ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 19, Number 1, Winter 1989

PROXIMINALITY OF CERTAIN SUBSPACES OF $C_b(S; E)$

JOÃO B. PROLLA AND ARY O. CHIACCHIO

Throughout this paper, S is a completely regular Hausdorff space and E is a Banach space. The vector space of all continuous and bounded functions $f : S \to E$, denoted by $C_b(S; E)$, is equipped with the sup-norm

$$||f|| = \sup\{||f(x)||; x \in S\}.$$

Recall that a closed subpsace V of a Banach space E is said to be *proximinal* if every $a \in E$ admits a best approximant from V, i.e., a point $v \in V$ for which

$$||v - a|| = \inf\{||w - a||; w \in V\} = \operatorname{dist}(a; V).$$

The set of best approximants to a from V is denoted by $P_V(a)$, and the set-valued mapping $a \to P_V(a)$ is called the *metric projection.* If V is proximinal, then $a \to P_V(a) \neq \emptyset$ for every $a \in E$. If $P_V(a)$ is a singleton for each $a \in E$, then V is called a *Chebyshev subspace* of E. If V is a proximinal subspace of E, then a map $s : E \to V$ such that $s(a)$ belongs to $P_V(a)$, for each $a \in E$, is called a *metric selection* or a *proximity map* for V.

The following notations are standard and will be used throughout this paper. If $a \in E$ and $r > 0$, $B(a; r) = \{v \in E; ||v - a|| < r\}$ and $\overline{B}(a; r) = \{v \in E : ||v - a|| \leq r\}$. For any $s \in S$, the bounded linear operator $\delta_s : C_b(S; E) \to E$ is defined by $\delta_s(f) = f(s)$, for all $f \in C_b(S; E)$. If W is a closed vector subspace of $C_b(S; E)$, then $\delta_s|W$ denotes the restriction of δ_s to W. Notice that $0 \leq ||\delta_s|W|| \leq 1$.

Given a proximinal subspace V of a Banach space E, then clearly $C_b(S; V)$ is a closed subspace of $C_b(S; E)$. In this paper we shall study the following questions.

QUESTION 1. Under what assumptions is $C_b(S; V)$ proximinal in $C_b(S; E)$?

Received by the editors on September 16, 1986.

Copyright ©1989 Rocky Mountain Mathematics Consortium

QUESTION 2. If V admits a continuous proximity map, under what conditions is the same true for $C_b(S;V)$?

Notice that, if V admits a continuous metric selection, say s, then the mapping s^*, defined by $s^*(f) = s \circ f$, for any $f \in C_b(S;E)$, clearly maps $C_b(S;E)$ onto $C_b(S;V)$ and

$$\begin{aligned} ||s^*(f) - f|| &= \sup_{x \in S} ||s(f(x)) - f(x)|| \\ &= \sup_{x \in S} \operatorname{dist}(f(x);V) \le \operatorname{dist}(f;C_b(S;V)). \end{aligned}$$

Therefore s^* is a proximity map for $C_b(S;V)$, and $C_b(S;V)$ is a proximinal subspace of $C_b(S;E)$, but the question of continuity of s^* remains. One case in which s^* is continuous occurs when S is compact. For a proof, see Lemma 11.8 of Light and Cheney [**7**]. Another case in which s^* is continuous happens when s is uniformly continuous on bounded sets. This follows from Lemma 1.

LEMMA 1. *Let s be a continuous map of a Banach space E into a Banach space V which is uniformly continuous on bounded sets. Then the map defined by $s^*(f) = s \circ f$, for any $f \in C_b(S;E)$, is continuous from $C_b(S;E)$ into $C_b(S;V)$.*

PROOF. Let $f \in C_b(S;E)$ and $\varepsilon > 0$ be given. The set $B = \{a \in E; ||a - f(t)|| < \varepsilon$, for some $t \in S\}$ is bounded. By our assumption, there is some $\delta > 0$, and we may assume $\delta \le \varepsilon$, such that $||x' - x''|| < \delta$ implies $||s(x') - s(x'')|| < \varepsilon$ for any x' and x'' in B. Take now $g \in C_b(S;E)$ with $||g - f|| < \delta$. If $t \in S$, then $||g(t) - f(t)|| < \delta < \varepsilon$, and therefore $g(t) \in B$. Clearly, $f(t)$ belongs to B. Hence $||s(g(t)) - s(f(t))|| < \varepsilon$. This shows that $||s^*(g) - s^*(f)|| \le \varepsilon$, and s^* is continuous. □

The remarks preceding Lemma 1 establish the following easy answer to Question 2.

THEOREM 1. *If $s : E \to V$ is a continuous proximity map, which is uniformly continuous on bounded sets, then $C_b(S;V)$ is proximinal in*

$C_b(S;E)$ and, in fact, the mapping s^* defined by $s^*(f) = s \circ f$, for any $f \in C_b(S;E)$, is a continuous proximity map for $C_b(S;V)$.

COROLLARY 1. *If the Banach space E is uniformly convex with respect to V, then $C_b(S;V)$ is proximinal in $C_b(S;E)$ and has a continuous proximity map.*

PROOF. By Lemma 2.1, Amir and Deustch [**1**], V is a Chebyshev subspace of E and P_V is uniformly continuous in the set $\{a \in E;\ \text{dist}\ (a;V) \leq R\}$, for any $R > 0$. □

COROLLARY 2. *If E is uniformly convex, then $C_b(S;V)$ is proximinal in $C_b(S;E)$ and admits a continuous proximity map, for any closed vector subspace V of E.*

PROOF. For any closed subspace $V \subset E$, the space E is uniformly convex with respect to V. □

To state our next result we need to recall the definition of the 1 1/2-ball property: a closed subspace V of E has the *1 1/2-ball property* in E if $V \cap \overline{B}(v;\varepsilon) \cap \overline{B}(f;r) \neq \emptyset$, whenever $v \in V$, $f \in E$, $\varepsilon > 0$ and $r > 0$ are such that $||f - v|| < r + \varepsilon$ and $V \cap \overline{B}(f;r) \neq \emptyset$. This notion was introduced by D.T. Yost [**9**], who proved that when V has the 1 1/2-ball property in E, then V is proximinal and admits a continuous homogeneous proximity map s satisfying $s(a+v) = s(a) + v$, for all $a \in E$, $v \in V$.

Examples of subspaces with the 1 1/2-ball property include: M-ideals, any closed subalgebra of $C(S;\mathbf{R})$, for compact S; the space $K(\ell^1,\ell^1)$ of compact operators in ℓ^1 as a subspace of the space $L(\ell^1,\ell^1)$ of all bounded linear operators in ℓ^1.

COROLLARY 3. *If V has the 1 1/2-ball property in E, then $C_b(S;V)$ is proximinal in $C_b(S;E)$ and, for compact S, it admits a continuous proximity map.*

THEOREM 2. *Let E be a real Lindenstrauss space, S, T and U compact Hausdorff spaces $\pi : T \to U$ a continuous surjection; $V = \{g \circ \pi; g \in C(U; E)\}$. Then $C(S; V)$ is proximinal in $C(S \times T; E)$, and admits a continuous proximity map.*

PROOF. By Theorem 2.1, Yost [**9**], V has the 1 $1/2$-ball property in $C(T; E)$. It remains to apply Corollary 3 and the identification $C(S; C(T; E)) = C(S \times T; E)$. Notice that, under this identification $C(S; V)$ is the set of all continuous functions $f : S \times T \to E$ such that, for each $s \in S$, the map $f_s : T \to E$ (defined by $f_s(t) = f(s; t)$, for all $t \in T$), factors through π, i.e., there exists $g_s \in C(U; E)$ such that $f_s = g_s \circ \pi$. □

THEOREM 3. *Let S, T, U and π be as in Theorem 2, and $V = \{g \circ \pi; g \in C(U; \mathbf{C})\}$. Then $C(S; V)$ is proximinal in $C(S \times T; \mathbf{C})$ and admits a continuous proximity map.*

PROOF. By Proposition 3.2, Fakhoury [**5**], V is proximinal in $C(T; \mathbf{C})$ and admits a continuous homogeneous metric selection. □

DEFINITION 1. Let V be a closed vector subspace of a Banach space E. We say that V has property (A) if, for every $\varepsilon > 0$ and $R > 0$, there exists $\delta > 0$ such that, given $f \in E$ with $\text{dist}(f; V) \leq R$ and $w \in V$ such that $||f - w|| < R + \delta$, there exists $v \in V$ such that $||f - v|| \leq R$ and $||v - w|| \leq \varepsilon$.

Notice that, when proving that a subspace V has property (A), it suffices to consider $w = 0$ and $R = 1$.

EXAMPLE 1. If V has the 1 $1/2$-ball property in E, then V has property (A).

PROOF. Indeed, let $\varepsilon > 0$ and $R > 0$ be given. Choose $\delta = \varepsilon$. Let $f \in E$ and $w \in V$ be such that $\text{dist}(f; V) \leq R$ and $||f - w|| < R + \varepsilon$. Since V is proximinal (Yost [**9**]), $V \cap \overline{B}(f; R) \neq \emptyset$. By the 1 $1/2$-ball property, it follows that $V \cap \overline{B}(w;\varepsilon) \cap \overline{B}(f;R) \neq \emptyset$. Hence V has a property (A). □

DEFINITION 2. A Banach space E is said to be *quasi-uniformly convex* (q.u.c) *with respect to a closed subspace* V if, for every $0 < \varepsilon < 1$, there exists $0 < \tilde{\delta} = \tilde{\delta}(\varepsilon) \leq \varepsilon$ such that, given $v \in V$, there exists $w \in V$ with $||w|| \leq \varepsilon$ and such that $\overline{B}(v; 1-\tilde{\delta}) \cap \overline{B}(0;1) \subset \overline{B}(w; 1-\tilde{\delta})$.

This notion is due to Calder, Coleman and Harris [**4**].

EXAMPLE 2. If E is quasi-uniformly convex with respect to V, then V has property (A) in E.

PROOF. Let $\varepsilon > 0$ and $R > 0$ be given. Without loss of generality we may assume $R = 1$ and $\varepsilon < 1$. Choose $\varepsilon' > 0$ such that $\varepsilon' \leq \varepsilon/2$. Then $\varepsilon' < 1/2$. By Definition 2 there exists $\eta = \delta(\varepsilon')$ satisfying $\eta \leq \varepsilon'$ and the q.u.c. condition. Take $\delta = \eta/(1-\eta)$. Let $f \in E$ be given with $\mathrm{dist}(f;V) \leq 1$ and $||f|| < 1+\delta$. Since V is proximinal in E [**2**] Proposition 2.4, there is some $v \in V$ such that $||f - v|| \leq 1$. Notice that $1 = (1+\delta)(1-\eta)$. Hence $u = f/(1+\delta)$ and $y = v/(1+\delta)$ are such that $y \in V$, $u \in E$, $||u|| < 1$ and $||u - y|| \leq 1 - \eta$. By q.u.c. there exists $z \in V$ with $||z|| \leq \varepsilon'$ and $||u - z|| \leq 1 - \eta$. Let $w = (1+\delta)z$. Then $w \in V$, and $||f - w|| \leq (1+\delta)(1-\eta) = 1$. On the other hand $||w|| \leq (1+\delta)\varepsilon' = \varepsilon'/(1-\eta) \leq 2\varepsilon' \leq \varepsilon$. Hence V has property (A) in E. □

REMARK. Since, for any Banach space E, $V = E$ always has property (A) in E, any Banach space which is not quasi-uniformly convex (with respect to itself) gives a counter-example to (A) $\Rightarrow$ q.u.c. infinite-dimensional $L^1(\mu)$-spaces are not quasi-uniformly convex [**2**]. Corollary 2.7. Example 2.5 of [**2**] gives a 3-dimensional space which is not quasi-uniformly convex.

EXAMPLE 3. If E is uniformly convex, then any closed vector subspace of E has property (A).

PROOF. If E is uniformly convex, and $V \subset E$ is any closed subspace, then E is uniformly convex with respect to V, and then by Proposition 2.2 of [**2**], E is q.u.c. with respect to V. □

EXAMPLE 4. Let T be a compact Hausdorff space, and let V be a closed vector sublattice of $E = C(T; \mathbf{R})$ such that

$$\lambda = \inf\{||\delta_x|V||; x \in T\} > 0.$$

Then V has property (A).

PROOF. Let $R > 0$ and $\varepsilon > 0$ be given. Choose $\eta > 0$ such that $\eta < \lambda$ and then choose $\delta = \eta\varepsilon$. Notice that we have $\delta < \varepsilon$, because $\eta < \lambda \leq 1$. Let $f \in E$ with $\mathrm{dist}(f; V) \leq R$ and $||f|| < R + \delta$ be given. Choose $h \in V$ such that $||f - h|| \leq R$. Since V is proximinal (see Blatter [**3**]), this can be done.

For each $t \in T$, there is some $g_t \in V$ such that $0 \leq g_t \leq 1$ and $\eta < g_t(t)$. Let $V_t = \{x \in T; \eta < g_t(x)\}$. By compactness, there are $t_1, \dots, t_n$ such that T is contained in the union of the V_{t_i} $(i = 1, \dots, n)$. Let $g = \max\{g_{t_1}, \dots, g_{t_n}\}$. Then $g \in V$ and, for each $t \in T$, we have $0 < \eta < g(t) \leq 1$. Define $v = \varepsilon g$. Then $v \in V$, and $0 < \delta < v(t) \leq \varepsilon$, for all $t \in T$. Let $w = (v \wedge h) \vee (-v)$. Then $w \in V$ and $||w|| \leq ||v|| \leq \varepsilon$. We claim that $||f - w|| \leq R$. Let $x \in T$ be given.

Case 1. $|h(x)| \leq v(x)$. Then $w(x) = h(x)$ and therefore $|f(x) - w(x)| = |f(x) - h(x)| \leq ||f - h|| \leq R$.

Case 2. $h(x) > v(x)$. Then $w(x) = v(x)$ and we have $-R \leq f(x) - h(x) < f(x) - v(x) < R + \delta - v(x) < R$.

Case 3. $h(x) < -v(x)$. Then $w(x) = -v(x)$ and we have $-R = -(R + \delta) + \delta < f(x) + \delta < f(x) + v(x) < f(x) - h(x) \leq R$. □

If X is any set, we denote by $\ell_\infty(X; \mathbf{R})$ the Banach space of all bounded functions $f : X \to \mathbf{R}$ equipped with the sup-norm $||f|| = \sup\{|f(x)|; x \in X\}$, for $f \in \ell_\infty(X; \mathbf{R})$.

EXAMPLE 5. Let V be a closed vector subspace of $\ell_\infty(T; \mathbf{R})$ such that, for each $h \in V$ and $r > 0$, the function $(r \wedge h) \vee (-r)$ belongs to V. Then V has the 1 ½-ball property in $\ell_\infty(T;\mathbf{R})$. In particular, $C_b(T;\mathbf{R})$ has the 1 ½-ball property in $\ell_\infty(T;\mathbf{R})$, for any topological space T.

PROOF. Let $f \in \ell_\infty(T;\mathbf{R}), R > 0$ and $r > 0$ be given with $V \cap \overline{B}(f;R) \neq \emptyset$ and $||f|| < R + r$. Choose $h \in V$ with $||f - h|| \leq R$. Let $w = (r \wedge h) \vee (-r)$. Then $w \in V$ and $||w|| \leq r$. An argument similar to that of Example 4 shows that $||f - w|| \leq R$. (Just make $v(x) = r$ there, for all $x \in T$). □

REMARK. Let $T = [-1,1] \subset \mathbf{R}$. Then $V = \{f \in C[0,1]; f(x) = -f(-x)\}$ satisfies the hypothesis of Example 5 but not of Example 4.

DEFINITION 3. (LAU [**6**]). A closed subspace V of a Banach space E is said to be *U-proximinal* if there exists a positive function $\delta(\varepsilon)$, defined for $\varepsilon > 0$, with $\delta(\varepsilon) \to 0$ as $\varepsilon \to 0$, satisfying

$$((1+\varepsilon)B) \cap (V + B) \subset B + \delta(\varepsilon)(B \cap V)$$

for all $\varepsilon > 0$, where B denotes the closed unit ball of E.

EXAMPLE 6. If V is a U-proximinal subspace of E, then V has property (A).

PROOF. Let $\varepsilon > 0$ and $R > 0$ be given. Choose $\eta > 0$ such that $R \cdot \delta(\eta) < \varepsilon$ and then choose $\delta > 0$ such that $\delta < \eta \cdot R$.

Let $f \in E$ and $v \in V$ be given with $\text{dist}(f;V) \leq R$ and $||f - v|| < R + \delta$. By Proposition 2.3, Lau [**6**], V is proximinal. Hence $\text{dist}(f - v; V) = \text{dist}(f;V) \leq R$ implies that $f - v$ belongs to $V + RB$. Therefore $(f - v)/R$ belongs to $((1+\eta)B) \cap (V + B)$.

Since V is U-proximinal it follows that $f - v$ belongs to $R{\cdot}B + R{\cdot}\delta(\eta) \cdot (B \cap V)$, which is contained in $R \cdot B + \varepsilon(B \cap V)$. Hence $f - v = u + z$ where $||u|| \leq R$ and $z \in V$ with $||z|| \leq \varepsilon$. Let $w = v + z$. Then $w \in V$, $||f - v|| \leq R$ and $||v - w|| \leq \varepsilon$. □

Examples of U-proximinal subspace include: every closed subspace of a uniformly convex space; M-ideals; $C_b(S;\mathbf{R})$ as a subspace of $\ell_\infty(S;\mathbf{R})$; the space $K(L^1(\mu),\ell^1)$ of compact operators is a U-proximinal subspace of the space $L(L^1(\mu),\ell^1)$ of all bounded linear operators, for any σ-finite measure space (Ω,Σ,μ).

THEOREM 4. *If the subspace V has property* (A) *in E, then $C_b(S;V)$ is proximinal in $C_b(S;E)$, for any paracompact Hausdorff space S.*

PROOF. Let $f \in C_b(S;E)$, with $R = \operatorname{dist}(f;C_b(S;V)) > 0$ be given. Define a set-valued mapping φ on S by $\varphi(s) = V \cap \overline{B}(f(s);R)$, for all $s \in S$. Clearly, $\varphi(s)$ is closed and convex, for each $s \in S$. Since $\operatorname{dist}(f(s);V) \leq \operatorname{dist}(f;C_b(S;V)) = R$, and since proprerty (A) implies proximinality, it follows that $\varphi(s) \neq \emptyset$, for each $s \in S$.

We claim that φ is lower-semicontinuous. Let $s_0 \in S$, $a \in E$ and $r > 0$ be given such that $\varphi(s_0)\cap B(a;r) \neq \emptyset$. Choose w in $\varphi(s_0)\cap B(a;r)$ and then choose $t > 0$ such that $||w-a|| < t < r$. Let $\varepsilon = r-t > 0$, and let $\delta > 0$ be given by property (A) applied to $\varepsilon > 0$ and $R > 0$. Notice that $\operatorname{dist}(f(s_0);V) \leq R$ and $||f(s_0) - w|| \leq R < R+\delta$. By continuity there is a neighborhood N of s_0 in S such that $||f(s) - w|| < R+\delta$ for all $s \in N$. Fix $s \in N$. By property (A) there exists $v_s \in V$ such that $||f(s)-v_s|| \leq R$ and $||v_s - w|| \leq \varepsilon$. Then $||v_s - a|| = ||w-a+v_s-w|| \leq ||w-a|| + \varepsilon < t+\varepsilon = r$. Hence $v_s \in \varphi(s)\cap B(a;r)$. Consequently, $\varphi(s)\cap B(a;r) \neq \emptyset$ for all $s \in N$, and φ is lower-semicontinuous. By Michael's selection theorem [8], there is some $g \in C_b(S;V)$ such that $||g(s) - f(s)|| \leq R$ for all $s \in S$. □

COROLLARY 4. *If S is a paracompact Hausdorff space, then $C_b(S;V)$ is a proximinal subspace of $C_b(S;E)$ in the following cases*:

(a) *V has the 1 1/2-ball property E*;

(b) *V is an M-ideal of E*;

(c) *$V = K(\ell^1,\ell^1)$ and $E = L(\ell^1,\ell^1)$*;

(d) *V is a U-proximinal subspace of E*;

(e) *$V = C_b(T;\mathbf{R})$ and $E = \ell_\infty(T;\mathbf{R})$*;

(f) *$V = K(L^1(\mu),\ell^1)$ and $E = L(L^1(\mu),\ell^1)$, for any σ-finite measure* *space* (Ω, Σ, μ); *and*

(g) E *is quasi-uniformly convex with respect to* V.

THEOREM 5. *Let* S *and* T *be compact Hausdorff spaces, and let* V *be a closed vector sublattice of* $C(T; \mathbf{R})$ *such that*

$$\lambda = \inf\{||\delta_x|V||;\ x \in T\} > 0.$$

Then $C(S; V)$ *is proximinal in* $C(S \times T; \mathbf{R})$.

PROOF. Identify the Banach spaces $C(S \times T; \mathbf{R})$ and $C(S; C(T; \mathbf{R}))$. The result follows from Example 4 and Theorem 4. □

COROLLARY 5. *Let* S *and* T *be compact Hausdorff spaces, and let* A *be a closed subalgebra of* $C(T; \mathbf{R})$ *such that, given* $t \in T$, *there is* $v \in A$ *such that* $v(t) \neq 0$. *Then* $C(S; A)$ *is proximinal in* $C(S \times T; \mathbf{R})$.

PROOF. Since A is an algebra, $||\delta_x|A|| = 0$ or $||\delta_x|A|| = 1$, for every $x \in T$. By hypothesis $||\delta_x|A|| \neq 0$ for all $x \in T$. It is well known that any closed subalgebra of $C(T; \mathbf{R})$ is a closed sublattice. □

THEOREM 6. *Let* S *and* T *be compact Hausdorff spaces, and let* V *be a closed subspace of* $C(T; \mathbf{R})$ *as in Example* 5. *Then* $C(S; V)$ *is proximinal in* $C(S \times T; \mathbf{R})$.

Acknowledgements. This paper was written while the first Author was a Fellow of the John Simon Guggenheim Memorial Foundation.

REFERENCES

1. D. Amir and F. Deutsch, *Approximation by certain subspaces in the Banach space of continuous vector valued functions*, J. Approx. Theory **27** (1979), 254-270.

2. ——— , J. Mach and K. Saatkamp, *Existence of Chebyshev centers, best n-nets and best compact approximants*, Trans. Amer. Math. Soc. **271** (1982), 513-524.

3. J. Blatter, *Grothendieck spaces in approximation theory*, Memoirs Amer. Math. Soc. **120**, Amer. Math. Soc., Providence, Rhode Island, 1972.

4. J.R. Calder, W.P. Coleman and R.L. Harris, *Centers of infinite bounded sets in a normed space*, Canad. J. Math. **25** (1973), 986-999.

5. H. Fakhoury, *Existence d'une projection continue de meilleure approximation dans certain espaces de Banach*. J. Math. Pures et Appl. **53** (1974), 1-16.

6. K.S. Lau, *On a sufficient condition for proximity*, Trans. Amer. Math. Soc. **251** (1979), 343-356.

7. W.A. Light and E.W. Cheney, *Approximation theory in tensor product spaces*, Lecture Notes in Mathematics **1169**, Springer Verlag, Berlin 1985.

8. E. Michael, *Selected selection theorems*, Amer. Math. Monthly **63** (1956), 233-238.

9. D.T. Yost, *Best approximation and intersections of balls in Banach spaces*, Bull. Austral. Math. Soc. **20** (1979), 285-300.

Departamento de Matemática, Universidade Estadual de Campinas, Campinas, SP Brazil