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ON NORM AND ZERO ASYMPTOTICS OF ORTHOGONAL POLYNOMIALS FOR GENERAL MEASURES, I

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ABSTRACT. Let μ be a positive unit Borel measure with infinite support in the interval $I=[-1,1]$, and let $\{p_n(x)\}_{n\geq 0}$ be the monic orthogonal polynomials associated to μ. Let ν_n denote the unit measure having mass $1/n$ at each zero of $p_n(x)$, and let $\lambda_n = (\int (p_n(x))^2 d\mu)^{1/2n}$. It is known that, subject to the condition that no set of unit μ-measure has capacity zero, when a subsequence of the sequence $\{\nu_n\}_{n\geq 1}$ converges, the corresponding subsequence of $\{\lambda_n\}_{n\geq 1}$ converges as well. We give an example which shows that the converse statement is not true in general.

Introduction and Definitions. We say that μ is a *weight measure* if it is a positive unit Borel measure with infinite support in the interval $I=[-1,1]$, where the *support* $S(\mu)$ is the smallest closed set of unit μ-measure. Given a weight measure μ, for $n=0,1,2,\ldots,$ let $p_n(x) = x^n + \cdots,$ denote the n^{th} orthogonal polynomial associated to μ, so that

$$\left(\int p_m(x)p_n(x)d\mu\right)^{1/2} = N_n\delta_{m,n},$$

where $\delta_{m,n} = \{0 \text{ if } m\neq n,\ 1 \text{ if } m=n\}$. Let ν_n denote the *zero measure* for $p_n(x)$, i.e., the unit measure with mass $1/n$ at each zero of $p_n(x)$, and let $\lambda_n = (N_n)^{1/n}$ be the n^{th} *linearized norm*. It is known that, subject to the condition that no set of unit μ-measure has capacity zero, when a subsequence of the sequence $\{\nu_n\}_{n\geq 1}$ converges, the corresponding subsequence of $\{\lambda_n\}_{n\geq 1}$ converges as well. Here we give an example which shows that the converse statement is not true in general.

If, for a unit Borel measure μ defined on I, we have $\lim_{n\to\infty}\int f d\nu_n = \int f d\mu$ for all functions f continuous on I,then we say that the sequence $\{\nu_n\}_{n\geq 1}$ *converges weakly* to μ or that μ is a *weak limit* of $\{\nu_n\}_{n\geq 1}$, and we write $\lim_{n\to\infty}\nu_n = \mu$. By a theorem of Helly, $\{\nu_n\}_{n\geq 1}$ always has weakly convergent subsequences.

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The capacity of a bounded Borel set B is taken as its inner logarithmic capacity, i.e., $C(B) = \sup_{K\subset B} C(K)$, where K runs through all compact subsets of B, and $C(K)$ is the capacity of K as derived from the logarithmic potential function $U(z,\mu) = \int \log(|z-t|^{-1})d\mu(t)$ [**4**]. The following are properties of capacity [**5**]: For Borel sets B_1 and B_2,

(i) $B_1 \subset B_2 \Rightarrow C(B_1) \leq C(B_2)$,

(ii) $C(B_1) = 0 \Rightarrow C(B_2 \cup B_1) = C(B_2 \backslash B_1) = C(B_2)$,

(iii) if K_n is compact, and $K_n \subset K_{n+1} \subset I (n = 1, 2, \ldots)$, then $\lim_{n\to\infty} C(K_n) = C(\cup_{n\geq 1} K_n)$.

A *carrier* of a weight measure μ is a Borel subset B of $S(\mu)$ such that $\mu(B) = 1$. Associated with μ are two numbers, $\underline{C} = \inf C(B)$, where the infimum is taken over all carriers, and $\overline{C} = C(S(\mu))$. There exist weight measures μ for which $\underline{C} < \overline{C}$ and these measures are called *undetermined.* The sequences of zero measures and linearized norms of an undetermined measure may or may not converge [**7, 8, 9**].

Let B be a bounded Borel subset of the complex plane $\mathbf{C}$ having positive capacity. The *equilibrium measure for* B is the unique unit measure μ which satisfies the following [**3**, p. 171]:

(i) $U(z,\mu) \leq \log(C(B)^{-1})$ for $z \in \mathbf{C}$,

(ii) $U(z,\mu) = \log(C(B)^{-1})$ for $z \in B\backslash Z,\ C(Z) = 0$,

(iii) $S(\mu) \subset \overline{B}$, where $\overline{B}$ denotes the closure of B.

We denote the equilibrium measure for B by μ_B.

When $B \subset I$, condition (iii) can be changed to (iii)* $S(\mu) \subset I$. To see this let μ be any unit measure satisfying (i), (ii), and (iii)*. Let $K_n \subset K_{n+1} (n = 1, 2, \ldots)$ be compact subsets of $B\backslash Z$ such that $C(K_1) > 0$ and $\lim_{n\to\infty} C(K_n) = C(B\backslash Z) = C(B)$. Let μ_n be the equilibrium measure for K_n, and let

$$v_n(z) = (U(z,\mu) - \log(C(B)^{-1})) - (U(z,\mu_n) - \log(C(K_n)^{-1})).$$

Then $v_n(z)$ is superharmonic on $\mathbf{C}\backslash K_n$ and has a finite limit as z tends to infinity. Since $U(z,\mu)$ is lower-semicontinuous we have, for $\zeta \in K_n$,

$$\begin{aligned}\liminf_{z\to\zeta} v_n(z) &\geq \liminf_{z\to\zeta}(U(z,\mu) - \log(C(B)^{-1})) \\ &\quad - \limsup_{z\to\zeta}(U(z,\mu_n) - \log(C(K_n)^{-1})) \\ &\geq U(\zeta,\mu) - \log(C(B)^{-1}) = 0.\end{aligned}$$

Hence, by the minimum principle for superharmonic functions [**1**, p. 59] we have $v_n(z) \geq 0$ for $z \in \mathbf{C}\backslash K_n$ and a fortiori for $z \in \mathbf{C}\backslash I$. Thus, for some weakly convergent subsequence of $\{\mu_n\}_{n\geq 1}$, say $\lim_{k\to\infty} \mu_{n_k} = \nu$, we have $U(z,\mu) - U(z,\nu) \geq 0$ for $z \in \mathbf{C}\backslash I$. Thus $U(z,\mu) - U(z,\nu) = 0$ for $z \in \mathbf{C}\backslash I$, and hence $\mu = \nu$ [**4**, pp. 34, 50]. Since all this is true for $\mu = \mu_B$ we have $\nu = \mu_B$. Thus, when $B \subset I$, any measure μ satisfying (i), (ii), and (iii)* must equal μ_B.

Results and proofs. We first state a result on the relationship between the sequences of zero measures and linearized norms which is due to Ullman and whose proof will appear elsewhere.

PROPOSITION. *Let μ be an undetermined weight measure having no carrier of capacity zero. If a subsequence $\{\nu_{n_k}\}_{k\geq 1}$ of $\{\nu_n\}_{n\geq 1}$ has weak limit ν_0, then the corresponding subsequence $\{\lambda_{n_k}\}_{k\geq 1}$ of $\{\lambda_n\}_{n\geq 1}$ converges to a limit λ_0. Furthermore, λ_0 is associated with ν_0 in the sense that, if $\{\nu_{m_k}\}_{k\geq 1}$ is any other subsequence of $\{\nu_n\}_{n\geq 1}$ converging weakly to ν_0, then the subsequence $\{\lambda_{m_k}\}_{k\geq 1}$ also converges to λ_0.*

It is our intention here to show that the converse statement to the proposition is not true in general by constructing an example of a weight function $w(x)$ on I such that, for $d\mu = w(x)\,dx$, the subsequence $\{\lambda_{n_m}\}_{m\geq 1}$ converges while the subsequences $\{\nu_{n_{2k-1}}\}_{k\geq 1}$ and $\{\nu_{n2k}\}_{k\geq 1}$ converge weakly to unequal limits.

We now give four lemmas to be used in the construction. The first lemma extends the "transfer function" lemma of Ullman.

LEMMA 1. [**6, 8**]. *Let K be a finite disjoint union of nondegenerate closed subintervals in I. Then, for each positive integer n, there is a sequence $\{c_n\}_{n\geq 1}$ of positive numbers depending only on K such that $\lim_{n\to\infty} c_n^{1/n} = 1$, and there is a family $\{\tau_{n,\theta}(x)\}_{n\geq 1, 0<\theta\leq 1}$ of nonnegative Borel measurable functions such that, for each θ, $0 < \theta \leq 1$, and each polynomial $q_n(x)$ of degree n:*

(i) $\int \tau_{n,\theta}(x)dx = n^{-2}$,

(ii) $A_n(\theta) = \{x : \tau_{n,\theta}(x) > 0\}$ *is a compact subset of K, and*

(iii) $\int (q_n(x))^2 \tau_{n,\theta}(x)dx \geq c_n(||q_n(x)||_K)^2$, *where* $||q_n(x)||_K = \max_{x\in K} |q_n(x)|$.

We say that $\tau_{n,\theta}(x)$ *is a transfer function and* $A_n(\theta)$ *a transfer set for* K. *Moreover, the transfer functions* $\tau_{n,\theta}(x)$ *can be chosen in such a way that* $C(A_n(\theta))$ *is a continuous and nondecreasing function of* θ *so that, for any compact subset* J *of* I,

$$(C(J), C(J \cup K)] \subset \{C(J \cup A_n(\theta)) : 0 < \theta \leq 1\}.$$

PROOF. For $z \in \mathbf{C}$, let $G(z) = \log(C(K)^{-1}) - U(z, \mu_K)$, and let $G_\delta = \max_{z\in K_\delta} |G(z)|$ where $K_\delta = \{z \in \mathbf{C} : \min_{t\in K} |z - t| \leq \delta\}$. It has been shown [**6**] that if $x^* \in K$ is chosen such that $|q_n(x^*)| = ||q_n(x)||_K$ and we set $d_n = [n \exp(nG_{2/n})] + 1$ (brackets $[\cdot]$ indicating the least integer function), then, for x such that $|x-x^*| \leq (2d_n)^{-1}$, it follows that $|q_n(x)| \geq (1/2)||q_n(x)||_K$. Let $I_k = [-1+(k-1)(2d_n)^{-1}, \ -1+k(2d_n)^{-1}]$ and let $\{K_i : 1 \leq i \leq m_n\}$, where $m_n \leq 4d_n$, be the nonempty sets (excluding any isolated points) among $\{K \cap I_k : 1 \leq k \leq 4d_n\}$. We may express each K_i as a finite union of closed intervals, say $K_i = J_1 \cup \cdots \cup J_m$, and if we set $J_j = [a_j, b_j]$ for $1 \leq j \leq m$, then let $J_j(\theta) = [(a_j + b_j)/2 - \theta(b_j - a_j)/2, \ (a_j + b_j)/2 + \theta(b_j - a_j)/2]$ and $K_i(\theta) = J_1(\theta) \cup \cdots \cup J_m(\theta)$ for $0 < \theta \leq 1$. Then set $c_n = (4n^2 m_n)^{-1}$ and $\tau_{n,\theta}(x) = (n^2 m_n)^{-1} \sum_{1\leq i\leq m_n} \chi_{K_i(\theta)}(x) \ (\ell(K_i(\theta)))^{-1}$, where $\chi_{K_i(\theta)}(x)$ is the characteristic function of the set $K_i(\theta)$, and where ℓ indicates linear measure. Now x^* will be in one of the sets $K_i(1)$, say $x^* \in K_{i^*}(1)$, so we then have

$$\int (q_n(x))^2 \tau_{n,\theta}(x)\, dx \geq (n^2 m_n)^{-1} \int (q_n(x))^2 \chi_{K_{i^*}(\theta)}(x) \ (\ell(K^{-1}_{i^*(\theta)}))\, dx$$
$$\geq c_n(||q_n(x)||_K)^2.$$

The fact that $\lim_{n\to\infty} c_n^{1/n} = 1$ follows since $\lim_{\delta\to 0} G_\delta = 0$ [**4**, p. 7]. Now let $A_n(\theta) = K_1(\theta) \cup \cdots \cup K_{m_n}(\theta)$. The function $C(A_n(\theta))$ is nondecreasing by monotonicity of capacity. To see that $C(A_n(\theta))$ is continuous we will prove the general statement that, for a given Borel subset B of I and for a given positive ε, there exists a positive δ such that, for any Borel subset A of I where $C(A) < \delta$, we have

$$C(B) - \varepsilon \leq C(B\backslash A) \leq C(B \cup A) \leq C(B) + \varepsilon.$$

We may assume that $C(B) > 0$, $C(B)^2 \geq C(A) > 0$, $C(B\backslash A) > 0$, and that B and A are subsets of [-1/2, 1/2] [**4**, p. 56]. From the inequality [**4**, p. 63] $(\log C(B))^{-1} \geq (\log C(B\backslash A))^{-1} + (\log C(A))^{-1}$, it follows that $C(B\backslash A) \geq C(B)^2$ and also that

$$C(B) \leq C(B\backslash A)^{\left(\frac{\log C(A)}{\log C(A)+\log C(B\backslash A)}\right)}.$$

Hence $C(B) \leq C(B\backslash A) + \varepsilon$ provided we choose $C(A)$ sufficiently small so as to satisfy

$$1 + \frac{\log(1 + \frac{\varepsilon}{C(B\backslash A)})}{\log C(B\backslash A)} \leq 1 + \frac{\log(1 + \frac{\varepsilon}{C(B)^2})}{\log(C(B))} \leq \frac{\log(C(A)}{\log C(A) + \log C(B\backslash A)}.$$

Similarly, it follows from the inequality [**4**, p. 63] $(\log C(B \cup A))^{-1} \geq (\log C(B))^{-1} + (\log C(A))^{-1}$ that

$$C(B \cup A) \leq C(B)^{\left(\frac{\log C(A)}{\log C(A)+\log C(B)}\right)},$$

and hence that $C(B \cup A) \leq C(B) + \varepsilon$ provided we choose $C(A)$ sufficiently small so as to satisfy

$$1 + \frac{\log(1 + \frac{\varepsilon}{C(B)})}{\log C(B)} \leq \frac{\log C(A)}{\log C(A) + \log C(B)}. \ \square$$

LEMMA 2. (SZEGÖ and TONELLI) [**4**, p. 73]. *Let K be a compact subset of I and let $M_n(K) = \inf ||q_n(x)||_K$, where the infimum is taken over all monic polynomials $q_n(x)$ of degree n.*

(i) *If K contains at least n points, then there is a unique monic Chebychev polynomial of degree n for K, denoted $T_n(x, K)$, such that $||T_n(x, K)||_K = M_n(K)$.*

(ii) *If K is infinite, then $\lim_{n\to\infty}(M_n(K))^{1/n} = C(K)$.*

(iii) *For $x \in I, |T_n(x, K)| \leq 2^n$.*

LEMMA 3. [**7**]. *Let μ be a weight measure with subsequence of monic orthogonal polynomials $\{p_{n_m}(x)\}_{m\geq 1}$ and zero measures $\{\nu_{n_m}\}_{m\geq 1}$.*

Let B be a carrier of μ of positive capacity with equilibrium measure μ_B. Then the following two statements are equivalent:

(i) $\lim_{m\to\infty} \nu_{n_m} = \mu_B$,

(ii) $\limsup_{m\to\infty} |P_{n_m}(x)|^{1/n_m} \leq C(B)$

for all x in some Borel subset B^* of B of the same capacity.

Let B be a bounded Borel set of positive capacity with equilibrium measure μ_B. We denote by $A(\mu_B)$ the set $\{z \in \mathbf{C} : U(z,\mu_B) = \log(C(B)^{-1})\}$, and we will use the fact that $C(A(\mu_B)) = C(B)$ [**2**, p. 189].

LEMMA 4. (ULLMAN). *Let B_1 and B_2 be two Borel subsets of I, each having positive capacity C. Then $\mu_{B_1} = \mu_{b_2}$ if and only if $C(B_1 \cup B_2) = C$.*

PROOF. Assuming $C(B_1 \cup B_2) = C$, it follows that $\mu_{B_1 \cup B_2}$ is an equilibrium measure for B_1 and B_2, so $\mu_{B_1} = \mu_{B_2}$. Conversely, if $\mu_{B_1} = \mu_{B_2}$, then $U(z,\mu_{B_1}) = U(z,\mu_{B_2}) = \log(C^{-1})$ for $z \in B_1 \cup B_2 \backslash Z$ where $C(Z) = 0$. Thus $C \leq C(B_1 \cup B_2) = C(B_1 \cup B_2 \backslash Z) \leq C(A(\mu_{B_1})) = C$, so $C(B_1 \cup B_2) = C$. □

We now proceed with the construction of $w(x)$. Choose intervals I_1 and I_2 in I and some number λ such that $\max\{C(I_1), C(I_2)\} < \lambda < C(I_1 \cup I_2)$. For a compact set K, let $T_n(x,K)$ be the Chebychev polynomial of degree n for K, and let $||T_n(x,K)|| = ||T_n(x,K)||_K$. Let $\{\delta_m\}_{m\geq 1}$ be a sequence of positive numbers decreasing to zero, and let $n_0 = 1$. Then n_1 can be chosen such that $||T_{n_1}(x,I_1)||^{1/n_1} < \lambda$ (Lemma 2) and $\sum_{n\geq n_1} n^{-2} \leq (\lambda/2)^{2n_0}$, and there exists θ_1 such that $\lambda - \delta_1 < C(I_2 \cup A_{n_1}) < \lambda$ where $A_{n_1} = A_{n_1}(\theta_1)$ is a transfer set for I_1 (Lemma 1); set $\tau_{n_1} = \tau_{n_1.\theta_1}$. Similarly, n_2 can be chosen such that $||T_{n_2}(x, I_2 \cup A_{n_1})||^{1/n_2} < \lambda$, $\sum_{n\geq n_2} n^{-2} \leq (\lambda/2)^{2n_1}$, and there exists θ_2 such that $\lambda - \delta_2 < C(I_1 \cup A_{n_2}) < \lambda$, where $A_{n_2} = A_{n_2}(\theta_2)$ is a transfer set for $I_2 \cup A_{n_1}$; set $\tau_{n_2} = \tau_{n_2.\theta_2}$.

Generally, having chosen $n_1, n_2, \ldots, n_{2k-2}$ and $A_{n_1}, A_{n_2}, \ldots, A_{n_{2k-2}}$, choose n_{2k-1} such that $||T_{n_{2k-1}}(x, I_1 \cup A_{n_2} \cup A_{n_4} \cup \cdots \cup A_{n_{2k-2}})||^{1/n_{2k-1}}$

$< \lambda$ and $\sum_{n\geq n_{2k-1}} n^{-2} \leq (\lambda/2)^{2n_{2k-2}}$, and let $A_{n_{2k-1}}$ (with $\tau_{n_{2k-1}}$) be a transfer set for $I_1 \cup A_{n_2} \cup A_{n_4} \cup \cdots \cup A_{n_{2k-2}}$ such that $\lambda - \delta_{2k-1} < C(I_2 \cup A_{n_1} \cup A_{n_3} \cup \cdots \cup A_{n_{2k-1}}) < \lambda$. (We can do this because $\lambda - \delta_{2k-3} < C(I_2 \cup A_{n_1} \cup A_{n_3} \cup \cdots \cup A_{n_{2k-3}}) < \lambda$.) Similarly, choose n_{2k} such that $||T_{n_{2k}}(x, I_2 \cup A_{n_1} \cup A_{n_3} \cup \cdots \cup A_{n_{2k-1}})||^{1/n_{2k}} < \lambda$ and $\sum_{n\geq n_{2k}} n^{-2} \leq (\lambda/2)^{2n_{2k-1}}$, and let $A_{n_{2k}}$ (with $\tau_{n_{2k}}$) be a transfer set for $I_2 \cup A_{n_1} \cup A_{n_3} \cup \cdots \cup A_{n_{2k-1}}$ such that $\lambda - \delta_{2k} < C(I_1 \cup A_{n_2} \cup A_{n_4} \cup \cdots \cup A_{n_{2k}}) < \lambda$.

Let $w(x) = c\sum_{k\geq 1}(\tau_{n_{2k-1}}(x) + \tau_{n_{2k}}(x))$, $c = (\sum_{m\geq 1}(n_m)^{-2})^{-1}$, and let $d\mu = w(x)dx$. Let $p_n(x)$ be the n$^{\text{th}}$ orthogonal polynomial associated to μ, λ_n be the n^{th} linearized norm, and ν_n the zero measure for $p_n(x)$. First, it follows from Lemma 1 and the fact $||p_n(x)||_K \geq (C(K))^n$ [**4**, p. 62] that $\liminf_{k\to\infty} \lambda_{n_{2k-1}} \geq \lambda$ and $\liminf_{k\to\infty} \lambda_{n_{2k}} \geq \lambda$. Next, we have

$$\begin{aligned}
(N_{n_{2k-1}})^2 &= \int (p_{n_{2k-1}}(x))^2 w(x)dx \\
&\leq \int (T_{n_{2k-1}}(x, I_1 \cup A_{n_2} \cup A_{n_4} \cup \cdots \cup A_{n_{2k-2}}))^2 w(x)dx \quad ([\mathbf{5}]) \\
&\leq \int\limits_{I_1 \cup A_{n_2} \cup A_{n_4} \cup \cdots \cup A_{n_{2k-2}}} (T_{n_{2k-1}}(x, I_1 \cup A_{n_2} \cup A_{n_4} \cup \cdots \cup A_{n_{2k-2}}))^2 w(x)dx \\
&\quad + \int\limits_{I\backslash(I_1 \cup A_{n_2} \cup A_{n_4} \cup \cdots \cup A_{n_{2k-2}})} (T_{n_{2k-1}}(x, I_1 \cup A_{n_2} \cup A_{n_4} \cup \cdots \cup A_{n_{2k-2}}))^2 w(x)dx \\
&\leq \lambda^{2n_{2k-1}} + c2^{2n_{2k-1}}((n_{2k})^{-2} + (n_{2k+1})^{-2} + \cdots) \quad \text{(Lemma 2)} \\
&\leq (1+c)\lambda^{2n_{2k-1}},
\end{aligned}$$

so $\limsup_{k\to\infty} \lambda_{n_{2k-1}} \leq \lambda$. Similarly,

$$\begin{aligned}
(N_{n_{2k}})^2 &= \int (p_{n_{2k}}(x))^2 w(x)dx \\
&\leq \int (T_{n_{2k}}(x, I_2 \cup A_{n_1} \cup A_{n_3} \cup \cdots \cup A_{n_{2k-1}}))^2 w(x)dx \\
&\leq \int\limits_{I_2 \cup A_{n_1} \cup A_{n_3} \cup \cdots \cup A_{n_{2k-1}}} (T_{n_{2k}}(x, I_2 \cup A_{n_1} \cup A_{n_3} \cup \cdots \cup A_{n_{2k-1}}))^2 w(x)dx
\end{aligned}$$

$$
\begin{aligned}
&+ \int\limits_{I \backslash (I_2 \cup A_{n_1} \cup A_{n_3} \cup \cdots \cup A_{n_{2k-1}})} (T_{n_{2k}}(x, I_2 \cup A_{n_1} \cup A_{n_3} \cup \cdots \cup A_{n_{2k-1}}))^2 w(x) dx \\
&\leq \lambda^{2n_{2k}} + c2^{2n_{2k}}((n_{2k+1})^{-2} + (n_{2k+2})^{-2} + \cdots) \\
&\leq (1+c)\lambda^{2n_{2k}},
\end{aligned}
$$

so $\limsup_{k\to\infty} \lambda_{n_{2k}} \leq \lambda$. Hence $\lim_{m\to\infty} \lambda_{n_m} = \lambda$. If we let $A_1 = A_{n_1} \cup A_{n_3} \cup \cdots$, and $A_2 = A_{n_2} \cup A_{n_4} \cup \cdots$, then it will follow that $\lim_{k\to\infty} \nu_{n_{2k-1}} = \mu_{I_1 \cup A_2}$, and $\lim_{k\to\infty} \nu_{n_{2k}} = \mu_{I_2 \cup A_1}$. The arguments for these limits are identical so we demonstrate the first. By Lemma 3 if suffices to show that $\limsup_{k\to\infty} |p_{n_{2k-1}}(x)|^{1/n_{2k-1}} \leq C(I_1 \cup A_2)$ for $x \in I_1 \cup A_2$. Should this be fals there would exist $x^* \in I_1 \cup A_2$, some positive ε, and an increasing sequence of positive integers $\{n_j\}_{j\geq 1} \subset \{n_{2k-1}\}_{k\geq 1}$ such that $|p_{n_j}(x^*)| > (C(I_1 \cup A_2) + \varepsilon)^{n_j}$ $(j = 1, 2, \ldots)$. Then

$$(N_{n_j})^2 \geq c_{n_j}(||p_{n_j}(x)||_{I_1 \cup A_2})^2 > c_{n_j}(C(I_1 \cup A_2) + \varepsilon)^{2n_j}$$

which would imply that $\limsup_{j\to\infty} \lambda_{n_j} > C(I_1 \cup A_2)$, a contradiction. Finally, we have $C(I_1 \cup A_2) = C(I_2 \cup A_1) = \lambda < C(I_1 \cup I_2)$, so it follows from Lemma 4 that $\mu_{I_1 \cup A_2} \neq \mu_{I_2 \cup A_1}$.

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