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EXTENSIONS OF THE HEISENBERG GROUP AND COAXIAL COUPLING OF TRANSVERSE EIGENMODES

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1. Optical fiber communication. Lightwave electronics, including optical communication via silica fibers and optoelectronic devices, has become one of the most promising fields of applied physics and electrical engineering since the laser first appeared.

The advantages of optical fiber communication are among others:

- extremely low loss of the optical signals over a wide range of wavelengths (less than 1dB/km, corresponding to a 25% loss per km)

- immense bandwith (1 and 100 GHz, respectively, for multimode and single-mode fibers over 1 km) that makes it possible to use extremely short pulses.

Characteristic of the progress in lightwave communication technology is the enormous reduction of the transmission loss of optical fibers accomplished in the last decade as illustrated by the diagram below.

One of the most important factors that helped make optical fiber communication a reality is the invention of the light-emitting diode (LED) and the semiconductor injection laser. The coupling between lasers and optical fibers causes some power loss which is described by the coupling coefficients of the various modes. The main purpose of the present paper is to calculate the coupling coefficients of the quantized transverse eigenmodes excited in coaxial circular and rectangular laser resonators and optical waveguides in terms of Krawtchouk polynomials evaluated at Gaussian beam parameters. The method we will present is based on non-commutative harmonic analysis, specifically, on the representation theory of various (three-step nilpotent and solvable) group extensions of the real Heisenberg two-step nilpotent Lie group $\tilde{A}(\mathbf{R})$.

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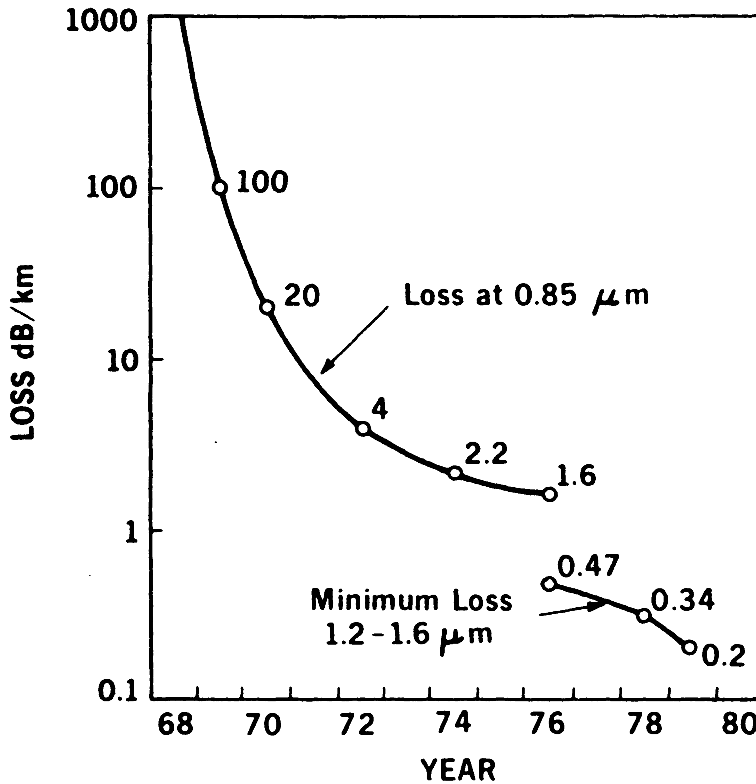

2. Extensions of the Heisenberg nilpotent Lie group. The real Heisenberg nilpotent Lie group $\tilde{A}(\mathbf{R})$ is the universal covering group of the reduced Heisenberg group and can be realized by the unipotent matrices

$$\begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} = (x, y, z) \in \mathbf{R}^3.$$

The Lie group $\tilde{A}(\mathbf{R})$ has a representation theory that is at once simple and rich in structure. Its topologically irreducible, continuous, unitary, linear representations are classified by the Stone-von Neumann theorem of basic quantum mechanics [**10**]: Any such representation of $\tilde{A}(\mathbf{R})$ having $e_\lambda : z \rightsquigarrow e^{2\pi i\lambda z} (\lambda \in \mathbf{R}, \lambda \neq 0)$ as central character is unitarily isomorphic to the representation $(U_\lambda, L^2(\mathbf{R}))$ which acts on the wavefunction $\psi \in \mathcal{S}(\mathbf{R}) \hookrightarrow L^2(\mathbf{R})$ according to the prescription

$$U_\lambda(x, y, z)\psi(t) = e^{2\pi i\lambda(z+yt)}\psi(t+x) \quad (t \in \mathbf{R});$$

cf. the monograph [**7**] for details.

Let G denote the maximal unipotent subgroup of $\mathbf{Sp}(1,\mathbf{R}) = \mathbf{SL}(2,\mathbf{R})$ which fixes pointwise the closed subgroup

$$\{(0, y, 0) \in \tilde{A}(\mathbf{R}) | y \in \mathbf{R}\}$$

of $\tilde{A}(\mathbf{R})$. Form the external semi-direct product

$$B(\mathbf{R}) = G \ltimes \tilde{A}(\mathbf{R})$$

with the convention that the open end of the symbol $\ltimes$ always points to the normal subgroup. Then $B(\mathbf{R})$ represents the smallest real three-step nilpotent Lie group (cf. Ratcliff [**5**]). The Kirillov correspondence [**7**] shows that, for all pairs $(a, \lambda) \in \mathbf{R} \times \mathbf{R}^\times$, the unitary linear representation

$$U_{(a.\lambda)}(v, x, y, z) = e^{2\pi i(a - \frac{\lambda}{2}t^2 - \frac{\lambda}{2}xt - \frac{\lambda}{6}x^2)v} U_\lambda(x, y, z)$$

of $B(\mathbf{R})$ acting in $L^2(\mathbf{R})$ for all elements $(v, x, y, z) \in B(\mathbf{R})$ is topologically irreducible. Obviously $U_{(a.\lambda)}$ restricts to U_λ and admits therefore the central character e_λ.

Finally, let us identify the one-dimensional torus group $\mathbf{T}$ with the maximal compact subgroup $\mathbf{SO}(2,\mathbf{R})$ of $\mathbf{Sp}(1,\mathbf{R})$ and let the elements of $\mathbf{SO}(2,\mathbf{R})$ act as automorphisms of $\tilde{A}(\mathbf{R})$ by rotating the first two coordinates of $(x, y, z) \in \tilde{A}(\mathbf{R})$ and leaving the central coordinate fixed. Then

$$D(\mathbf{R}) = \mathbf{T} \ltimes \tilde{A}(\mathbf{R})$$

forms a real, connected, non-exponential, solvable Lie group which is called the diamond group (cf. [**8**]). In order to classify all the topologically irreducible, continuous, unitary, linear representations of $D(\mathbf{R})$ which restrict to $U_\lambda (\lambda \in \mathbf{R}, \lambda \neq 0)$ let

$$\mathcal{H}_\lambda = \frac{1}{\lambda}\frac{d^2}{dt^2} - 4\pi^2\lambda t^2$$

denote the Hermite operator acting on the Sobolev space $\{\psi \in L^2(\mathbf{R}) | t^2\psi \in L^2(\mathbf{R})\}$. Then $\mathcal{H}_\lambda$ is an essentially self-adjoint linear operator in $L^2(\mathbf{R})$ and forms the infinitesimal generator of $\mathbf{SO}(2,\mathbf{R})$ in $\mathbf{Sp}(1,\mathbf{R})$. The closure of $\mathcal{H}_\lambda$ has a pure point spectrum formed by the simple eigenvalues $\{-2\pi(\mathrm{sign}\lambda)(2n+1) | n \in \mathbf{N}\}$ and the associated eigenfunctions are given by the scaled Hermite functions $t \rightsquigarrow H_n(\sqrt{|\lambda|}t)$, $n \in \mathbf{N}$.

Let $\tilde{\mathbf{T}} = \{e^{\pi i\theta} | \theta \in \mathbf{R}\}$ denote the double covering of the torus group $\mathbf{T} = \{e^{2\pi i\theta} | \theta \in \mathbf{R}\}$ ad define a unitary linear representation S_λ of $\tilde{\mathbf{T}} \times \mathbf{R}$ by

$$S_\lambda(e^{\pi i\theta}, z) = e^{i\theta\mathcal{H}_\lambda}$$

and the characters of $\tilde{\mathbf{T}} \times \mathbf{R}$ by

$$\chi_\lambda^n(e^{\pi i\theta}, z) = e^{2\pi i((2n+1)\theta + \lambda z)}\mathrm{id}_{L^2\mathbf{R})} \quad (n \in \mathbf{Z}).$$

Then

$$S_\lambda \circ U_\lambda \otimes \chi_\lambda^n \quad (n \in \mathbf{Z})$$

is a family of topologically irreducible, continuous, unitary, linear representations of $D(\mathbf{R})$ having U_λ as their restrictions to $\tilde{A}(\mathbf{R})$. Conversely, each representation of $D(\mathbf{R})$ having these properties is unitarily isomorphic to exactly one of the representations $S_\lambda \circ U_\lambda \otimes \chi_\lambda^n (n \in \mathbf{Z})$; see, for instance, Lion [**2**].

In the following sections we will use the Lie groups $\tilde{A}(\mathbf{R}), B(\mathbf{R})$, and $D(\mathbf{R})$ to study the transverse eigenmodes of optical systems in terms of classical orthogonal polynomials and to calculate the coupling coefficients in terms of the Krawtchouk polynomials.

3. Transverse eigenmodes. If ψ and φ are wavefunctions belonging to the Schwartz space $\mathcal{S}(\mathbf{R})$, then

$$H(\psi, \varphi; x, y) = \int_{\mathbf{R}} \psi(t+x)\overline{\varphi}(t)e^{2\pi iyt}\,dt$$

is the cross-ambiguity function associated with ψ and φ. See [**6**] where a symmetrized version of $H(\psi, \varphi; .\ , .)$ is employed. It follows that $H(\psi, \varphi; .\ , .) \in \mathcal{S}(\mathbf{R} \oplus \mathbf{R})$ and

$$H(\psi, \varphi; x, y) = \langle U_1(x, y, 0)\psi | \varphi \rangle$$

for all $(x, y) \in \mathbf{R}\oplus\mathbf{R}$. Thus the cross-ambiguity function $H(\psi, \varphi; ., .)$ is given by the coefficient function of the linear Schrödinger representation U_1 of $\tilde{A}(\mathbf{R})$ with respect to ψ and φ modulo the center of $\tilde{A}(\mathbf{R})$. From this we conclude the orthogonality relations

$$\iint_{\mathbf{R}\oplus\mathbf{R}} H(\psi, \varphi; x, y)\overline{H}(\psi', \varphi'; x, y)\, dx\, dy = \langle\psi|\psi'\rangle\langle\varphi'|\varphi\rangle$$

which holds for all wavefunctions ψ, ψ' and φ, φ' in $\mathcal{S}(\mathbf{R})$; cf. Moore-Wolf [**3**].

A wavefunction $\psi \in \mathcal{S}(\mathbf{R})$ is called a transverse eigenmode of a circular optical waveguide if the auto-ambiguity function

$$H(\psi; ., .) = H(\psi, \psi; ., .)$$

is radial on $\mathbf{R}\oplus\mathbf{R}$. The following theorem furnishes a characterization of these eigenmodes.

THEOREM 1. *Let $(H_n)_{n\geq 0}$ denote the sequence of Hermite functions. The waveform $\psi \in \mathcal{S}(\mathbf{R})$ is a transverse eigenmode of a circular optical waveguide if and only if*

$$\psi = \zeta_n H_n \quad (\zeta_n \in \mathbf{C})$$

for an integer $n \geq 0$.

PROOF. The topologically irreducible, continuous, unitary, linear representation $S_\lambda \circ U_\lambda \otimes \chi_\lambda^n (n \in \mathbf{Z})$ of the diamond solvable Lie group $D(\mathbf{R})$ acts on $L^2(\mathbf{R})$ according to the prescription

$$\begin{aligned} &S_\lambda \circ U_\lambda \otimes \chi_\lambda^n(e^{2\pi i\theta}, x, y, z) \\ &= e^{i(\mathcal{H}_\lambda + 2\pi(2n+1))\theta} U_\lambda(x\cos 2\pi\theta + y\sin 2\pi\theta, -x\sin 2\pi\theta + y\cos 2\pi\theta, z). \end{aligned}$$

It follows that $\psi \in \mathcal{S}(\mathbf{R}), ||\psi|| = 1$, is an eigenmode of a circular optical waveguide if and only if there is a number $n \in \mathbf{N}$ such that the unitary linear representation

$$\mathbf{T} \ni e^{2\pi i\theta} \rightsquigarrow \chi_1^n \cdot S_1(e^{\pi i\theta}, 0)$$

acts trivially on ψ. From this the theorem follows. □

REMARK. The geometric optics reason behind the fact that the elements $e^{\pi i\theta} \in \tilde{\mathbf{T}}$ occur as arguments of the representation $\chi_1^n \cdot S_1$ in the preceding proof and not the underlying elements $e^{2\pi i\theta} \in \mathbf{T}$ is the phase shift (which actually turns out to be a phase retardation) of $\frac{1}{2}\pi$ suffered by the beam at each of the two caustics of the graded-index optical fiber [**8**].

COROLLARY 1. *Let $(L_n^{(\alpha)})_{n\geq 0}$ denote the sequence of Laguerre functions of order $\alpha > -1$. Then Schwinger's formula*

$$H(H_m, H_n; x, y) = \sqrt{\frac{n!}{m!}}(\sqrt{\pi}(x+iy))^{m-n} L_n^{(m-n)}(\pi(x^2+y^2))$$

$(m \geq n \geq 0)$ *holds for* $(x, y) \in \mathbf{R} \oplus \mathbf{R}$.

COROLLARY 2. *The wavefunction $\psi \in \mathcal{S}(\mathbf{R}\oplus\mathbf{R}), \psi \neq 0$, is a transverse eigenmode of a rectangular optical waveguide if and only if*

$$\psi = \zeta_{n,m} H_n \otimes H_m \quad (\zeta_{n,m} \in \mathbf{C})$$

for integers $n \geq 0, m \geq 0$.

4. Coaxial coupling coefficients. If the wave functions ψ', φ' and ψ, φ belonging to the Schwartz space $\mathcal{S}(\mathbf{R})$ represent two transverse eigenmodes of two coaxial optical devices like laser resonators or dielectric waveguides, their coupling coefficient is defined according to the prescription

$$C(\psi', \varphi', \psi, \varphi) = \iint_{\mathbf{R}\oplus\mathbf{R}} H(\psi', \varphi'; x, y).\overline{H}(\psi, \varphi; x, y)\, dx\, dy.$$

The integral has to be evaluated at the coupling plane $\mathbf{R}\oplus\mathbf{R}$ transverse to the common beam axis of the optical devices. In the present section we will calculate the coaxial coupling coefficients explicitly in the circular as well as in the rectangular case (cf. Kogelnik [**1**]).

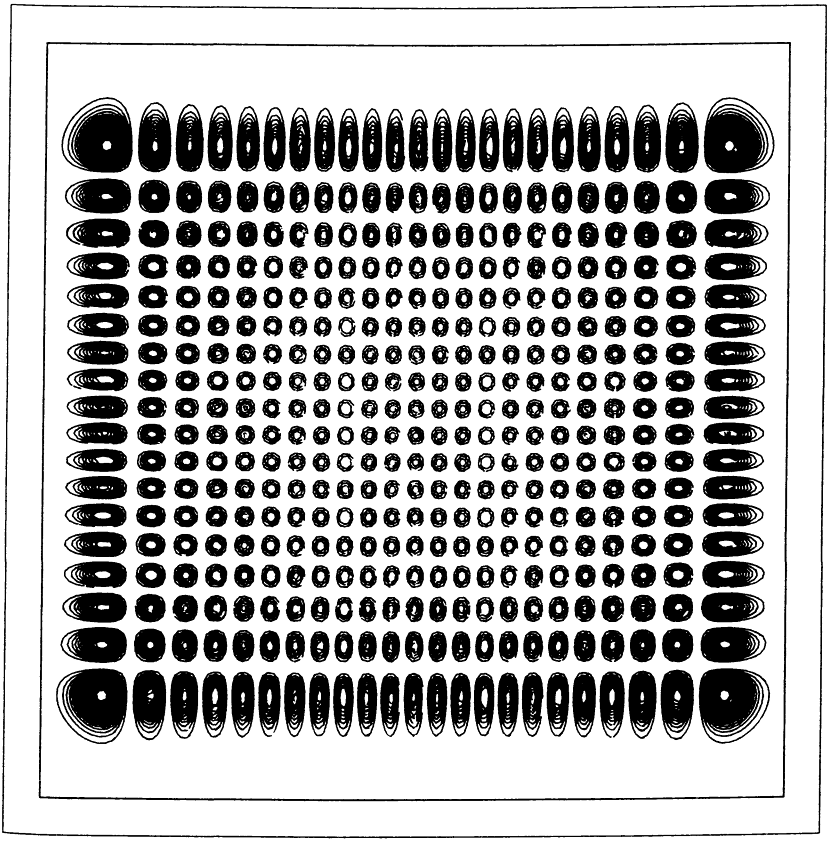

Figure I. Transverse Eigenmodes of a Rectangular Laser. Computer Plot.

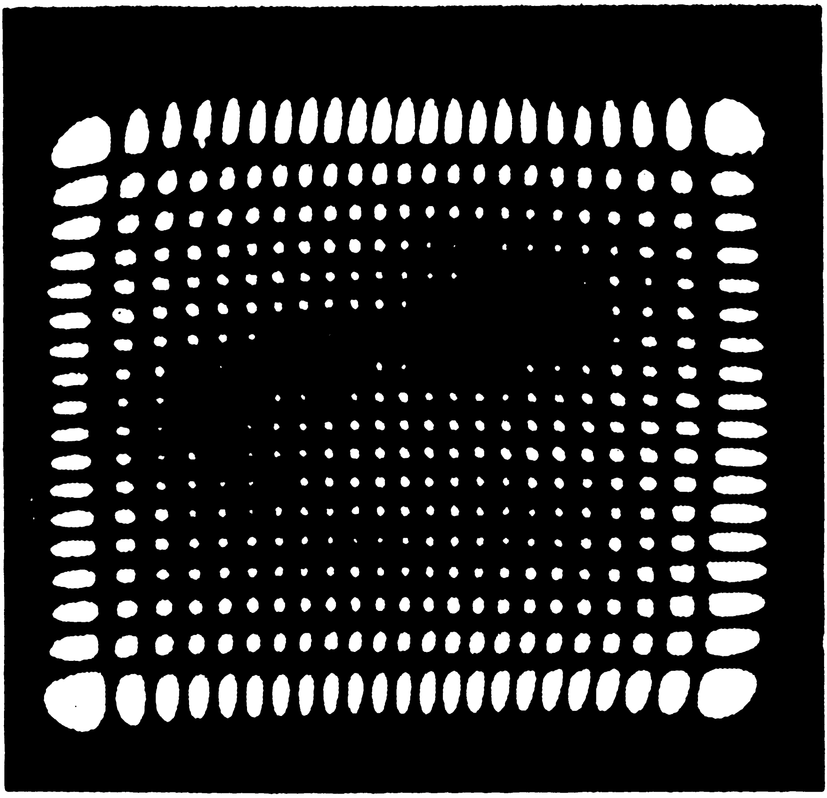

Figure II. Transverse Eigenmodes of a Rectangular Laser. Experiment Plot.

Lasers emit a very narrow cone of monochromatic radiation referred to as a laser beam. Denote its wavelength by λ. The transverse electromagnetic eigenmode output of a laser is a beam with a Gaussian wavefront. Define the Gaussian beam parameters by

$$\alpha' = \frac{2}{w'^2}, \quad \alpha = \frac{2}{w^2}$$

and

$$q = \left(\frac{1}{w'^2} + \frac{1}{w^2} + \frac{ik}{2}\left(\frac{1}{r'} - \frac{1}{r}\right)\right),$$

where

$$k = \frac{1}{\lambda}$$

denotes the wave number, w', w the beam radii and r', r the radii of curvature of the phase fronts at the coupling plane of the optical devices. Then the symplectic reference planes of the two beams are transferred to the coupling plane $\mathbf{R} \oplus \mathbf{R}$ by the symplectic mappings with matrices

$$\begin{pmatrix} \frac{1}{\sqrt{\alpha'}} & 0 \\ 0 & \sqrt{\alpha'} \end{pmatrix}, \begin{pmatrix} \frac{1}{\sqrt{\alpha}} & 0 \\ 0 & \sqrt{\alpha} \end{pmatrix} \in \mathbf{Sp}(1, \mathbf{R})$$

associated with the beam radii w' and w, respectively, and by the symplectic mappings with matrices

$$\begin{pmatrix} 1 & 0 \\ +\frac{k}{r'} & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ -\frac{k}{r} & 1 \end{pmatrix} \in \mathbf{Sp}(1, \mathbf{R})$$

associated with the radii of curvature r' and r, respectively, of the phase fronts.

Next recall the definition of the Krawtchouk polynomials $K_n(x; p, N)$. For all integers $n \geq 0$ these hypergeometric polynomials are given by

$$K_n(x; p, N) = {}_2F_1(-n, -x; -N; p^{-1}),$$

where $N \geq 0,\ 0 \leq x \leq N, p \in]0, 1[$. In terms of shifted factorials, $K_n(x; p, N)$ admits the expression

$$K_n(x; p, N) = \sum_{0 \leq k \leq n} \frac{(-n)_k (-x)_k}{(-N)_k k!} \left(\frac{1}{p}\right)^k \quad (0 \leq n \leq N).$$

The Krawtchouk polynomials are orthogonal on $\{0, 1, \ldots, N\}$ with respect to the binomial distribution. Thus their orthogonality relations take the form

$$\sum_{0 \le x \le N} K_n(x; p, N) K_m(x; p, N) \binom{N}{x} p^x (1-p)^{N-x} = 0 \quad (n \neq m)$$

for $n \le N, m \le N$ and $p \in]0, 1[$.

THEOREM 2. *Let $m \ge n \ge 0, m' \ge n' \ge 0$ be integers. Keep to the preceding notations.*

a) *In the circular case set $m = p, m - n = l, m' = p', m' - n' = l'$. Then*

$$C_{p.1.p'.1'} = 0 \textit{ for } l = l',$$

i.e., there is no coupling between transverse eigenmodes of different angular moments. Moreover, in terms of Krawtchouk polynomials,

$$C_{p.l.p'.l} = \left(\frac{2}{ww'q}\right)^{l+1} \frac{(p+p'+l)!}{\sqrt{p!p'!(p+l)!(p'+l')!}} \left(1 - \frac{\alpha}{q}\right)^p \left(1 - \frac{\alpha'}{q}\right)^{p'}.$$

$$K_p\left(p'; \frac{(q-\alpha)(q-\alpha')}{q(q-\alpha-\alpha')}, p+p'+l\right).$$

b) *In the rectangular case we have*

$$C_{m.n.m'.n'} = 0 \textit{ for } \begin{cases} m + m' \equiv 1 \bmod 2, \\ n + n' \equiv 1 \bmod 2, \end{cases}$$

i.e., there is no coupling between even and odd transverse eigenmodes. In other words, the parity is preserved under the coupling of eigenmodes. In the case

$$m' = 2\mu', \quad m = 2\mu, \quad n' = 2\nu', \quad n = 2\nu$$

we get, in terms of the Krawtchouk polynomials,

$$C_{m,n,m',n'} = \left(-\frac{1}{2}\right)^{\mu+\nu+\mu'+\nu'} \frac{2}{ww'q} \frac{(2\mu+2\mu')!(2\nu+2\nu')!}{(\mu+\mu')!(\nu+\nu')!\sqrt{2\mu!2\mu'!2\nu!2\nu'!}}$$
$$\cdot\left(1-\frac{\alpha}{q}\right)^{\mu+\nu}\left(1-\frac{\alpha'}{q}\right)^{\mu'+\nu'}$$
$$\cdot K_\mu\left(\mu'; \frac{(q-\alpha)(q-\alpha')}{q(q-\alpha-\alpha')}, \mu+\mu'-\frac{1}{2}\right).$$
$$\cdot K_\nu\left(\nu'; \frac{(q-\alpha)(q-\alpha')}{q(q-\alpha-\alpha')}; \nu+\nu'-\frac{1}{2}\right).$$

A similar result holds in the case

$$m' = 2\mu'+1, \quad m = 2\mu+1, \quad n' = 2\nu'+1, \quad n = 2\nu+1.$$

PROOF. In terms of the representations $(U_{(a,\lambda)})_{(a,\lambda)\in\mathbf{R}\times\mathbf{R}^\times}$ of the nilpotent Lie group $B(\mathbf{R})$ we get, for the ambiguity functions at the coupling plane,

$$\left\langle U_{(0,\sqrt{\alpha'})}\psi'\left(\frac{k}{r'}, x, y, 0\right)|\varphi'\right\rangle, \quad \left\langle U_{\left(0,\sqrt{\alpha}\right)}\phi'\left(-\frac{k}{r}, x, y, 0\right)|\varphi\right\rangle.$$

By virtue of Corollary 1 and Corollary 2 of Theorem 1 the result follows by taking Laplace transforms; see Oberhettinger-Badii [**4**]. □

5. Concluding remarks. An extention of the group theoretical method used in this paper, more specifically the spectral theory of reductive dual pairs, can be applied to establish via the oscillator representation [**9**] of the metaplectic group $\mathbf{Mp}(n, \mathbf{R})$ a singular value decomposition of the classical Radon transform in $\mathbf{R}^n$. As a consequence, the inversion formulae of computerized tomography and Hankel's formula which expresses the Laguerre functions in terms of Bessel functions are popping up. For more details, see the forthcoming paper [**8**].

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