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ASYMPTOTICS FOR ORTHOGONAL POLYNOMIALS WITH REGULARLY VARYING RECURRENCE COEFFICIENTS

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1. Introduction. Let $\{p_n(x); n = 0, 1, 2, \ldots\}$ be orthogonal polynomials with recurrence relation

$$(1.1)\qquad \begin{aligned} xp_n(x) &= a_{n+1}p_{n+1}(x) + b_n p_n(x) + a_n p_{n-1}(x), \\ n &= 0, 1, 2, \ldots, \quad p_{-1} = 0, \quad p_0 = 1. \end{aligned}$$

We want to investigate the case where the recurrence coefficients a_n and b_n are unbounded. We will use the notion of regular variation to specify the behavior of a_n and b_n as n tends to infinity.

DEFINITION. (SENETA [**20**, p. 46]) A sequence $\{c_n : n = 0, 1, 2, \ldots\}$ is regularly varying at infinity if there exists a sequence $\{\lambda_n : n = 0, 1, 2, \ldots\}$ such that

$$\lim_{n\to\infty} \frac{c_n}{\lambda_n} = 1$$

and

$$(1.2)\qquad \lim_{n\to\infty} n\left(\frac{\lambda_{n+1}}{\lambda_n} - 1\right) = \alpha$$

The number α is called the index of regular variation.

One can show that a regularly varying sequence $\{c_n : n = 0, 1, 2, \ldots\}$ with index α can be written as $c_n = n^\alpha L(n)$ where L is a positive and measurable function on $[0, \infty)$ such that, for every $y > 0$,

$$\lim_{x\to\infty} \frac{L(xy)}{L(x)} = 1$$

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(Seneta [20, p. 47]). We will assume that the recurrence coefficients a_n and b_n in (1.1) satisfy

$$\lim_{n\to\infty} \frac{a_n}{\lambda_n} = a > 0, \quad \lim_{n\to\infty} \frac{b_n}{\lambda_n} = b \in \mathbf{R} \tag{1.3}$$

and

$$\lim_{n\to\infty} n\left(\frac{a_{n+1}-a_n}{\lambda_n}\right) = a\alpha, \quad \lim_{n\to\infty} n\left(\frac{b_{n+1}-b_n}{\lambda_n}\right) = b\alpha \tag{1.4}$$

for some $\alpha > 0$. These conditions imply that both $\{a_n\}$ and $\{b_n\}$ are regularly varying with index α (if $b \neq 0$). We will investigate the asymptotic behavior of the rescaled polynomials $p_n(\lambda_n x)$ when x is in the complex plane away from the oscillatory region on the real line. The class of orthogonal polynomials with regularly varying recurrence coefficients (in particular $\lambda_n = n^\alpha$) has become very important now that Freud's conjecture has been proven (Freud [4, 5], Magnus [8, 9], Bessis, et al. [2,3], Lubinsky, et al. [6]). We will prove a generalization of the Plancherel-Rotach formula outside the oscillatory region for the Hermite polynomials (Plancherel-Rotach [18], Szegö [22]) using techniques and ideas of Máté-Nevai-Totik [10, 11] and Bauldry [1]. Asymptotics for orthogonal polynomials with exponential weights (Freud weights) have been studied earlier by some authors treating both n^{th} root asymptotics (Mhaskar-Saff [12], Rakhmanov [19]) and strong asymptotics (Plancherel-Rotach [18], Moecklin [13], Nevai [14, 15, 16] Bauldry [1] and Sheen [21]).

2. Asymptotic analysis. Define an array of functions by

$$\begin{aligned} &\phi_{k,n}(x) = p_k(\lambda_n x) - \frac{1}{z_{k,n}} p_{k-1}(\lambda_n x), \\ &k = 1, 2, \ldots, n+1, \quad n = 1, 2, 3, \ldots, \end{aligned} \tag{2.1}$$

where

$$z_{k,n} = \frac{\lambda_n x - b_k}{2a_k} + \sqrt{\left(\frac{\lambda_n x - b_k}{2a_k}\right)^2 - 1} \tag{2.2}$$

(we always define the square root such that $|x+\sqrt{x^2-1}|>1$ when $x\in \mathbf{C}\backslash[-1,1]$). Notice that

$$\begin{aligned}\phi_{k+1,n}(x) &- \frac{a_k}{a_{k+1}} z_{k,n}\phi_{k,n}(x)\\ &= \left(\frac{\lambda_n x - b_k}{a_{k+1}} - \frac{a_k}{a_{k+1}} z_{k,n} - \frac{1}{z_{k+1,n}}\right) p_k(\lambda_n x),\end{aligned}$$

where we have used (1.1) to eliminate $p_{k+1}(\lambda_n x)$ and $p_{k-1}(\lambda_n x)$. Now use

$$z_{k,n} + \frac{1}{z_{k,n}} = \frac{\lambda_n x - b_k}{a_k}.$$

Then

$$\phi_{k+1,n}(x) - \frac{a_k}{a_{k+1}} z_{k,n}\phi_{k,n}(x) = \Delta_{k,n} p_k(\lambda_n x) \tag{2.3}$$

with

$$\Delta_{k,n} = \frac{a_k}{a_{k+1}}\frac{1}{z_{k,n}} - \frac{1}{z_{k+1,n}}. \tag{2.4}$$

From this one finds

$$\frac{a_{k+1}}{a_k}\frac{\phi_{k+1,n}}{z_{k,n}\phi_{k,n}} = 1 + \frac{\Delta_{k,n}}{\frac{a_k}{a_{k+1}}\left(z_{k,n} - \frac{p_{k-1}(\lambda_n x)}{p_k(\lambda_n x)}\right)},$$

and, if we take a product with k running from $k=1$ to $k=n$, then

$$\begin{aligned}&\frac{a_{n+1}}{a_1}\frac{\phi_{n+1,n}(x)}{\phi_{1,n}(x)\prod_{k=1}^n z_{k,n}}\\ &\quad = \prod_{k=1}^n\left(1 + \frac{\Delta_{k,n}}{\frac{a_k}{a_{k+1}}\left(z_{k,n} - \frac{p_{k-1}(\lambda_n x)}{p_k(\lambda_n x)}\right)}\right).\end{aligned} \tag{2.5}$$

Let us now establish the asymptotics for $\phi_{n+1,n}(x)$ as n tends to infinity:

THEOREM 1. *Suppose that* (1.4) *and* (1.3) *are valid, with* $\alpha > 0$. *Then*

$$\begin{aligned}(2.6)\qquad &\lim_{n\to\infty} \frac{\phi_{n+1,n}(x)}{\prod_{k=1}^{n} z_{k,n}} \\ &= \frac{x}{a}\left(\frac{(x-b)^2-4a^2}{x^2}\right)^{1/4} \exp\left(\frac{b}{2}\int_0^1 \frac{ds}{\sqrt{(x-bs)^2-4a^2s^2}}\right)\end{aligned}$$

uniformly on every compact set K *in* $\mathbf{C}\backslash[A,B]$, *where* $[A,B]$ *is the smallest interval containing* $\{0\}$ *and* $[b-2a, b+2a]$.

PROOF. We begin by observing that

$$\phi_{1,n}(x) = \frac{\lambda_n x - b_0}{a_1} - \frac{\lambda_n x - b_1}{2a_1} + \sqrt{\left(\frac{\lambda_n x - b_1}{2a_1}\right)^2 - 1}$$

so that

$$(2.7)\qquad \lim_{n\to\infty} \frac{a_1}{a_{n+1}}\phi_{1,n}(x) = \frac{x}{a}.$$

Now we analyze the right hand side of (2.5). The product can be written as

$$\begin{aligned}&\prod_{k=1}^{n}\left(1+\frac{\Delta_{k,n}}{\frac{a_k}{a_{k+1}}\left(z_{k,n}-\frac{p_{k-1}(\lambda_n x)}{p_k(\lambda_n x)}\right)}\right) \\ &\qquad = \exp\sum_{k=1}^{n}\log\left(1+\frac{\Delta_{k,n}}{\frac{a_k}{a_{k+1}}\left(z_{k,n}-\frac{p_{k-1}(\lambda_n x)}{p_k(\lambda_n x)}\right)}\right),\end{aligned}$$

where the branch of the logarithm is suitably chosen (the actual choice of the branch is irrelevant since we take the exponential of this expression). The sum can be written as an integral:

$$\begin{aligned}(2.8)\qquad &\sum_{k=1}^{n}\log\left(1+\frac{\Delta_{k,n}}{\frac{a_k}{a_{k+1}}\left(z_{k,n}-\frac{p_{k-1}(\lambda_n x)}{p_k(\lambda_n x)}\right)}\right) \\ &= n\int_0^1 \log\left(1+\frac{\Delta_{[nt]+1,n}}{\frac{a_{[nt]+1}}{a_{[nt]+2}}\left(z_{[nt]+1,n}-\frac{p_{[nt]}(\lambda_n x)}{p_{[nt]+1}(\lambda_n x)}\right)}\right)dt,\end{aligned}$$

where $[nt]$ is the integer part of nt. The integrand converges for every $t \in (0,1)$ since

$$\begin{aligned} \lim_{n\to\infty} z_{[nt]+1,n} &= \frac{x-bt^\alpha}{2at^\alpha} + \sqrt{\left(\frac{x-bt^\alpha}{2at^\alpha}\right)^2 - 1}, \\ \lim_{n\to\infty} \frac{p_{[nt]}(\lambda_n x)}{p_{[nt]+1}(\lambda_n x)} &= \frac{x-bt^\alpha}{2at^\alpha} - \sqrt{\left(\frac{x-bt^\alpha}{2at^\alpha}\right)^2 - 1}. \end{aligned} \tag{2.9}$$

The latter is a result in Van Assche [**23**] and is true uniformly for x on compact sets of $\mathbf{C}\backslash[A,B]$. A straightforward analysis shows that

$$\lim_{n\to\infty} n\Delta_{[nt]+1,n} = \frac{\alpha}{2at}\left(b - \frac{b(x-bt^\alpha)+4a^2t^\alpha}{\sqrt{(x-bt^\alpha)^2-4a^2t^{2\alpha}}}\right)$$

holds for every $t \in (0,1)$. Since

$$\lim_{n\to\infty} nc_n = c(t) \Rightarrow \lim_{n\to\infty} n\log(1+c_n(t)) = c(t),$$

the integrand of the right hand side of (2.8) converges for every $t \in (0,1)$. The interchanging of limit and integral can easily be justified and it follows that

$$\begin{aligned} \lim_{n\to\infty} \sum_{k=1}^{n} \log\left(1 + \frac{\Delta_{k,n}}{\frac{a_k}{a_{k+1}}\left(z_{k,n} - \frac{p_{k-1}(\lambda_n x)}{p_k(\lambda_n x)}\right)}\right) \\ = \frac{b}{2}\int_0^1 \frac{ds}{\sqrt{(x-bs)^2-4a^2s^2}} + \frac{1}{4}\int_0^1 \frac{d((x-bs)^2-4a^2s^2)}{(x-bs)^2-4a^2s^2}. \end{aligned} \tag{2.10}$$

Notice that the second term on the right side is analytic in $\mathbf{C}\backslash[A,B]$ and can be expressed as

$$\frac{1}{4}\log\frac{(x-b)^2-4a^2}{x^2}$$

with a suitable choice of the logarithm. The result of the lemma now follows by combining (2.5), (2.7), and (2.10). □

THEOREM 2. *Suppose that* (1.4) *and* (1.3) *are valid, with* $\alpha > 0$. *Then*

$$\lim_{n\to\infty} \frac{p_n(\lambda_n x)}{\prod_{k=1}^n z_{k,n}} = \left(\frac{(x-b)^2-4a^2}{x^2}\right)^{-1/4} \exp\left(\frac{b}{2}\int_0^1 \frac{ds}{\sqrt{(x-bs)^2-4a^2s^2}}\right) \tag{2.11}$$

uniformly on compact subsets of $\mathbf{C}\setminus[A,B]$.

PROOF. Observe that

$$\frac{\phi_{n+1,n}(x)}{p_n(\lambda_n x)} = \frac{p_{n+1}(\lambda_n x)}{p_n(\lambda_n x)} - \frac{1}{z_{n+1,n}}.$$

If we set

$$z = \frac{x-b}{2a} + \sqrt{\left(\frac{x-b}{2a}\right)^2 - 1},$$

then, by (2.9), we find

$$\lim_{n\to\infty} \frac{\phi_{n+1,n}(x)}{p_n(\lambda_n x)} = z - \frac{1}{z} = 2\sqrt{\left(\frac{x-b}{2a}\right)^2 - 1}.$$

Combining this with Theorem 1, (2.11) follows. □

3. A special case. The asymptotic behavior given in Theorems 1 and 2 is in terms of the product $\prod_{k=1}^n z_{k,n}$. If one assumes that the recurrence coefficients are very regular, then one can analyze that product in more detail.

THEOREM 3. *Suppose that* $a_n = an^\alpha$ *and* $b_n = bn^\alpha (\alpha > 0)$ *and let* K *be a compact set in* $\mathbf{C}\setminus[A,B]$. *Then* (*taking* $\lambda_n = n^\alpha$)

$$\prod_{k=1}^n z_{k,n} \sim \frac{1}{(2\pi n)^{\alpha/2}} z^n H^n \left(\frac{az}{x}\right)^{1/2}, \tag{3.1}$$

where

$$z = \frac{x-b}{2a} + \sqrt{\left(\frac{x-b}{2a}\right)^2 - 1}, \tag{3.2}$$

$$H = \exp\left(\alpha x \int_0^1 \frac{dt}{\sqrt{(x-bt^\alpha)^2 - 4a^2t^{2\alpha}}}\right) \tag{3.3}$$

and $f_n(x) \sim g_n(x)$ *means that the ratio* $f_n(x)/g_n(x)$ *converges to* 1 *uniformly on* K. *As a consequence*

$$\begin{aligned} \lim_{n\to\infty} \frac{(2\pi n)^{\alpha/2} p_n(n^\alpha x)}{z^n H^n} =& (az)^{1/2}((x-b)^2 - 4a^2)^{-1/4} \\ & \times \exp \frac{b}{2} \int_0^1 \frac{ds}{\sqrt{(x-bs)^2 - 4a^2 s^2}} \end{aligned} \tag{3.4}$$

uniformly on K.

PROOF. First of all we rewrite the product as

(3.5)

$$\prod_{k=1}^n z_{k,n} \left(\prod_{k=1}^n \frac{n^\alpha}{2ak^\alpha}\right) \prod_{k=1}^n \left(x - b\left(\frac{k}{n}\right)^\alpha + \sqrt{\left(x - b\left(\frac{k}{n}\right)^\alpha\right)^2 - 4a^2\left(\frac{k}{n}\right)^{2\alpha}}\right).$$

The first product on the right hand side can be approximated by Stirling's formula and gives

$$\prod_{k=1}^n \frac{n^\alpha}{2ak^\alpha} = \frac{n^{n\alpha}}{(2a)^n (n!)^\alpha} = \frac{1}{(2a)^n} \frac{e^{n\alpha}}{(2\pi n)^{\alpha/2}}(1 + o(1)). \tag{3.6}$$

To estimate the second product on the right hand side of (3.5) we take a logarithm and use the Euler-McLaurin formula (Olver [**17**, p. 281]) to obtain

$$\begin{aligned} &\sum_{k=1}^n \log\left(x - b\left(\frac{k}{n}\right)^\alpha + \sqrt{(x - b\left(\frac{k}{n}\right)^\alpha)^2 - 4a^2\left(\frac{k}{n}\right)^{2\alpha}}\right) \\ &= n \int_0^1 \log f(t)dt + \frac{1}{2}\log\frac{az}{x} \\ &+ \frac{1}{n}\int_0^1 \frac{B_2 - B_2(nt - [nt])}{2} \frac{d^2}{dt^2}(\log f(t))dt, \end{aligned} \tag{3.7}$$

where we have set

$$f(t) = x - bt^\alpha + \sqrt{(x-bt^\alpha)^2 - 4a^2t^{2\alpha}}.$$

Integration by parts yields

$$\int_0^1 \log f(t)dt = \log f(1) - \int_0^1 t\frac{f'(t)}{f(t)}dt$$

and easy calculus gives

$$f'(t) = -f(t)\frac{\alpha bt^{\alpha-1}}{\sqrt{(x-bt^\alpha)^2-4a^2t^{2\alpha}}} - \frac{4\alpha a^2t^{2\alpha-1}}{\sqrt{(x-bt^\alpha)^2-4a^2t^{2\alpha}}}.$$

Using

$$\frac{1}{f(t)} = \frac{x-bt^\alpha - \sqrt{(x-bt^\alpha)^2-4a^2t^{2\alpha}}}{4a^2t^{2\alpha}}$$

we find

$$t\frac{f'(t)}{f(t)} = \frac{-\alpha x}{\sqrt{(x-bt^\alpha)^2-4a^2t^{2\alpha}}} + \alpha,$$

and since $f(1) = 2az$ we obtain the formula

$$\int_0^1 \log f(t)dt = \log z + \log H - \alpha + \log 2a. \tag{3.8}$$

Let us now estimate the remainder term in the Euler-McLaurin formula. We break the interval of integration into two parts:

$$\begin{aligned}
&\int_0^1 \frac{B_2 - B_2(nt-[nt])}{2}(\log f(t))''dt \\
&= \int_{1/n}^1 \frac{B_2 - B_2(nt-[nt])}{2}(\log f(t))''dt + \int_0^{1/n} \frac{B_2 - B_2(nt)}{2}(\log f(t))''dt \\
&= I_1 + I_2.
\end{aligned}$$

Some elementary calculus gives

$$(\log f(t))'' = \frac{\alpha t^{\alpha-2}}{\sqrt{(x-bt^\alpha)^2-4a^2t^{2\alpha}}}\left(\frac{4a^2t^\alpha}{f(t)} + b + \alpha x\frac{(b^2-4a^2)t^\alpha - bx}{(x-bt^\alpha)^2-4a^2t^{2\alpha}}\right).$$

Define

$$\delta = \inf\{|y - x| : y \in [A, B],\ x \in K\} > 0.$$

Then, for $x \in K$,

$$\begin{aligned}
&\left|\frac{1}{\sqrt{(x - bt^\alpha)^2 - 4a^2t^{2\alpha}}}\right| \\
&= \left|\frac{1}{\pi}\int_{(b-2a)t^\alpha}^{(b+2a)t^\alpha} \frac{1}{\sqrt{4a^2t^{2\alpha} - (y - bt^\alpha)^2}}\frac{dy}{x - y}\right| \\
&\le \frac{1}{\delta}, \left|\frac{2}{f(t)}\right| = \left|\frac{1}{2a^2t^{2\alpha}\pi}\int_{(b-2a)t^\alpha}^{(b+2a)t^\alpha} \sqrt{4a^2t^{2\alpha} - (y - bt^\alpha)^2}\frac{dy}{x - y}\right| \le \frac{1}{\delta},
\end{aligned}$$

so that we obtain the bound ($x \in K,\ 0 \le t \le 1$)

$$\begin{aligned}
|(\log f(t)'')| &\le \frac{\alpha t^{\alpha-2}}{\delta}\left(\frac{2a^2}{\delta} + |b| + \frac{\alpha|x|}{\delta^2}(|b^2 - 4a^2| + |bx|)\right) \\
&\le ct^{\alpha-2}
\end{aligned}$$

where c is some positive constant. For I_1 we use the bound

$$|B_2 - B_2(x)| \le \frac{1}{4}, \quad 0 \le x \le 1,$$

to obtain

$$|I_1| \le \frac{c}{8}\int_{1/n}^1 t^{\alpha-2}dt = \begin{cases} \frac{c}{8}\frac{1}{\alpha-1}(1 - \frac{1}{n^{\alpha-1}}) & \text{if } \alpha \ne 1 \\ \frac{c}{8}\log n & \text{if } \alpha = 1. \end{cases}$$

For I_2 we use

$$|B_2 - B_2(x)| = x - x^2 \le x, \quad 0 \le x \le 1,$$

and find

$$|I_2| \le \frac{cn}{2}\int_0^{1/n} t^{\alpha-1}dt = \frac{cn}{2\alpha}\frac{1}{n^\alpha}.$$

Hence
(3.9)

$$\left|\frac{1}{n}\int_0^1 \frac{B_2 - B_2(nt - [nt])}{2}(\log f(t))''dt\right| = \begin{cases} O\left(\frac{1}{n}\right) & \text{if } \alpha > 1 \\ O\left(\frac{\log n}{n}\right) & \text{if } \alpha = 1\ . \\ O\left(\frac{1}{n^\alpha}\right) & \text{if } \alpha < 1 \end{cases}$$

The result in (3.1) follows by combining (3.5)-(3.9). The result in (3.4) is a consequence of (3.1) and Theorem 2. □

Note that the special case $\alpha = 1/2$ and $b = 0$ in (3.4) corresponds exactly to the result for Hermite polynomials in Maejima - Van Assche [**7**] (a factor $2^{-1/2}$ has accidently been left out of Theorem 3 in [**7**]).

4. Concluding remarks. In this paper we have obtained an asymptotic formula for the polynomials $p_n(\lambda_n x)$ which satisfy a recurrence formula with regularly varying recurrence coefficients. The asymptotic formula holds in the complex plane, away from the smallest interval $[A, B]$ that contains $\{0\}$ and $[b-2a, b+2a]$. A natural question next is to ask for the asymptotic behavior of $p_n(\lambda_n x)$ when x is on the interval $[A, B]$. The zeros of the polynomial $p_n(\lambda_n x)$ are dense in $[A, B]$ so that we expect oscillatory behavior on that interval. We hope to return soon to this particular problem.

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