan Math. J. **34** (1987), 99-104.

5. B.J. Cole and T.W. Gamelin, *Tight uniform algebras and algebras of analytic functions*, J. Funct. Anal. **46** (1982), 158-220.

6. T.W. Gamelin, *Uniform Algebras*, Chelsea, New York (2nd Ed.), 1984.

7. ———, *Uniform algebras on plane sets*, in Approximation Theory (Proc. Conf. Austin, Texas), Academic Press, New York, 1973, 100-149.

8. ———, *Rational Approximation Theory*, University of California, Los Angeles Course Notes for Math 285g, Fall 1975.

9. ———, Private Communication, 1986.

10. A. Sakai, *Localization theorem for holomorphic approximation on open Riemann surfaces*, J. Math. Soc. Japan **24** (1972), 189-197.

11. S. Scheinberg, *Uniform approximation by meromorphic functions having prescribed poles*, Math. Ann. **243** (1979), 83-93.

THE UNIVERSITY OF WESTERN ONTARIO, LONDON, CANADA, N6A 5B7